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Best Proximity Point Results in Non-Archimedean Modular Metric Space

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Abstract: In this paper, we introduce the new notion of Suzuki-type $(\alpha, \beta, \theta, \gamma)$-contractive mapping and investigate the existence and uniqueness of the best proximity point for such mappings in non-Archimedean modular metric space using the weak P_λ-property. Meanwhile, we present an illustrative example to emphasize the realized improvements. These obtained results extend and improve certain well-known results in the literature.

Keywords: best proximity point; fixed point; modular metric space; weak P_1-property

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1. Introduction and Preliminaries

Modular metric spaces are a natural and interesting generalization of classical modulars over linear spaces, like Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii spaces and others. The concept of modular metric spaces was introduced in [1,2]. Here, we look at modular metric spaces as the nonlinear version of the classical one introduced by Nakano [3] on vector spaces and modular function spaces introduced by Musielak [4] and Orlicz [5].

Recently, many authors studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (e.g., lithium polymethacrylate). A perfect model for these fluids is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case that p is a function [6].

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function; for simplicity, we will write:

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1. *[1,2] A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:*

(i) $x = y$ *if and only if* $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$ *and* $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$ *and* $x, y, z \in X$.

If in the above definition, we utilize the condition:

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i), then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied:

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, ω is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$, the inequality holds:

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y).$$

Remark 1. *Note that if ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x, y)$ is decreasing on $(0, +\infty)$ for all $x, y \in X$. That is, if $0 < \mu < \lambda$, then:*

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 2. *References [1,2] suppose that ω be a pseudomodular on X and $x_0 \in X$ and fixed. Therefore, the two sets:*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad as \quad \lambda \to +\infty\}$$

and:

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \quad such\ that \quad \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X_ω^* are called modular spaces (around x_0).

It is evident that $X_\omega \subset X_\omega^*$, but this inclusion may be proper in general. Assume that ω is a modular on X; from [1,2], we derive that the modular space X_ω can be equipped with a (nontrivial) metric, induced by ω and given by:

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \quad \text{for all} \quad x, y \in X_\omega.$$

Note that if ω is a convex modular on X, then according to [1,2], the two modular spaces coincide, i.e., $X_\omega^* = X_\omega$, and this common set can be endowed with the metric d_ω^* given by:

$$d_\omega^*(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \quad \text{for all} \quad x, y \in X_\omega.$$

Such distances are called Luxemburg distances.

Example 2.1 presented by Abdou and Khamsi [7] is an important motivation for developing the modular metric spaces theory. Other examples may be found in [1,2].

Definition 3. *Reference [8] assume X_ω to be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Therefore:*

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n, x) \to 0$, as $n \to +\infty$ for all $\lambda > 0$. x will be called the ω-limit of (x_n).
(2) $(x_n)_{n\in\mathbb{N}}$ is called ω-Cauchy if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to +\infty$ for all $\lambda > 0$.
(3) M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.
(4) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.
(5) M is called ω-bounded if for all $\lambda > 0$, we have $\delta_\omega(M) = \sup\{\omega_\lambda(x, y); x, y \in M\} < +\infty$.

Recently Paknazar et al. [9] introduced the following concept.

Definition 4. *If in Definition 1, we replace (iii) by:*

$$(iv)\ \omega_{\max\{\lambda,\mu\}}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$$

for all $\lambda, \mu > 0$ *and* $x, y, z \in X$

Then, X_ω *is called the non-Archimedean modular metric space. Since (iv) implies (iii), every non-Archimedean modular metric space is a modular metric space.*

One of the most important generalizations of Banach contraction mappings was given by Geraghty [10] in the following form.

Theorem 1 (Geraghty [10])**.** *Suppose that* (X, d) *is a complete metric space and* $T : X \to X$ *is self-mapping. Suppose that there exists* $\beta : [0, +\infty) \to [0, 1)$ *satisfying the condition:*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty.$$

If T satisfies the following inequality:

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{ for all } x, y \in X, \tag{1}$$

hence T has a unique fixed point.

Moreover, Kirk [11] explored some significant generalizations of the Banach contraction principle to the case of non-self mappings. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is called a k-contraction if there exists $k \in [0, 1)$, such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in A$. Evidently, k-contraction coincides with Banach contraction mapping if we take $A = B$.

Furthermore, a non-self contractive mapping may not have a fixed point. In this case, we try to find an element x such that $d(x, Tx)$ is minimum, i.e., x and Tx are in close proximity to each other. It is clear that $d(x, Tx)$ is at least $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. We are interested in investigating the existence of an element x such that $d(x, Tx) = d(A, B)$. In this case, x is a best proximity point of the non-self-mapping T. Evidently, a best proximity point reduces to a fixed point T as a self-mapping.

The reader can refer to [12–16]. Note that best proximity point theorems furnish an approximate solution to the equation $Tx = x$, when there are not any fixed points for T.

Here, we collect some notions and concepts that will be utilized throughout the rest of this work. We denote by A_0 and B_0 the following sets:

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned} \tag{2}$$

In 2003, Kirk et al. [12] established sufficient conditions for determining when the sets A_0 and B_0 are nonempty.

Furthermore, in [14], the authors proved that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space satisfies the P-property. Clearly for any nonempty subset A of (X, d), the pair (A, A) has the P-property.

Recently, Zhang et al. [16] introduced the following notion and showed that it is weaker than the P-property.

Definition 5. *Let* (A, B) *be a pair of nonempty subsets of a metric space* (X, d) *with* $A_0 \neq \emptyset$*. Then, the pair* (A, B) *is said to have the weak P-property if and only if for any* $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0$:

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2). \tag{3}$$

Finally, we recall the following result of Caballero et al. [17].

Theorem 2. *Assume that (A, B) is a pair of nonempty closed subsets of a complete metric space (X, d), such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then, there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.*

Recently, Kumam et al. [18] introduced the useful notion of triangular α-proximal admissible mapping as follows. See also [19]:

Definition 6 (Reference [18]). *Let A and B be two nonempty subsets ofa metric space (X, d) and $\alpha : A \times A \to [0, +\infty)$ be a function. We say that a non-self-mapping $T : A \to B$ is triangular α-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$:*

$$\text{(T1)} \begin{cases} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \implies \alpha(u_1, u_2) \geq 1,$$

$$\text{(T2)} \begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \implies \alpha(x, y) \geq 1.$$

Let Θ denote the set of all functions $\theta : R^{+^4} \to R^+$ satisfying:
(Θ_1) θ is continuous and increasing in all of its variables;
(Θ_2) $\theta(t_1, t_2, t_3, t_4) = 0$ iff $t_1.t_2.t_3.t_4 = 0$.
For more details on Θ, see [20].

Let $\mathcal{F}$ denote the set of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty.$$

2. Best Proximity Point Results

At first, we introduce the following concept, which will be suitable for our main Theorem.

