Article

Nonlinear Gronwall–Bellman Type Inequalities and Their Applications

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Abstract: In this paper, some nonlinear Gronwall–Bellman type inequalities are established. Then, the obtained results are applied to study the Hyers–Ulam stability of a fractional differential equation and the boundedness of solutions to an integral equation, respectively.

1. Introduction

The study of Gronwall–Bellman inequalities has been paid much attention and developed at a high rate in the last three decades. These inequalities play an important role in many fields. They are applied to the investigation of the stability, boundedness, global existence, uniqueness, and continuous dependence on the initial or boundary value and parameters of solutions to differential equations, integral equations, as well as difference equations [1–11]. They are also used to study the regularized family of models for homogeneous incompressible two-phase flows [12], the state of the high nonlinear circuit [13], the Cousin problems and the emergence of the Sheaf Concept [14].

Recently, Willett [15] discussed the linear inequality:

$$u(t) \leq w_0(t) + \sum_{i=1}^{n} w_i(t) \int_0^t v_i(s)u(s)\mathrm{d}s \ \ (t \in I) \tag{1}$$

and Lin [16] extended the study to the linear inequality as follows:

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i-1}u(s)\mathrm{d}s \tag{2}$$

Several generalizations of the Gronwall inequality were established and then applied to prove the uniqueness of solutions for fractional differential equations with various derivatives.

In this paper, we are concerned with the following nonlinear Gronwall–Bellman-type inequality:

$$u^p(x) \leq a(x) + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t)g_i(t,u(t))\mathrm{d}t$$
$$+ \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i-1} f_i(t,u(t))\mathrm{d}t \tag{3}$$

Some new results are obtained and then applied to investigate the qualitative properties of differential and integral equations.

This paper is organized as follows: In Section 2, we introduce some definitions and notations. Some nonlinear Gronwall–Bellman-type inequalities are presented in Section 3. In Section 4, the obtained results are applied to prove the Hyers–Ulam stability of a fractional differential equation and the boundedness of the solutions to a integral equation.

2. Preliminaries

In this section, we recall some standard definitions and notations.

Lemma 1. *(see [17]). Assume that $a \geq 0, p \geq q \geq 0$ with $p \neq 0$. Then, for any $b > 0$, we have:*

$$a^{\frac{q}{p}} \leq \frac{q}{p} b^{\frac{q-p}{p}} a + \frac{p-q}{p} b^{\frac{q}{p}} \tag{4}$$

Lemma 2. *(see [15]) Suppose that:*

$$u(t) \leq w_0(t) + \sum_{i=1}^{n} w_i(t) \int_0^t v_i(s) u(s) \mathrm{d}s \quad (t \in I) \tag{5}$$

where $v_i w_j (i = 1, 2 \cdots, n; j = 0, 1, \cdots, n)$ and $v_i u (i = 1, 2, \cdots, n)$ are locally integrable on I, all functions are assumed non-negative. Then:

$$u \leq E_n w_0 \tag{6}$$

where $E_i (i = 0, 1, \cdots, n)$ is defined inductively as the composition of $i + 1$ functional operators; i.e., $E_i = D_i D_{i-1} \cdots D_0$, where:

$$D_0 w = w \tag{7}$$

$$D_j w = w + (E_{j-1} w_j)(exp \int_0^t v_j E_{j-1} w_j) \int_0^t v_j w \mathrm{d}s \quad (j = 1, 2, \cdots, n) \tag{8}$$

3. Main Results

In this section, we will establish some nonlinear Gronwall–Bellman-type inequalities. The first result is the following:

Theorem 1. *Suppose that $p \geq 1, \beta_i > 0$ are constants, $f_i \in C(R_+ \times R_+, R_+)$ with $0 \leq f_i(x, u) - f_i(x, v) \leq l_i(u - v)$ for $u \geq v \geq 0$, where l_i is the Lipschitz constants, and all the functions are nonnegative and continuous on [0,T), with $b_i(i = 1, 2, \cdots, n)$ are bounded and nondecreasing functions. If the following inequality is satisfied:*

$$u^p(x) \leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, u(t)) \mathrm{d}t \tag{9}$$

then:

$$
\begin{aligned}
u(x) \leq \Big(\tilde{a}(x) + \sum_{k=1}^{\infty} \Big(\sum_{1', 2', \cdots k'=1}^{n} \big(\frac{1}{p} b^{\frac{1-p}{p}} \big)^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \cdot \\
\int_0^x (x-t)^{\sum_{i=1}^{k} \beta_{i'} - 1} \tilde{a}(t) \mathrm{d}t \big) \Big)^{\frac{1}{p}}
\end{aligned}
\tag{10}
$$

for any $b > 0$ and $t \in [0, T]$, where:

$$\tilde{a}(x) = a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \tag{11}$$

