On Optimal Fuzzy Best Proximity Coincidence Points of Proximal Contractions Involving Cyclic Mappings in Non-Archimedean Fuzzy Metric Spaces

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Academic Editor: Pasquale Vetro
Received: 30 March 2016; Accepted: 27 March 2017; Published: 1 April 2017

Abstract: The main objective of this paper is to deal with some properties of interest in two types of fuzzy ordered proximal contractions of cyclic self-mappings T integrated in a pair (g, T) of mappings. In particular, g is a non-contractive fuzzy self-mapping, in the framework of non-Archimedean ordered fuzzy complete metric spaces and T is a p-cyclic proximal contraction. Two types of such contractions (so called of type I and of type II) are dealt with. In particular, the existence, uniqueness and limit properties for sequences to optimal fuzzy best proximity coincidence points are investigated for such pairs of mappings.

Keywords: fixed points; best proximity points; fuzzy set; fuzzy metric; optimal fuzzy best proximity coincidence points; proximal ψ-contractions of types I and II

1. Introduction

Concepts and related results on fuzzy sets in several research disciplines are abundant in the background literature. From a mathematical context, studies are available, for instance, in [1–18] and the references therein, following its introduction and characterization by Zadeh [4]. Among the research performed on the subject, effort has been devoted to the investigation of the existence and uniqueness of fixed points, best proximity points, fuzzy fixed points, fuzzy best proximity points, common fuzzy fixed points and optimal fuzzy coincidence points [15–28]. Also, research has been devoted to related properties of convergence of sequences to the abovementioned relevant points. Fixed Point Theory is also relevant to the stability properties of some iterative schemes of that of dynamic systems [29–33], as an alternative tool to other classical techniques like Lyapunov stability. (See, for instance, [33–37].) There are also abundant studies on all such topics in classical metric spaces and Banach spaces, either in the fuzzy formalism or not necessarily under the fuzzy formalism, including a lot of research on contractive and non-expansive mappings, self-mappings and, in particular, cyclic proximal mappings. (See, for instance, [26–29,38–43] and the references therein concerning different iterative schemes and their relations to proximal split feasibility, variational inequalities and fixed point problems. There are also recent studies on the generalizations of several types of contractions in [31] with an introduction of the so-called simulation function.
Recent work in fuzzy metric spaces and probabilistic metric spaces can be found in [22–25]. Also, the so-called simulation function is introduced and discussed in [31] related to a new special generalization of contraction that generalizes the Banach contraction and unifies several previously known types of contractions.

There are certain real-life problems for which fixed points, best proximity points, optimal coincidence points or optimal best proximity coincidence points do not exist, so their approximate counterparts are looked for in order to have an approximate solution of the problem at hand. We recall the following basic concepts:

If \((X, d)\) is a metric space, \(A, B \subseteq X\) are non-empty then:

1. \(x \in A\) is a fixed point of \(T : A \rightarrow A\) if \(d(x, Tx) = 0\);
2. \(x \in A\) is an approximate fixed point of \(T : A \rightarrow A\) if \(d(x, Tx) = \inf \{d(y, Tx) : y \in A\}\);
3. \(x \in A\) is a best proximity point of \(T : A \rightarrow B\) in \(A\) if \(d(x, Tx) = d(A, B) = \inf \{d(z, y) : z \in A, y \in B\}\);
4. \(x \in A\) is an approximate best proximity point of \(T : A \rightarrow B\) in \(A\) if \(d(x, Tx) = \inf \{d(y, Tx) : y \in A\}\);

Note that a fixed point of \(T : A \rightarrow A\) is an approximate fixed point of \(T : A \rightarrow A\) while the converse is not true, in general. Also, a best proximity point of \(T : A \rightarrow B\), which is also a fixed point if \(A\) and \(B\) intersect, is an approximate best proximity point of \(T : A \rightarrow B\) while the converse is not true, in general. If we have two mappings \(g : A \rightarrow A\) and \(T : A \rightarrow B\) then:

5. \(x \in A\) is an optimal best proximity coincidence point of the pair \((g, T)\) if \(d(gx, Tx) = d(A, B)\);
6. \(x \in A\) is an approximate optimal best proximity coincidence point of the pair \((g, T)\) if \(d(gx, Tx) = \inf \{d(gy, Tx) : y \in A\}\).

Note that optimal best proximity coincidence points are also approximate optimal best proximity coincidence points but the converse is not true, in general. Note also that, if \(A\) and \(B\) intersect, then an optimal best proximity coincidence point of the pair \((g, T)\) is also a coincidence point of \((g, T)\). The above concepts can be extended to the “fuzzy” framework formalism when dealing with fuzzy metric spaces. The purpose of this paper is to investigate some relevant properties of two types of fuzzy ordered proximal contractions of cyclic self-mappings \(T\) integrated in a pair \((g, T)\) of mappings, where \(g\) is a non-contractive fuzzy self-mapping and \(T\) is a cyclic proximal contraction, in the framework of non-Archimedean ordered fuzzy complete metric spaces. In particular, the existence, uniqueness and limit properties for sequences of optimal fuzzy best proximity coincidence points are investigated for such pairs of mappings.