Definition 7. *Suppose that (A, B) is a pair of nonempty subsets of a modular metric space X_ω with $A_0^\lambda \neq \emptyset$ for all $\lambda > 0$. We say the pair (A, B) has the weak P_λ-property if and only if for any $x_1, x_2 \in A_0$, $y_1, y_2 \in B_0$ and $\lambda > 0$:*

$$\omega_\lambda(x_1, y_1) = \omega_\lambda(A, B) \text{ and } \omega_\lambda(x_2, y_2) = d(A, B) \Rightarrow \omega_\lambda(x_1, x_2) \leq \omega_\lambda(y_1, y_2), \tag{4}$$

where:

$$\omega_\lambda(A, B) =: \inf\{\omega_\lambda(x, y)|\ x \in A \text{ and } y \in B\},$$

$$A_0^\lambda =: \{x \in A : \omega_\lambda(x, y) = \omega_\lambda(A, B) \text{ for some } y \in B\}.$$

Now, let us introduce the concept of Suzuki-type $(\alpha, \beta, \theta, \gamma)$-contractive mapping.

Definition 8. *Let A and B be two nonempty subsets of a modular metric space X_ω where $A_0^\lambda \neq \emptyset$ for all $\lambda > 0$ and $\alpha : X_\omega \times X_\omega \to [0, \infty)$ is a function. A mapping $T : A \to B$ is said to be a Suzuki-type $(\alpha, \beta, \theta, \gamma)-$contractive mapping if there exists $\beta \in \mathcal{F}$ and $\theta \in \Theta$, such that for all $x, y \in A$ and $\lambda > 0$ with $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $\alpha(x, y) \geq 1$, one has:*

$$\omega_\lambda(Tx, Ty) \leq \beta\big(M(x, y)\big)M(x, y) + \gamma\big(N(x, y, \theta)\big)N(x, y, \theta) \tag{5}$$

where $\gamma : [0, \infty) \to [0, \infty)$ is a bounded function, $\omega_\lambda^*(x, y) = \omega_\lambda(x, y) - \omega_\lambda(A, B)$,

$$M(x,y) = \max\left\{\omega_\lambda(x,y), \frac{\omega_\lambda(x,Tx)+\omega_\lambda(y,Ty)}{2} - \omega_\lambda(A,B), \right.$$
$$\left. \frac{\omega_\lambda(x,Ty)+\omega_\lambda(y,Tx)}{2} - \omega_\lambda(A,B)\right\}$$

and:

$$N(x,y,\theta) = \theta\Big(\omega_\lambda(x,Tx) - \omega_\lambda(A,B), \omega_\lambda(y,Ty) - \omega_\lambda(A,B),$$
$$\omega_\lambda(x,Ty) - \omega_\lambda(A,B), \omega_\lambda(y,Tx) - \omega_\lambda(A,B)\Big).$$

Now, we are ready to prove our main result.

Theorem 3. *Let A and B be two nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular, such that A is $\omega-$complete and A_0^λ is nonempty for all $\lambda > 0$. Assume that T is a Suzuki-type $(\alpha, \beta, \theta, \gamma)$-contractive mapping satisfying the following assertions:*

(i) $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, and the pair (A, B) satisfies the weak P_λ-property,
(ii) T is a triangular α-proximal admissible mapping,
(iii) there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$, such that:

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B) \text{ and } \alpha(x_0, x_1) \geq 1$$

(iv) if $\{x_n\}$ is a sequence in A, such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then, there exists an x^* in A, such that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$. Further, the best proximity point is unique if, for every $x, y \in A$, such that $\omega_\lambda(x,Tx) = \omega_\lambda(A,B) = \omega_\lambda(y,Ty)$, we have $\alpha(x,y) \geq 1$.

Proof. By (iii), there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$, such that:

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B) \text{ and } \alpha(x_0, x_1) \geq 1.$$

On the other hand, $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$. Therefore, there exists $x_2 \in A_0$, such that:

$$\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A,B).$$

Now, since T is triangular α-proximal admissible, we have $\alpha(x_1, x_2) \geq 1$. That is:

$$\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A,B) \text{ and } \alpha(x_1, x_2) \geq 1.$$

Again, since $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, there exists $x_3 \in A_0^\lambda$, such that:

$$\omega_\lambda(x_3, Tx_2) = \omega_\lambda(A,B).$$

Thus, we have:

$$\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A,B) \text{ and } \omega_\lambda(x_3, Tx_2) = \omega_\lambda(A,B) \text{ and } \alpha(x_1, x_2) \geq 1.$$

Again, since T is triangular α-proximal admissible, $\alpha(x_2, x_3) \geq 1$. Hence:

$$\omega_\lambda(x_3, Tx_2) = \omega_\lambda(A,B) \text{ and } \alpha(x_2, x_3) \geq 1.$$

Continuing this process, we get:

$$\omega_\lambda(x_{n+1}, Tx_n) = \omega_\lambda(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{6}$$

Since (A, B) has the weak P_λ-property, we derive that:

$$\omega_\lambda(x_n, x_{n+1}) \leq \omega_\lambda(Tx_{n-1}, Tx_n) \quad \text{for any} \quad n \in \mathbb{N}. \tag{7}$$

Now, by (6), we get:

$$\omega_\lambda(x_{n-1}, Tx_{n-1}) \leq \omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(x_n, Tx_{n-1}) = \omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(A, B). \tag{8}$$

Clearly, if there exists $n_0 \in \mathbb{N}$, such that $\omega_\lambda(x_{n_0}, x_{n_0+1}) = 0$, then we have nothing to prove. In fact:

$$0 = \omega_\lambda(x_{n_0}, x_{n_0+1}) = \omega_\lambda(Tx_{n_0-1}, Tx_{n_0}).$$

Since ω is regular, we get, $Tx_{n_0-1} = Tx_{n_0}$. Thus, we conclude that:

$$\omega_\lambda(A, B) = \omega_\lambda(x_{n_0}, Tx_{n_0-1}) = \omega_\lambda(x_{n_0}, Tx_{n_0}).$$

For the rest of the proof, we suppose that $\omega_\lambda(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$. Now, from (8), we deduce that:

$$\frac{1}{2}\omega^*_\lambda(x_{n-1}, Tx_{n-1}) \leq \omega^*_\lambda(x_{n-1}, Tx_{n-1}) \leq \omega_\lambda(x_n, x_{n-1}). \tag{9}$$