Proof. Denote $u^p(x) = v(x)$, then we have $u(x) = v^{\frac{1}{p}}(x)$:

$$v(x) \leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, v^{\frac{1}{p}}(t)) \mathrm{d}t \tag{12}$$

According to Lemma 1:

$$
\begin{aligned}
v(x) &\leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&= a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} [f_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) \\
&\quad - f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) + f_i(t, \frac{p-1}{p} b^{\frac{1}{p}})] \mathrm{d}t \\
&\leq a(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} v(t) \mathrm{d}t \\
&\quad + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&= \tilde{a}(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} v(t) \mathrm{d}t
\end{aligned}
\tag{13}
$$

Now let:

$$B\varphi(x) = \sum_{i=1}^{n} \frac{1}{p} b^{\frac{1-p}{p}} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} \varphi(t) \mathrm{d}t \tag{14}$$

$x \in [0, T)$, for locally integrable functions φ. Then:

$$v(x) \leq \tilde{a}(x) + Bv(x). \tag{15}$$

By mathematical induction method, we have:

$$
\begin{aligned}
v(x) &\leq \tilde{a}(x) + B(\tilde{a}(x) + Bv(x)) \\
&= \tilde{a}(x) + B\tilde{a}(x) + B^2 v(x) \\
&\leq \tilde{a}(x) + B\tilde{a}(x) + B^2(\tilde{a}(x) + Bv(x)) \\
&= \tilde{a}(x) + B\tilde{a}(x) + B^2 \tilde{a}(x) + B^3 v(x) \\
&\leq \cdots \\
&\leq \sum_{m=0}^{k-1} B^m \tilde{a}(x) + B^k v(x).
\end{aligned}
\tag{16}
$$

Let $r = \frac{1}{p} b^{\frac{1-p}{p}}$. We assert that:

$$B^k v(x) \leq \sum_{1', 2', \cdots k'=1}^{n} r^k \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \int_0^x (x-t)^{\sum_{i=1}^{k} \beta_{i'} - 1} v(t) \mathrm{d}t \tag{17}$$

for any $k \in N$ and $B^k v(x) \to 0$ as $k \to \infty$ for each x in $0 \leq x < T$.

In fact, (i) If $k = 1$, then:

$$
\begin{aligned}
Bv(x) &\leq \sum_{1'=1}^{n} r \frac{[l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\beta_{i'})} \int_0^x (x-t)^{\beta_{i'}-1} v(t) \mathrm{d}t \\
&= \sum_{1'=1}^{n} r l_{i'} b_{i'}(x) \int_0^x (x-t)^{\beta_{i'}-1} v(t) \mathrm{d}t \\
&= Bv(x)
\end{aligned}
\tag{18}
$$

So we know the inequality (17) holds for $k = 1$.

(ii) Assume that the inequality (17) holds for $k = j$; that is:

$$
B^j v(x) \leq \sum_{1',2',\cdots j'=1}^{n} r^j \frac{\prod_{i=1}^{j}[l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{j} \beta_{i'})} \int_0^x (x-t)^{\sum_{i=1}^{j} \beta_{i'}-1} v(t) \mathrm{d}t
\tag{19}
$$

(iii) For $k = j + 1$, we have:

$$
\begin{aligned}
B^{j+1} v(x) &= B(B^j v(x)) \\
&\leq \sum_{i=1}^{n} r l_i b_i(x) \int_0^x (x-t)^{\beta_i-1} \cdot \Big(\\
&\quad \sum_{1',2',\cdots j'=1}^{n} r^j \frac{\prod_{i=1}^{j}[l_{i'} b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{j} \beta_{i'})} \cdot \int_0^t (t-s)^{\sum_{i=1}^{j} \beta_{i'}-1} v(s) \mathrm{d}s \Big) \mathrm{d}t
\end{aligned}
\tag{20}
$$