Applying (6) and (7), we obtain:

$$\begin{aligned}
M(x_{n-1}, x_n) = \max\Big\{&\omega_\lambda(x_{n-1}, x_n), \frac{\omega_\lambda(x_{n-1}, Tx_{n-1}) + \omega_\lambda(x_n, Tx_n)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_{n-1}, Tx_n) + \omega_\lambda(x_n, Tx_{n-1})}{2} - \omega_\lambda(A, B)\Big\} \\
\leq \max\Big\{&\omega_\lambda(x_{n-1}, x_n), \\
&\frac{\omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(x_n, Tx_{n-1}) + \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_{n-1}, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) + \omega_\lambda(x_n, Tx_{n-1})}{2} - \omega_\lambda(A, B)\Big\} \\
= \max\Big\{&\omega_\lambda(x_{n-1}, x_n), \\
&\frac{\omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(A, B) + \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(A, B)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_{n-1}, x_{n+1}) + \omega_\lambda(A, B) + \omega_\lambda(A, B)}{2} - \omega_\lambda(A, B)\Big\} \\
= \max\Big\{&\omega_\lambda(x_{n-1}, x_n), \frac{\omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(x_n, x_{n+1})}{2}, \frac{\omega_\lambda(x_{n-1}, x_{n+1})}{2}\Big\} \\
\leq \max\{&\omega_\lambda(x_{n-1}, x_n), \frac{\omega_\lambda(x_{n-1}, x_n) + \omega_\lambda(x_n, x_{n+1})}{2}\} \\
\leq \max\{&\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1})\}.
\end{aligned}$$

Thus:

$$M(x_{n-1}, x_n) \leq \max\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1})\}. \tag{10}$$

Furthermore:

$$N(x_{n-1},x_n,\theta)=\theta\Big(\omega_\lambda(x_{n-1},Tx_{n-1})-\omega_\lambda(A,B),\omega_\lambda(x_n,Tx_n)-\omega_\lambda(A,B),\\ \omega_\lambda(x_{n-1},Tx_n)-\omega_\lambda(A,B),\omega_\lambda(x_n,Tx_{n-1})-\omega_\lambda(A,B)\Big)\\ =\theta\Big(\omega_\lambda(x_{n-1},Tx_{n-1})-\omega_\lambda(A,B),\omega_\lambda(x_n,Tx_n)-\omega_\lambda(A,B),\\ \omega_\lambda(x_{n-1},Tx_n)-\omega_\lambda(A,B),0\Big)=0. \tag{11}$$

Since T is a Suzuki-type $(\alpha,\beta,\theta,\gamma)$-contractive mapping, we have:

$$\begin{aligned}\omega_\lambda(x_n,x_{n+1}) &\le \omega_\lambda(Tx_{n-1},Tx_n)\\ &\le \beta(M(x_{n-1},x_n))M(x_{n-1},x_n)+\gamma(N(x_{n-1},x_n,\theta))N(x_{n-1},x_n,\theta)\\ &< M(x_{n-1},x_n)+\gamma(N(x_{n-1},x_n,\theta))N(x_{n-1},x_n,\theta).\end{aligned} \tag{12}$$

From (10) to (12), we deduce:

$$\omega_\lambda(x_n,x_{n+1})<\max\{\omega_\lambda(x_{n-1},x_n),\omega_\lambda(x_n,x_{n+1})\}.$$

Now if, $\max\{\omega_\lambda(x_{n-1},x_n),\omega_\lambda(x_n,x_{n+1})\}=\omega_\lambda(x_n,x_{n+1})$ then,

$$\omega_\lambda(x_n,x_{n+1})<\omega_\lambda(x_n,x_{n+1}),$$

which is a contradiction. Hence:

$$\omega_\lambda(x_{n-1},x_n)\le M(x_{n-1},x_n)\le\max\{\omega_\lambda(x_{n-1},x_n),\omega_\lambda(x_n,x_{n+1})\}=\omega_\lambda(x_{n-1},x_n),$$

and so:

$$M(x_{n-1},x_n)=\omega_\lambda(x_{n-1},x_n), \tag{13}$$

for all $n\in\mathbb{N}$. Now, by (12), we get:

$$\begin{aligned}\omega_\lambda(x_n,x_{n+1}) &= \omega_\lambda(Tx_{n-1},Tx_n)\\ &\le \beta(\omega_\lambda(x_{n-1},x_n))\omega_\lambda(x_{n-1},x_n)\\ &< \omega_\lambda(x_{n-1},x_n),\end{aligned} \tag{14}$$

for all $n\in\mathbb{N}$. Consequently, $\{\omega_\lambda(x_n,x_{n+1})\}$ is a non-increasing sequence, which is bounded from below, and so, $\lim_{n\to\infty}\omega_\lambda(x_n,x_{n+1}):=L$ exists. Let $L>0$. Then, from (14), we have:

$$\frac{\omega_\lambda(x_n,x_{n+1})}{\omega_\lambda(x_{n-1},x_n)}\le\beta(\omega_\lambda(x_{n-1},x_n))\le 1,$$

for each $n\ge 1$, which implies:

$$\lim_{n\to\infty}\beta(\omega_\lambda(x_n,x_{n+1}))=1.$$

On the other hand, since $\beta\in\mathcal{F}$, we conclude:

$$L=\lim_{n\to\infty}\omega_\lambda(x_n,x_{n+1})=0. \tag{15}$$

Since, $\omega_\lambda(x_n,Tx_{n-1})=\omega_\lambda(A,B)$ holds for all $n\in\mathbb{N}$ and (A,B) satisfies the weak P_λ-property, so for all $m,n\in\mathbb{N}$ with $n<m$, we obtain, $\omega_\lambda(x_m,x_n)\le\omega_\lambda(Tx_{m-1},Tx_{n-1})$. Note that:

$$\begin{aligned}
M(x_m, x_n) = \max\bigg\{&\omega_\lambda(x_m, x_n), \frac{\omega_\lambda(x_m, Tx_m) + \omega_\lambda(x_n, Tx_n)}{2} - \omega_\lambda(A, B),\\
&\frac{\omega_\lambda(x_m, Tx_n) + \omega_\lambda(x_n, Tx_m)}{2} - \omega_\lambda(A, B)\bigg\}\\
&\leq \max\bigg\{\omega_\lambda(x_m, x_n),\\
&\frac{\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(x_{m+1}, Tx_m) + \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n)}{2} - \omega_\lambda(A, B),\\
&\frac{\omega_\lambda(x_m, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) + \omega_\lambda(x_n, x_{m+1}) + \omega_\lambda(x_{m+1}, Tx_m)}{2} - \omega_\lambda(A, B)\bigg\}\\
&= \max\bigg\{\omega_\lambda(x_m, x_n), \frac{\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(x_n, x_{n+1})}{2}, \omega_\lambda(x_m, x_{n+1})\bigg\}\\
&\leq \max\bigg\{\omega_\lambda(x_m, x_n), \frac{\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(x_n, x_{n+1})}{2},\\
&\omega_\lambda(x_m, x_n) + \omega_\lambda(x_n, x_{n+1})\bigg\}.
\end{aligned}$$

As $\lim_{n\to\infty} \omega_\lambda(x_n, x_{n+1}) = 0$, we have:

$$\lim_{m,n\to\infty} \omega_\lambda(x_m, x_n) \leq \lim_{m,n\to\infty} M(x_m, x_n) \leq \lim_{m,n\to\infty} \omega_\lambda(x_m, x_n),$$

that is:

$$\lim_{m,n\to\infty} M(x_m, x_n) = \lim_{m,n\to\infty} \omega_\lambda(x_m, x_n). \tag{16}$$