Owing to the monotonicity of $b_i (i = 1, 2, \cdots, n)$, we get:

$$
\begin{aligned}
B^{j+1} v(x) &\leq \sum_{i=1}^{n} r l_i b_i(x) \cdot \sum_{1',2',\cdots j'=1}^{n} r^j \frac{\prod_{i=1}^{j}[l_{i'} b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{j} \beta_{i'})} \cdot \\
&\quad \int_0^x (x-t)^{\beta_i-1} \cdot \int_0^t (t-s)^{\sum_{i=1}^{j} \beta_{i'}-1} v(s) \mathrm{d}s \mathrm{d}t
\end{aligned}
\tag{21}
$$

Interchanging the order of integration, we have:

$$
\begin{aligned}
B^{j+1} v(x) &\leq r^{j+1} \sum_{i=1}^{n} l_i b_i(x) \cdot \sum_{1',2',\cdots j'=1}^{n} \frac{\prod_{i=1}^{j}[l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{j} \beta_{i'})} \cdot \\
&\quad \frac{\Gamma(\beta_i) \Gamma(\sum_{i=1}^{j} \beta_{i'})}{\Gamma(\beta_i + \sum_{i=1}^{j} \beta_{i'})} \cdot \int_0^x (x-s)^{\sum_{i=1}^{j} \beta_{i'} + \beta_i - 1} v(s) \mathrm{d}s \\
&= r^{j+1} \sum_{i=1}^{n} \sum_{1',2',\cdots j'=1}^{n} \frac{[l_i b_i(x) \Gamma(\beta_i)] \cdot \prod_{i=1}^{j}[l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\beta_i + \sum_{i=1}^{j} \beta_{i'})} \cdot \\
&\quad \int_0^x (x-s)^{\sum_{i=1}^{j} \beta_{i'} + \beta_i - 1} v(s) \mathrm{d}s
\end{aligned}
\tag{22}
$$

Let $i = (j+1)'$, then:

$$B^{j+1}v(x) \le r^{j+1} \sum_{(j+1)'=1}^{n} \sum_{1',2',\cdots j'=1}^{n} \frac{[l_{(j+1)'}b_{(j+1)'}(x)\Gamma(\beta_{(j+1)'})] \cdot \prod_{i=1}^{j}[b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\beta_{(j+1)'} + \sum_{i=1}^{j}\beta_{i'})} \cdot$$

$$\int_0^x (x-s)^{\sum_{i=1}^{j}\beta_{i'} + \beta_{(j+1)'} - 1}v(s)\mathrm{d}s \tag{23}$$

$$= r^{j+1} \sum_{1',2',\cdots(j+1)'=1}^{n} \frac{\prod_{i=1}^{j+1}[l_{i'}b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{j+1}\beta_{i'})} \int_0^x (x-s)^{\sum_{i=1}^{j+1}\beta_{i'} - 1}v(s)\mathrm{d}s$$

This implies that the inequality (17) holds for $k = j + 1$. Hence, it holds for any $k \in N$.

Since $b_i(i = 1, 2, \cdots, n)$ are bounded, $b_i < M_i(M_i > 0)(i = 1, 2, \cdots, n)$:

$$B^k v(x) \le \sum_{1',2',\cdots k'=1}^{n} \frac{[l_{1'}b_{1'}(t)\Gamma(\beta_{1'})][b_{2'}(x)\Gamma(\beta_{2'})] \cdots [b_{k'}(x)\Gamma(\beta_{k'})]}{\Gamma(\beta_{1'} + \beta_{2'} + \cdots + \beta_{k'})} \cdot$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k}\beta_{i'} - 1}v(t)\mathrm{d}t$$

$$\le \sum_{1',2',\cdots k'=1}^{n} \frac{(l_{1'}l_{2'}\cdots l_{k'})(M_{1'}M_{2'}\cdots M_{k'}) \cdot (\Gamma(\beta_{1'})\Gamma(\beta_{2'})\cdots\Gamma(\beta_{k'}))}{\Gamma(\beta_{1'} + \beta_{2'} + \cdots + \beta_{k'})} \cdot \tag{24}$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k}\beta_{i'} - 1}v(t)\mathrm{d}t$$

and according to the property of the Gamma function, $B^k v(x) \to 0$ as $n \to \infty$ for $x \in [0, T)$, then:

$$v(x) \le \tilde{a}(x) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p}b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k}[l_{i'}b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\right) \cdot$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k}\beta_{i'} - 1}\tilde{a}(t)\mathrm{d}t). \tag{25}$$

Hence:

$$u(x) \le (\tilde{a}(x) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p}b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k}[l_{i'}b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\right) \cdot$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k}\beta_{i'} - 1}\tilde{a}(t)\mathrm{d}t)^{\frac{1}{p}} \tag{26}$$

This completes the proof. □

By Theorem 1, the main result in [16] is a special case of Theorem 1 for $p = 1$, $f_i(t, u(t)) = u(t)$.

Corollary 1. *(see [16]). For any $t \in [0, T)$:*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i - 1}u(s)\mathrm{d}s \tag{27}$$

where all the functions are non-negative and continuous. The constants $\beta_i > 0$, $b_i(i = 1, 2, \cdots, n)$ are the bounded and monotonic increasing functions on [0,T). Then:

$$u(t) \le a(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})} \int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'} - 1}a(s)\mathrm{d}s\right) \tag{28}$$

Corollary 2. *Under the hypothesis of Theorem 1, if $a(x)$ is increasing on $[0.T]$, then:*

$$u(x) \leq (\tilde{a}(T) \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot$$

$$\frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot T^{\sum_{i=1}^{k} \beta_{i'}}))^{\frac{1}{p}} \tag{29}$$

Proof. Since $a(x)$ is increasing, $\tilde{a}(x)$ is also increasing:

$$u(x) \leq (\tilde{a}(x) + \sum_{k=1}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \cdot$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k} \beta_{i'} - 1} \tilde{a}(t) \mathrm{d}t))^{\frac{1}{p}}$$

$$\leq \tilde{a}^{\frac{1}{p}}(x) (1 + \sum_{k=1}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \cdot$$

$$\int_0^x (x-t)^{\sum_{i=1}^{k} \beta_{i'} - 1} \mathrm{d}t))^{\frac{1}{p}} \tag{30}$$

$$= (\tilde{a}(x) \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}}))^{\frac{1}{p}}$$

$$\leq (\tilde{a}(T) \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot T^{\sum_{i=1}^{k} \beta_{i'}}))^{\frac{1}{p}}$$

The proof is completed. □

Theorem 2. *Under the conditions of Corollary 2, if $\omega_i(x)(i = 1, 2, \cdots, n)$ are bounded and monotonic increasing. $g_i \in C(R_+ \times R_+, R_+)$ with $0 \leq g_i(x, u) - g_i(x, v) \leq T_i(u - v)$ for $u \geq v \geq 0$, where T_i is the Lipschitz constant. If the following inequality is satisfied:*

$$u^p(x) \leq a(x) + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) g_i(t, u(t)) \mathrm{d}t$$

$$+ \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, u(t)) \mathrm{d}t \tag{31}$$

then:

$$u(x) \leq (E_n \tilde{a}(x) \cdot \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot$$

$$\frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}}))^{\frac{1}{p}} \tag{32}$$

where $E_i(i = 0, 1, \cdots, n)$ is defined inductively as the composition of $i + 1$ functional operators; i.e., $E_i = D_i D_{i-1} \cdots D_0$, where:

$$D_0 w = w \tag{33}$$

$$D_j w = w + (E_{j-1} w_j)(exp \int_0^t v_j E_{j-1} w_j) \int_0^t v_j w \mathrm{d}s \quad (j = 1, 2, \cdots, n) \tag{34}$$

Proof. Denote $u^p(x) = v(x), u(x) = v^{\frac{1}{p}}(x)$. Then:

$$
\begin{aligned}
v(x) &\leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, v^{\frac{1}{p}}(t)) \mathrm{d}t \\
&\quad + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) g_i(t, v^{\frac{1}{p}}(t)) \mathrm{d}t \\
&\leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&\quad + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) g_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&\leq a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} [f_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) \\
&\quad - f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) + f_i(t, \frac{p-1}{p} b^{\frac{1}{p}})] \mathrm{d}t + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) \cdot \\
&\quad [g_i(t, \frac{1}{p} b^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} b^{\frac{1}{p}}) - g_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) + g_i(t, \frac{p-1}{p} b^{\frac{1}{p}})] \mathrm{d}t \\
&\leq a(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} v(t) \mathrm{d}t \\
&\quad + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&\quad + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} T_i \omega_i(x) \int_0^x h_i(t) v(t) \mathrm{d}t + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) g_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&= \tilde{a}(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} v(t) \mathrm{d}t \\
&\quad + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} T_i \omega_i(x) \int_0^x h_i(t) v(t) \mathrm{d}t
\end{aligned}
\tag{35}
$$