Furthermore:

$$\begin{aligned}
0 &\leq N(x_m, x_n, \theta)\\
&= \theta\Big(\omega_\lambda(x_m, Tx_m) - \omega_\lambda(A, B), \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A, B),\\
&\qquad \omega_\lambda(x_m, Tx_n) - \omega_\lambda(A, B), \omega_\lambda(x_n, Tx_m) - \omega_\lambda(A, B)\Big)\\
&\leq \theta\Big(\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(A, B) - \omega_\lambda(A, B), \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A, B),\\
&\qquad \omega_\lambda(x_m, Tx_n) - \omega_\lambda(A, B), \omega_\lambda(x_n, Tx_m) - \omega_\lambda(A, B)\Big)\\
&\leq \theta\Big(\omega_\lambda(x_m, x_{m+1}), \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A, B), \omega_\lambda(x_m, Tx_n) - \omega_\lambda(A, B),\\
&\qquad \omega_\lambda(x_n, Tx_m) - \omega_\lambda(A, B)\Big).
\end{aligned}$$

Again, by $\lim_{n\to\infty} \omega_\lambda(x_n, x_{n+1}) = 0$, we have:

$$
\begin{aligned}
0 &\leq \lim_{m,n\to\infty} N(x_m, x_n, \theta) \\
&\leq \lim_{m,n\to\infty} \theta\Big(\omega_\lambda(x_m, x_{m+1}), \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A,B), \omega_\lambda(x_m, Tx_n) - \omega_\lambda(A,B), \\
&\omega_\lambda(x_n, Tx_m) - \omega_\lambda(A,B)\Big) \\
&\leq \lim_{m,n\to\infty} \theta\Big(0, \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A,B), \omega_\lambda(x_m, Tx_n) - \omega_\lambda(A,B), \\
&\omega_\lambda(x_n, Tx_m) - \omega_\lambda(A,B)\Big) = 0.
\end{aligned}
$$

That is:

$$\lim_{m,n\to\infty} N(x_m, x_n, \theta) = 0. \tag{17}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. On the contrary, assume that:

$$\varepsilon = \limsup_{m,n\to\infty} \omega_\lambda(x_n, x_m) > 0. \tag{18}$$

Now, since $\lim_{n\to+\infty} \omega_\lambda(x_n, x_{n+1}) = 0$, then:

$$
\begin{aligned}
\omega_\lambda(A,B) &\leq \lim_{m\to+\infty} \omega_\lambda(x_m, Tx_m) \\
&\leq \lim_{m\to+\infty} [\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(x_{m+1}, Tx_m)] \\
&= \lim_{m\to+\infty} [\omega_\lambda(x_m, x_{m+1}) + \omega_\lambda(A,B)] = \omega_\lambda(A,B),
\end{aligned}
$$

which implies that $\lim_{m\to+\infty} \omega_\lambda(x_m, Tx_m) = \omega_\lambda(A,B)$, that is:

$$\lim_{m\to+\infty} \frac{1}{2}\omega_\lambda^*(x_m, Tx_m) = \lim_{m\to+\infty} \frac{1}{2}[\omega_\lambda(x_m, Tx_m) - \omega_\lambda(A,B)] = 0.$$

On the other hand, from (18), it is follows that there exists $N \in \mathbb{N}$, such that, for all $m, n \geq N$, we have:

$$\frac{1}{2}\omega_\lambda^*(x_m, Tx_m) \leq \omega_\lambda(x_n, x_m).$$

Furthermore, we can show that:

$$\alpha(x_m, x_n) \geq 1, \text{ where } n > m. \tag{19}$$

Indeed, since T is a triangular α-proximal admissible mapping and:

$$\begin{cases} \alpha(x_m, x_{m+1}) \geq 1 \\ \alpha(x_{m+1}, x_{m+2}) \geq 1 \end{cases},$$

from Condition (T2) of Definition 6, we have:

$$\alpha(x_m, x_{m+2}) \geq 1.$$

Again, since T is a triangular α-proximal admissible mapping and:

$$\begin{cases} \alpha(x_m, x_{m+2}) \geq 1 \\ \alpha(x_{m+2}, x_{m+3}) \geq 1 \end{cases},$$

from Condition (T2) of Definition 6, we have:

$$\alpha(x_m, x_{m+3}) \geq 1.$$

Continuing this process, we get (19).

Now, using the triangle inequality, we have:

$$\omega_\lambda(x_n, x_m) \leq \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, x_{m+1}) + \omega_\lambda(x_{m+1}, x_m). \tag{20}$$

From (5) and (20) we have:

$$\begin{aligned} &\omega_\lambda(x_n, x_m) \\ &\quad \leq \quad \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(Tx_n, Tx_m) + \omega_\lambda(x_{m+1}, x_m) \\ &\quad \leq \quad \omega_\lambda(x_n, x_{n+1}) + \beta(M(x_n, x_m))M(x_n, x_m) + \gamma(N(x_n, x_m, \theta))N(x_n, x_m, \theta) \\ &\qquad\quad +\omega_\lambda(x_{m+1}, x_m). \end{aligned} \tag{21}$$

Now, (16), (17), (21) and: $\lim_{n\to\infty} \omega_\lambda(x_n, x_{n+1}) = 0$, imply:

$$\begin{aligned} \lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) &\leq \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} M(x_m, x_n) \\ &\quad + \lim_{m,n\to\infty} \gamma(N(x_n, x_m, \theta)) \lim_{m,n\to\infty} N(x_m, x_n, \theta) \\ &= \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} \omega_\lambda(x_m, x_n). \end{aligned}$$

By (18), we get:

$$1 \leq \lim_{m,n\to\infty} \beta(M(x_n, x_m)).$$

Therefore, $\lim_{m,n\to\infty} \beta(M(x_n, x_m)) = 1$, so $\lim_{m,n\to\infty} M(x_n, x_m) = 0$. This implies:

$$\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $(x_n) \subset A$ and (A, d) is a complete metric space, we can find $x^* \in A$, such that $x_n \to x^*$ as $n \to \infty$. From (iv), we know that, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Next, using (14), we have:

$$\begin{aligned} \omega^*_\lambda(x_n, Tx_n) &= \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A, B) \\ &\leq \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) - \omega_\lambda(A, B) \\ &= \omega_\lambda(x_n, x_{n+1}), \end{aligned} \tag{22}$$

and:

$$\begin{aligned} \omega^*_\lambda(x_{n+1}, Tx_{n+1}) &= \omega_\lambda(x_{n+1}, Tx_{n+1}) - \omega_\lambda(A, B) \\ &\leq \omega_\lambda(Tx_n, Tx_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) - \omega_\lambda(A, B) \\ &= \omega_\lambda(Tx_n, Tx_{n+1}) \\ &= \omega_\lambda(x_{n+1}, x_{n+2}) \\ &\leq \omega_\lambda(x_n, x_{n+1}). \end{aligned} \tag{23}$$