where:

$$
\begin{aligned}
\tilde{a}(x) &= a(x) + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} f_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t \\
&\quad + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t) g_i(t, \frac{p-1}{p} b^{\frac{1}{p}}) \mathrm{d}t
\end{aligned}
\tag{36}
$$

Let:

$$
z(x) = \tilde{a}(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} T_i \omega_i(x) \int_0^x h_i(t) v(t) \mathrm{d}t
\tag{37}
$$

Then:

$$
v(x) \leq z(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} l_i b_i(x) \int_0^x (x-t)^{\beta_i - 1} v(t) \mathrm{d}t
\tag{38}
$$

By Inequality (37), we derive that $z(x)$ is nonnegative and increasing. According to Corollary 1, we have:

$$
v(x) \leq z(x) \sum_{k=0}^{\infty} \left(\sum_{1',2',\cdots k'=1}^{n} \left(\frac{1}{p} b^{\frac{1-p}{p}} \right)^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot x^{\sum_{i=1}^{k} \beta_{i'}} \right)
\tag{39}
$$

Combing (39) with (37):

$$z(x) \leq \tilde{a}(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} T_i \omega_i(x) \int_0^x h_i(t) \cdot$$
$$(z(t) \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}})) \mathrm{d}t \tag{40}$$
$$\leq \tilde{a}(x) + \frac{1}{p} b^{\frac{1-p}{p}} \sum_{i=1}^{n} T_i \omega_i(x) \int_0^x \tilde{h}_i(t) z(t) \mathrm{d}t$$

where:

$$\tilde{h}_i(x) = h_i(t) \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} (\frac{1}{p} b^{\frac{1-p}{p}})^k \cdot \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}}) \tag{41}$$

By Lemma 2, we obtain:

$$z(x) \leq E_n \tilde{a}(x) \tag{42}$$

By inequalities (39) and (42), we obtain (32). The proof is completed. □

Corollary 3. *Under the hypothesis of Theorem 2, if $p = 1$, then:*

$$u(x) \leq E_n \tilde{a}(x) \cdot \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}}) \tag{43}$$

4. Applications

In this section, we present two examples as applications of our results.

Example 1. *The following initial value problems of fractional differential equation was considered in [16]:*

$$\sum_{i=1}^{n} D_R^{\beta_i} u(t) = f(t, u(t)),$$
$$\sum_{i=1}^{n} D_R^{1-\beta_i} u(t)|_{t=0} = \delta, \tag{44}$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1, t \in [0, T)$, f satisfies $|f(t, u(t)) - f(t, v(t))| \leq l|u(t) - v(t)|$; l is the Lipschitz constant. D_R^{β} and I_R^{β} denote the Riemann–Liouville fractional derivative and fractional integral operators, respectively. The uniqueness of solutions was proved by Gronwall–Bellman inequality.

In this section, we study the Hyers–Ulam stability of this initial value problem.

As we know, if $u(t)$ is a solution of the differential Equation (44), then $u(t)$ satisfies the following integral equation:

$$u(t) = I_R^{\beta_n} f(t, u(t)) - \sum_{i=1}^{n-1} I_R^{\beta_n - \beta_i} u(t) - \delta \cdot \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)} \tag{45}$$

Theorem 3. *If $u_\varepsilon(t) \in C[0, T]$ satisfies:*

$$|\sum_{i=1}^{n} D_R^{\beta_i} u_\varepsilon(t) - f_\varepsilon(t, u_\varepsilon(t))| \leq \varepsilon ,$$

then there exists a solution of (44) such that $|u(t) - u_\varepsilon(t)| \leq k\varepsilon$ for all $t \in [0, T]$. Where:

$$k = \frac{\Gamma(\beta_n)}{\Gamma(\beta_n + 1)} (1 + \sum_{k=1}^{\infty} \sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot T^{\sum_{i=1}^{k} \beta_{i'}} \tag{46}$$