Therefore, (22) and (23) imply that:

$$\frac{1}{2}[\omega^*_\lambda(x_n, Tx_n) + \omega^*_\lambda(x_{n+1}, Tx_{n+1})] \leq \omega_\lambda(x_n, x_{n+1}). \tag{24}$$

Now, suppose that:

$$\frac{1}{2}\omega_\lambda^*(x_n, Tx_n) > \omega_\lambda(x_n, x^*) \quad \text{and} \quad \frac{1}{2}\omega_\lambda^*(x_{n+1}, Tx_{n+1}) > \omega_\lambda(x_{n+1}, x^*),$$

for some $n \in \mathbb{N}$. Hence, using (24), we can write:

$$\begin{aligned}
\omega_\lambda(x_n, x_{n+1}) &\leq \omega_\lambda(x_n, x^*) + \omega_\lambda(x_{n+1}, x^*) \\
&< \frac{1}{2}[\omega_\lambda^*(x_n, Tx_n) + \omega_\lambda^*(x_{n+1}, Tx_{n+1})] \\
&\leq \omega_\lambda(x_n, x_{n+1}),
\end{aligned}$$

which is a contradiction. Then, for any $n \in \mathbb{N}$, either:

$$\frac{1}{2}\omega_\lambda^*(x_n, Tx_n) \leq \omega_\lambda(x_n, x^*) \quad \text{or} \quad \frac{1}{2}\omega_\lambda^*(x_{n+1}, Tx_{n+1}) \leq \omega_\lambda(x_{n+1}, x^*)$$

holds.

We shall show that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$. Suppose, to the contrary, that:

$$\omega_\lambda(x^*, Tx^*) \neq \omega_\lambda(A, B).$$

From (5) with $x = x_n$ and $y = x^*$, we get:

$$\omega_\lambda(Tx_n, Tx^*) \leq \beta\big(M(x_n, x^*)\big)M(x_n, x^*) + \gamma\big(N(x_n, x^*, \theta)\big)N(x_n, x^*, \theta). \tag{25}$$

On the other hand:

$$\begin{aligned}
&M(x_n, x^*) \\
&= \max\Big\{\omega_\lambda(x_n, x^*), \frac{\omega_\lambda(x_n, Tx_n) + \omega_\lambda(x^*, Tx^*)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_n, Tx^*) + \omega_\lambda(x^*, Tx_n)}{2} - \omega_\lambda(A, B)\Big\} \\
&\leq \max\Big\{\omega_\lambda(x_n, x^*), \frac{\omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) + \omega_\lambda(x^*, Tx^*)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_n, x^*) + \omega_\lambda(x^*, Tx^*) + \omega_\lambda(x^*, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n)}{2} - \omega_\lambda(A, B)\Big\} \\
&= \max\Big\{\omega_\lambda(x_n, x^*), \frac{\omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(A, B) + \omega_\lambda(x^*, Tx^*)}{2} - \omega_\lambda(A, B), \\
&\frac{\omega_\lambda(x_n, x^*) + \omega_\lambda(x^*, Tx^*) + \omega_\lambda(x^*, x_{n+1}) + \omega_\lambda(A, B)}{2} - \omega_\lambda(A, B)\Big\},
\end{aligned}$$

and so:

$$\lim_{k\to\infty} M(x_n, x^*) \leq \frac{\omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B)}{2}. \tag{26}$$

Furthermore, we have:

$$\begin{aligned}
\omega_\lambda(x^*, Tx^*) &\leq \omega_\lambda(x^*, Tx_n) + \omega_\lambda(Tx_n, Tx^*) \\
&\leq \omega_\lambda(x^*, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) + \omega_\lambda(Tx_n, Tx^*) \\
&\leq \omega_\lambda(x^*, x_{n+1}) + \omega_\lambda(A, B) + \omega_\lambda(Tx_n, Tx^*).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have:

$$\omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B) \leq \lim_{n\to\infty} \omega_\lambda(Tx_n, Tx^*). \tag{27}$$

Further, we get:

$$\omega_\lambda(x_n, Tx_n) \leq \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(x_{n+1}, Tx_n) = \omega_\lambda(x_n, x_{n+1}) + \omega_\lambda(A, B).$$

Taking the limit as $n \to \infty$ in the above inequality, we get:

$$\lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) \leq \omega_\lambda(A, B),$$

and so, $\lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) = \omega_\lambda(A, B)$. Now, we have:

$$\begin{aligned}
&\lim_{n\to\infty} N(x_n, x^*, \theta) \\
&\qquad = \theta\Big(\lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) - \omega_\lambda(A, B), \\
&\qquad \omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B), \lim_{n\to\infty} \omega_\lambda(x_n, Tx^*) - \omega_\lambda(A, B), \\
&\qquad \lim_{n\to\infty} \omega_\lambda(x^*, Tx_n) - \omega_\lambda(A, B) \Big) \\
&\qquad = \theta\Big(0, \omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B), \\
&\qquad \lim_{n\to\infty} \omega_\lambda(x_n, Tx^*) - \omega_\lambda(A, B), \lim_{n\to\infty} \omega_\lambda(x^*, Tx_n) - \omega_\lambda(A, B) \Big) = 0,
\end{aligned}$$

that is:

$$\lim_{n\to\infty} N(x_n, x^*, \theta) = 0. \tag{28}$$

From (25) to (28), we deduce that:

$$\begin{aligned}
\omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B) &\leq \lim_{n\to\infty} \omega_\lambda(Tx_n, Tx^*) \\
&\leq \lim_{n\to\infty} \beta(M(x_n, x^*)) \lim_{n\to\infty} M(x_n, x^*) \\
&+ \lim_{n\to\infty} \gamma(N(x_n, x^*, \theta)) \lim_{n\to\infty} N(x_n, x^*, \theta) \\
&= \lim_{n\to\infty} \beta(M(x_n, x^*)) (\frac{\omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B)}{2}) \\
&< \omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B),
\end{aligned}$$

which is a contradiction. Therefore, $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$, and x^* is a best proximity point of T. We now show the uniqueness of the best proximity point of T. Suppose that x^* and y^* are two distinct best proximity points of T. This implies:

$$\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B) = \omega_\lambda(y^*, Ty^*). \tag{29}$$

Using the weak P_1-property, we have:

$$\omega_\lambda(x^*, y^*) \leq \omega_\lambda(Tx^*, Ty^*). \tag{30}$$

Since:

$$
\begin{aligned}
&M(x^*, y^*) \\
&= \max\Big\{\omega_\lambda(x^*, y^*), \frac{\omega_\lambda(x^*, Tx^*) + \omega_\lambda(y^*, Ty^*)}{2} - \omega_\lambda(A, B), \\
&\qquad \frac{\omega_\lambda(x^*, Ty^*) + \omega_\lambda(y^*, Tx^*)}{2} - \omega_\lambda(A, B)\Big\} \\
&= \max\Big\{\omega_\lambda(x^*, y^*), 0, \frac{\omega_\lambda(x^*, Ty^*) + \omega_\lambda(y^*, Tx^*)}{2} - \omega_\lambda(A, B)\Big\} \\
&\leq \max\Big\{\omega_\lambda(x^*, y^*), 0, \\
&\qquad \frac{\omega_\lambda(x^*, Tx^*) + \omega_\lambda(Tx^*, Ty^*) + \omega_\lambda(y^*, Ty^*) + \omega_\lambda(Ty^*, Tx^*)}{2} - \omega_\lambda(A, B)\Big\} \\
&\leq \max\Big\{\omega_\lambda(x^*, y^*), 0, \\
&\qquad \frac{\omega_\lambda(A, B) + \omega_\lambda(x^*, y^*) + \omega_\lambda(A, B) + \omega_\lambda(y^*, x^*)}{2} - \omega_\lambda(A, B)\Big\} \\
&= \omega_\lambda(x^*, y^*).
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
&N(x^*, y^*, \theta) \\
&\qquad = \theta\Big(\omega_\lambda(x^*, Tx^*) - \omega_\lambda(A, B), \omega_\lambda(y^*, Ty^*) - \omega_\lambda(A, B), \\
&\qquad \omega_\lambda(x^*, Ty^*) - \omega_\lambda(A, B), \omega_\lambda(y^*, Tx^*) - \omega_\lambda(A, B)\Big) \\
&\qquad = \theta\Big(\omega_\lambda(A, B) - \omega_\lambda(A, B), \omega_\lambda(A, B) - \omega_\lambda(A, B), \\
&\qquad \omega_\lambda(x^*, Ty^*) - \omega_\lambda(A, B), \omega_\lambda(y^*, Tx^*) - \omega_\lambda(A, B)\Big) \\
&\qquad = \theta\Big(0, 0, \omega_\lambda(x^*, Ty^*) - \omega_\lambda(A, B), \omega_\lambda(y^*, Tx^*) - \omega_\lambda(A, B)\Big) = 0.
\end{aligned}
$$

As T is a Suzuki-type $(\alpha, \beta, \theta, \gamma)$-contractive mapping and $\frac{1}{2}\omega_\lambda^*(x^*, Tx^*) = 0 \leq \omega_\lambda(x^*, y^*)$ and $\alpha(x^*, y^*) \geq 1$, then, we obtain:

$$
\begin{aligned}
\omega_\lambda(x^*, y^*) &\leq \omega_\lambda(Tx^*, Ty^*) \\
&\leq \beta(M(x^*, y^*))M(x^*, y^*) + \gamma(N(x^*, y^*, \theta))N(x^*, y^*, \theta) \\
&= \beta(\omega_\lambda(x^*, y^*))\omega_\lambda(x^*, y^*) \\
&< \omega_\lambda(x^*, y^*),
\end{aligned}
$$

which is a contradiction. This completes the proof of the theorem. □

If in Theorem 3, we take $\beta(t) = r$ where $r \in [0, 1)$ and $\gamma(t) = L$ where $L \geq 0$, then we obtain the following best proximity point result.

Corollary 1. *Let (A, B) be a pair of nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular, such that A is complete and A_0^λ is nonempty for all $\lambda > 0$. Let $T : A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$ and for all $x, y \in A$ with $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $\alpha(x, y) \geq 1$; one has:*

$$
\omega_\lambda(Tx, Ty) \leq rM(x, y) + LN(x, y, \theta)
$$

where $r \in [0,1)$, $L \geq 0$ and $\theta \in \Theta$. Suppose that the pair (A,B) has the weak P_1-property and the following assertions hold:

(i) *T is a triangular α-proximal admissible mapping,*
(ii) *there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$, such that:*

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B) \text{ and } \alpha(x_0,x_1) \geq 1.$$

(iii) *if $\{x_n\}$ is a sequence in A, such that $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exists an x^* in A, such that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$. Further, the best proximity point is unique if, for every $x,y \in A$, such that $\omega_\lambda(x,Tx) = \omega_\lambda(A,B) = \omega_\lambda(y,Ty)$, we have: $\alpha(x,y) \geq 1$.

If in Corollary 1 we take, $\theta(t_1,t_2,t_3,t_4) = \min\{t_1,t_2,t_3,t_4\}$, we obtain the following best proximity result.

Corollary 2. *Let (A,B) be a pair of nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular, such that A is complete and A_0^λ is nonempty for all $\lambda > 0$. Let $T: A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$ and for all $x,y \in A$ with $\frac{1}{2}\omega_\lambda^*(x,Tx) \leq \omega_\lambda(x,y)$ and $\alpha(x,y) \geq 1$; we have:*

$$\omega_\lambda(Tx,Ty) \leq rM(x,y) + LN(x,y)$$

where $r \in [0,1)$, $L \geq 0$,

$$M(x,y) = \max\left\{\omega_\lambda(x,y), \frac{\omega_\lambda(x,Tx)+\omega_\lambda(y,Ty)}{2} - \omega_\lambda(A,B), \frac{\omega_\lambda(x,Ty)+\omega_\lambda(y,Tx)}{2} - \omega_\lambda(A,B)\right\}$$

and:

$$N(x,y) = \min\{\omega_\lambda(x,Tx), \omega_\lambda(y,Ty), \omega_\lambda(x,Ty), \omega_\lambda(y,Tx)\} - \omega_\lambda(A,B).$$

Suppose that the pair (A,B) has the weak P_λ-property and the following assertions hold:

(i) *T is a triangular α-proximal admissible mapping,*
(ii) *there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$, such that:*

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B) \text{ and } \alpha(x_0,x_1) \geq 1.$$

(iii) *if $\{x_n\}$ is a sequence in A, such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exists an x^* in A, such that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$. Further, the best proximity point is unique if, for every $x,y \in A$, such that $\omega_\lambda(x,Tx) = \omega_\lambda(A,B) = \omega_\lambda(y,Ty)$, we have $\alpha(x,y) \geq 1$.

The following example illustrates our results.

Example 1. Consider the space $X = \mathbb{R}^2$ endowed with the non-Archimedean modular metric ω: $X \times X \to (0,+\infty)$ given by:

$$\omega_\lambda((x_1,x_2),(y_1,y_2)) = \frac{1}{\lambda}\Big(|x_1-y_1| + |x_2-y_2|\Big),$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Define the sets:

$$A = \{(1,0), (4,5), (5,4)\} \cup (-\infty, -1] \times (-\infty, -1]$$

and:

$$B = \{(0,0), (0,4), (4,0)\} \cup [10, \infty) \times [10, \infty)$$

so that $\omega_\lambda(A, B) = \frac{1}{\lambda}$, $A_0^\lambda = \{(1,0)\}$, $B_0^\lambda = \{(0,0)\}$ for all $\lambda > 0$, and the pair (A, B) has the weak P_λ-property. Furthermore, let $T : A \to B$ be defined by:

$$T(x_1, x_2) = \begin{cases} (10x_1^2, 15x_2^4) & \text{if } x_1, x_2 \in (-\infty, -1], \\ (x_1, 0) & \text{if } x_1, x_2 \notin (-\infty, -1] \text{ with } x_1 \leq x_2, \\ (0, x_2) & \text{if } x_1, x_2 \notin (-\infty, -1] \text{ with } x_1 > x_2. \end{cases}$$

Notice that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$.