$b_{i'}(t) = \frac{1}{\Gamma(\beta_n - \beta_i)}.$

Proof. According to (45), we know that $u_\varepsilon(t)$ satisfies:

$$-I_R^{\beta_n}\varepsilon \le u_\varepsilon(t) - [I_R^{\beta_n}f_\varepsilon(t, u_\varepsilon(t)) - \sum_{i=1}^{n-1} I_R^{\beta_n - \beta_i}u_\varepsilon(t) - \delta \cdot \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)}] \le I_R^{\beta_n}\varepsilon. \tag{47}$$

Then:

$$\begin{aligned}
|u(t) - u_\varepsilon(t)| &\le |I_R^{\beta_n}f(t, u(t)) - \sum_{i=1}^{n-1} I_R^{\beta_n - \beta_i}u(t) \\
&\quad - I_R^{\beta_n}f_\varepsilon(t, u_\varepsilon(t)) + \sum_{i=1}^{n-1} I_R^{\beta_n - \beta_i}u_\varepsilon(t)| \\
&\le I_R^{\beta_n}\varepsilon + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n - \beta_i)} \int_0^t (t-s)^{\beta_n - \beta_i - 1}|u(s) - u_\varepsilon(s)|ds
\end{aligned} \tag{48}$$

By Theorem 1, we have:

$$\begin{aligned}
|u(t) - u_\varepsilon(t)| &\le \varepsilon \frac{t_n^\beta}{\Gamma(\beta_n + 1)} + \sum_{k=1}^{\infty} \sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})} \cdot \\
&\quad \int_0^t (t-s)^{\Sigma_{i=1}^{k}\beta_{i'} - 1)} \cdot I_R^{\beta_n}\varepsilon ds \\
&\le \varepsilon \frac{\Gamma(\beta_n)}{\Gamma(\beta_n + 1)}(1 + \sum_{k=1}^{\infty} \sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'} + 1)} \cdot T^{\Sigma_{i=1}^{k}\beta_{i'}})
\end{aligned} \tag{49}$$

where $b_{i'}(t) = \frac{1}{\Gamma(\beta_n - \beta_i)}$. This completes the proof. □

Example 2. *Consider the integral equation as follows:*

$$\begin{aligned}
u(x) = u(0) &+ \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t)g_i(t, u(t))dt \\
&+ \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1}f_i(t, u(t))dt
\end{aligned} \tag{50}$$

This includes the integer and fractional integral parts.

We assert the solution of this integral equation is bounded on $[0, T]$, provided $\omega_i, g_i, f_i(i = 1, 2, \cdots, n)$ satisfy the assumptions of Corollary 3.

Theorem 4. *Let u be a solution of (50) on $[0, T]$. If $\omega_i, g_i, f_i(i = 1, 2, \cdots, n)$ satisfy the assumptions of Corollary 7, then for $x \in [0, T]$:*

$$u(x) \le E_n u(0) \cdot \sum_{k=0}^{\infty} (\sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k}[l_{i'}b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'} + 1)} \cdot T^{\Sigma_{i=1}^{k}\beta_{i'}}) \tag{51}$$

Proof. If u is a solution of (50), then by Corollary 3:

$$
\begin{aligned}
|u(x)| &\leq |u(0)| + \sum_{i=1}^{n} \omega_i(x) \int_0^x h_i(t)|g_i(t, u(t))|\mathrm{d}t \\
&\quad + \sum_{i=1}^{n} b_i(x) \int_0^x (x-t)^{\beta_i - 1} |f_i(t, u(t))|\mathrm{d}t \\
&\leq E_n u(0) \cdot \sum_{k=0}^{\infty} \Big(\sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot t^{\sum_{i=1}^{k} \beta_{i'}} \Big) \\
&\leq E_n u(0) \cdot \sum_{k=0}^{\infty} \Big(\sum_{1',2',\cdots k'=1}^{n} \frac{\prod_{i=1}^{k} [l_{i'} b_{i'}(x)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'} + 1)} \cdot T^{\sum_{i=1}^{k} \beta_{i'}} \Big)
\end{aligned}
\tag{52}
$$

□

Remark 1. *The result of Corollary 3 also can be used to prove the uniqueness of solutions to fractional differential equations.*

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