Now, consider the function $\beta : [0, +\infty) \to [0, 1)$ given by:

$$\beta(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{\ln(1+t)}{t} & \text{if } 0 < t \leq 1, \\ \frac{8}{9} & \text{if } 1 < t \leq 10, \\ \frac{10}{11} & \text{if } t > 10, \end{cases}$$

and note that $\beta \in \mathcal{F}$. Furthermore, define $\alpha : X \times X \to [0, \infty)$ by:

$$\alpha(x, y) = \left\{ \begin{array}{ll} 2, & x, y \in \{(1,0), (4,5), (5,4)\} \\ \frac{1}{4}, & \text{otherwise.} \end{array} \right. .$$

Clearly, $\omega_\lambda((1,0), T(1,0)) = \omega_\lambda(A, B) = \frac{1}{\lambda}$ and $\alpha((1,0), (1,0)) \geq 1$.

Assume that $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $\alpha(x, y) \geq 1$, for some $x, y \in A$. Then:

$$\left\{ \begin{array}{ll} x = (1,0), \ y = (4,5) & \text{or} \\ x = (1,0), \ y = (5,4) & \text{or} \\ y = (1,0), \ x = (4,5) & \text{or} \\ y = (1,0), \ x = (5,4). & \end{array} \right.$$

Since $\omega_\lambda(Tx, Ty) = \omega_\lambda(Ty, Tx)$ and $M(x, y) = M(y, x)$ for all $x, y \in A$, without any loss of generality, we can assume that:

$$(x, y) = ((1,0), (4,5)) \text{ or } (x, y) = ((1,0), (5,4)).$$

Now, we want to distinguish the following cases:

(i) if $(x, y) = ((1,0), (4,5))$, then:

$$\omega_\lambda(T(1,0), T(4,5)) = \frac{4}{\lambda} \leq \frac{8}{9} \cdot \frac{8}{\lambda} = \beta(M((1,0), (4,5)))[M((1,0), (4,5))];$$

(ii) if $(x, y) = ((1,0), (5,4))$, then:

$$\omega_\lambda(T(1,0), T(5,4)) = 4 \leq \frac{8}{9} \cdot \frac{8}{\lambda} = \beta(M((1,0), (5,4)))[M((1,0), (5,4))].$$

Consequently, we have:

$$\frac{1}{2}\omega^*_\lambda(x,Tx) \leq \omega_\lambda(x,y) \text{ and } \alpha(x,y) \geq 1 \ \Rightarrow \omega_\lambda(Tx,Ty) \leq \beta(M(x,y))[M(x,y)]$$

and hence, T is a Suzuki-type $(\alpha,\beta,\theta,\gamma)$-contractive mapping with $\gamma(t) = 0$. Let:

$$\begin{cases} \alpha(x,y) \geq 1 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) = \frac{1}{\lambda} \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) = \frac{1}{\lambda}, \end{cases}$$

then:

$$\begin{cases} x,y \in \{(1,0),(4,5),(5,4)\} \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) = \frac{1}{\lambda} \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) = \frac{1}{\lambda}, \end{cases}$$

and so, $u = v = (1,0)$. i.e., $\alpha(u,v) \geq 1$. Furthermore, assume that $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$. Then, $x,y,z \in \{(1,0),(4,5),(5,4)\}$, i.e., $\alpha(x,z) \geq 1$. Therefore, T is a triangular $\alpha-$proximal admissible mapping. Moreover, if $\{x_n\}$ is a sequence, such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq \{(1,0),(4,5),(5,4)\}$, and hence, $x \in \{(1,0),(4,5),(5,4)\}$. Consequently, $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Hence, as you see, all of the conditions of Theorem 3 hold true, and T has a unique best proximity point. Here, $x = (1,0)$ is the unique best proximity point of T.

If in Theorem 3, we take $\alpha(x,y) = 1$ for all $x,y \in A$, then we can deduce the following corollary.

Corollary 3. *Let (A,B) be a pair of nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular, such that A is complete and A_0^λ is nonempty for all $\lambda > 0$. Let $T : A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, and there exists $\beta \in \mathcal{F}$ and $\theta \in \Theta$, such that $\frac{1}{2}\omega^*_\lambda(x,Tx) \leq \omega_\lambda(x,y)$ implies:*

$$\omega_\lambda(Tx,Ty) \leq \beta\big(M(x,y)\big)M(x,y) + \gamma\big(N(x,y,\theta)\big)N(x,y,\theta).$$

Suppose that the pair (A,B) has the weak P_λ-property. Then, there exists a unique x^* in A, such that $\omega_\lambda(x^*,Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$.

We investigate the Suzuki-type result of Zhang et al. [16] in the setting of non-Archimedean modular metric space as follows:

Corollary 4. *Let (A,B) be a pair of nonempty and closed subsets of a complete non-Archimedean modular metric space X_ω with ω regular, such that A_0^λ is nonempty for all $\lambda > 0$. Let $T : A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, and there exists $r \in [0,1)$, such that $\frac{1}{2}\omega^*_\lambda(x,Tx) \leq \omega_\lambda(x,y)$ implies:*

$$\omega_\lambda(Tx,Ty) \leq r\omega_\lambda(x,y)$$

for all $x,y \in A$. Suppose that the pair (A,B) has the weak P_λ-property. Then there exists a unique point x^* in A, such that $\omega_\lambda(x^*,Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$.

Corollary 5. *(Suzuki-type result of Suzuki [21]) Let (A,B) be a pair of nonempty and closed subsets of a complete non-Archimedean modular metric space X_ω with ω regular, such that A_0^λ is nonempty for all $\lambda > 0$. Let $T : A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, and there exists $r \in [0,1)$, such that $\frac{1}{2}\omega^*_\lambda(x,Tx) \leq \omega_\lambda(x,y)$ implies:*

$$\omega_\lambda(Tx,Ty) \leq r\left[\frac{\omega_\lambda(x,Tx) + \omega_\lambda(y,Ty)}{2} - \omega_\lambda(A,B)\right] \tag{31}$$

for all $x, y \in A$. Suppose that the pair (A, B) has the weak P_λ-property. Therefore, there exists a unique point x^* in A, such that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$ for all $\lambda > 0$.

Corollary 6. *Let (A, B) be a pair of nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular, such that A is complete and A_0^λ is nonempty for all $\lambda > 0$. Let $T : A \to B$ be a non-self mapping, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$, and there exists $r \in [0, 1)$, such that $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ implies:*

$$\omega_\lambda(Tx, Ty) \leq r\left[\frac{\omega_\lambda(x, Ty) + \omega_\lambda(y, Tx)}{2} - \omega_\lambda(A, B)\right] \tag{32}$$

for all $x, y \in A_0$. Suppose that the pair (A, B) has the weak P_λ-property. Then, there exists a unique point x^* in A, such that $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$ for all $\lambda > 0$.

3. Best Proximity Point Results in Metric Spaces Endowed with a Graph

Consistent with Jachymski [22], let X_ω be a modular metric space, and Δ denotes the diagonal of the Cartesian product $X_\omega \times X_\omega$. Assume that G is a directed graph, such that the set $V(G)$ of its vertices coincides with X_ω and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We suppose that G has no parallel edges. We identify G with the pair $(V(G), E(G))$. Furthermore, we may handle G as a weighted graph (see [23], p. 309) by assigning to every edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices, such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, N$. The foremost fixed point result in this area was given by Jachymski [22].

Definition 9 (Reference [22])**.** *Let (X, d) be a modular metric space endowed with a graph G.We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is:*

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0, 1) \text{ such that for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

We define the following notion for modular metric spaces.

Definition 10. *Let X_ω be a modular metric space endowed with a graph G. We say that a self-mapping T: $X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is:*

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0, 1) \text{ such that for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow \omega_\lambda(Tx, Ty) \leq \alpha \omega_\lambda(x, y).$$

Definition 11. *Let A and B be two nonempty subsets of a non-Archimedean modular metric space X_ω endowed with a graph G and $A_0 \neq \emptyset$. A mapping T: $A \to B$ is said to be a Suzuki-type $G - (\beta, \theta, \gamma)$-contractive mapping if there exists $\beta \in \mathcal{F}$ and $\theta \in \Theta$, such that for all $x, y \in A$ with $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $(x, y) \in E(G)$, one has:*

$$\omega_\lambda(Tx, Ty) \leq \beta\big(M(x, y)\big)M(x, y) + \gamma\big(N(x, y, \theta)\big)N(x, y, \theta) \tag{33}$$

and:

$$\begin{cases} (x,y) \in E(G) \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases} \Longrightarrow (u,v) \in E(G).$$

Theorem 4. *Let A and B be two nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular endowed with a graph G, such that A is complete and A_0^λ is nonempty for all $\lambda > 0$. Assume that T is a Suzuki-type $G - (\beta,\theta,\gamma)$-contractive mapping satisfying the following assertions:*

(i) $T(A_0^\lambda) \subseteq B_0^\lambda$ *for all $\lambda > 0$, and the pair (A,B) satisfies the weak P-property,*
(ii) $(x,y) \in E(G)$ *and* $(y,z) \in E(G)$ *implies* $(x,z) \in E(G)$,
(iii) there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$, such that:

$$\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B) \text{ and } (x_0,x_1) \in E(G).$$

(iv) if $\{x_n\}$ is a sequence in A, such that $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}\cup\{0\}$ with $x_n \to x \in A$ as $n \to \infty$, then $(x_n,x) \in E(G)$ for all $n \in \mathbb{N}$.

Then, there exists an x^* in A, such that $\omega_\lambda(x^*,Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$.

Proof. Define $\alpha : X \times X \to [0,+\infty)$ with:

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

At first, we show that T is a triangular α-proximal admissible mapping. For this goal, assume:

$$\begin{cases} \alpha(x,y) \geq 1 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B). \end{cases}$$

Therefore, we have:

$$\begin{cases} (x,y) \in E(G) \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B). \end{cases}$$

Since T is a Suzuki-type $G - (\beta,\theta,\gamma)$-contractive mapping, we get $(u,v) \in E(G)$, that is $\alpha(u,v) \geq 1$. Furthermore, let $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, then $(x,z) \in E(G)$ and $(z,y) \in E(G)$. Consequently, from (iii), we deduce that $(x,y) \in E(G)$, that is, $\alpha(x,y) \geq 1$. Thus, T is a triangular α-proximal admissible mapping with $T(A_0) \subseteq B_0$. Now, assume that, $\frac{1}{2}\omega_\lambda^*(x,Tx) \leq \omega_\lambda(x,y)$ and $\alpha(x,y) \geq 1$. Then, $\frac{1}{2}\omega_\lambda^*(x,Tx) \leq \omega_\lambda(x,y)$ and $(x,y) \in E(G)$. As T is a Suzuki-type $G - (\beta,\theta,\gamma)$-contraction, then we get:

$$\omega_\lambda(Tx,Ty) \leq \beta\big(M(x,y)\big)M(x,y) + \gamma\big(N(x,y,\theta)\big)N(x,y,\theta),$$

and so, T is a Suzuki-type $(\alpha,\beta,\theta,\gamma)$-contractive mapping. From (iii), there exist $x_0, x_1 \in A_0$, such that $\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B)$ and $(x_0,x_1) \in E(G)$, that is $\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B)$ and $\alpha(x_0,x_1) \geq 1$. Hence, all of the conditions of Theorem 3 are satisfied, and so, T has a best proximity point. □

4. Best Proximity Point Results in Partially-Ordered Metric Spaces

The existence of best proximity points in partially-ordered metric spaces has been investigated in recent years by many authors (see, [24] and the references therein). In this section, we introduce a new notion of Suzuki-type ordered (β,θ,γ)-contractive mapping and investigate the existence of the best proximity points for such mappings in partially-ordered non-Archimedean modular metric spaces by using the weak P_λ-property.

Definition 12. *Let X_ω be a partially-ordered modular metric space. We say that a non-self-mapping $T\colon A \to B$ is proximally ordered-preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$:*

$$\begin{cases} x_1 \preceq x_2 \\ \omega_\lambda(u_1, Tx_1) = \omega_\lambda(A,B) \\ \omega_\lambda(u_2, Tx_2) = \omega_\lambda(A,B) \end{cases} \quad \Longrightarrow u_1 \preceq u_2.$$

Definition 13. *Let A and B be two nonempty closed subsets of a partially-ordered modular metric space X_ω and $A_0 \neq \emptyset$. A mapping $T\colon A \to B$ is said to be a Suzuki-type ordered (β, θ, γ)-contractive mapping if there exists $\beta \in \mathcal{F}$ and $\theta \in \Theta$, such that for all $x, y \in A$ with $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $x \preceq y$, we have:*

$$\omega_\lambda(Tx, Ty) \leq \beta\big(M(x,y)\big)M(x,y) + \gamma\big(N(x,y,\theta)\big)N(x,y,\theta).$$

Theorem 5. *Let A and B be two nonempty closed subsets of a partially-ordered non-Archimedean modular metric space with ω regular, such that A is complete, A_0^λ is nonempty for all $\lambda > 0$ and the pair (A, B) has the weak P_λ-property. Assume that $T\colon A \to B$ satisfies the following conditions:*

(i) *T is proximally ordered-preserving, such that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$,*
(ii) *there exist elements $x_0, x_1 \in A_0$, such that:*

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B) \text{ and } x_0 \preceq x_1,$$

(iii) *T is a Suzuki-type ordered (β, θ, γ)-contractive mapping,*
(iv) *if $\{x_n\}$ is an increasing sequence in A converging to $x \in A$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then, T has a best proximity point.

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