**Notation**

- \(\mathbb{R}\) is the set of real numbers, \(\mathbb{R}_+ = \{z \in \mathbb{R} : z > 0\}\), \(\mathbb{R}_{0+} = \mathbb{R}_+ \cup \{0\}\);
- \(\mathbb{Z}\) is the set of integer numbers, \(\mathbb{Z}_+ = \{z \in \mathbb{Z} : z > 0\}\), \(\mathbb{Z}_{0+} = \mathbb{Z}_+ \cup \{0\}\);
- \(\mathcal{P} = \{1, \ldots, p\}\);
- \(\text{cl}(\cdot)\) is the closure of the \((\cdot)\)-set.

The subsequent equality holds for the t-norm \(* : [0, 1]^2 \rightarrow [0, 1]\) for a fuzzy set \(M\) on \(X \times X \times [0, \infty)\):

\[
\bigwedge_{n=m}^n M(x, y, t_n) = M(x, y_1, t_m) * M(z_0, z_1, t_{m+1}) * \ldots * M(z_0, z_1, t_n)
\]

Some useful technical definitions to be used are given below:

**Definition 1 [1].** A binary operation \(* : [0, 1]^2 \rightarrow [0, 1]\) is said to be a continuous t-norm if:

(i) \(*\) is continuous, commutative and associative;
(iii) \( a * 1 = 1 \) for all \( a \in [0, 1] \);

(iii) \( a * b \leq c * d \) if \( a \leq c \) and \( b \leq d \).

The formalism of fuzzy sets was proposed by Zadeh [4]. The following formal definition of fuzzy sets on non-Archimedean fuzzy metric spaces will be used throughout this manuscript.

**Definition 2** [2,3]. Let \( X \) be a non-empty set and \( * \) be a continuous \( t \)-norm. A fuzzy set \( M \) on \( X \times X \times [0, \infty) \) is said to be a fuzzy metric on the non-Archimedean fuzzy metric space \((X, M, *)\) if for any \( x, y, z \in X \), the following conditions hold:

(i) \( M(x, y, t) > 0 \);

(ii) \( x = y \) if and only if \( M(x, y, t) = 1 \); \( \forall t \in \mathbb{R}_+ \);

(iii) \( M(x, y, t) = M(y, x, t) \);

(iv) \( M(x, y, \max(t, s)) = M(x, z, t) * M(z, y, s) \); \( \forall t, s \in \mathbb{R}_+ \);

(v) \( M(x, y, \cdot) : [0, \infty) \to [0, 1] \) is left-continuous.

If the condition (iv) of Definition 2 is replaced with \( M(x, y, t + s) = M(x, z, t) * M(z, y, s) \); \( \forall t, s \in \mathbb{R}_+ \) then \((X, M, *)\) is a (Archimedean) fuzzy metric space and \( M(x, y, \cdot) \) is non-decreasing on \([0, \infty)\) and continuous on \( X^2 \times (0, \infty) \) [5]. If \( t = s \) then \( M(x, y, t) = M(x, z, t) * M(z, y, t) \);

\( \forall t \in \mathbb{R}_+ \) and \( M \) is said to be the strong metric on \( X \). Each fuzzy metric \( M \) on \( X \) generates a Hausdorff topology \( \tau_M \) whose base is the family of open balls of members \( B_M(x, \varepsilon, t) = \{ y \in X : M(x, y, t) > 1 - \varepsilon \} \) for \( \varepsilon \in (0, 1) \), \( t \in \mathbb{R}_+ \), and a sequence \( \{ x_n \} \) converges to \( x \in X \) with respect to \( \tau_M \) if and only if \( \lim_{n \to \infty} M(x_n, x, t) = 1 \); \( \forall t \in \mathbb{R}_+ \). Note that, since (iv) implies the above condition, any non-Archimedean fuzzy metric space is a fuzzy metric space.

**Definition 3** [13]. Let \( A \) and \( B \) be two non-empty subsets of a non-Archimedean fuzzy metric space \((X, M, *)\). Define the sets \( A_0(t) \) and \( B_0(t) \) as:

\[
A_0(t) = \{ x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B \}, \\
B_0(t) = \{ y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A \}.
\]

**Definition 4** [39]. Let \( \Psi \) be the set of all mappings \( \psi : [0, 1] \to [0, 1] \) satisfying the following properties:

(i) \( \psi(0) = 0, \psi(1) = 1 \) and \( \psi(t) > t \) for \( t \in (0, 1) \) and it is continuous in \((0, 1)\),

(ii) \( \lim_{n \to \infty} \psi^n(t) = 1 \) if and only if \( t = 1 \).

A point \( x \) in an abstract non-empty set \( X \) will be said to be an optimal fuzzy best proximity coincidence point of the pair of mappings \((g, T)\), where \( g : A \to A \) and \( T : A \to B \), where \( A \) and \( B \) are non-empty subsets of \( X \), if \( M(gx, Tx, t) = M(A, B, t) \), where \( X \) is a non-empty set and \( M \) is a fuzzy metric. The main paper body consists of other two sections. In such sections, some proximal contractions associated with pair \((g, T)\) where \( g \) is a non-contractive self-mapping and \( T \) is \( p \)-cyclic fuzzy ordered proximal contractive self-mapping are formalized and some of their properties, like convergence of sequences and existence and uniqueness of optimal fuzzy best proximity coincidence points are investigated. The obtained results and their discussion are split into two sections as follows. Section 2 introduces some necessary concepts in the fuzzy framework while some results are obtained and proved concerning the so-called optimal fuzzy best proximity coincidence points in partially ordered non-Archimedean fuzzy metric spaces for cyclic fuzzy order preserving proximal \( \psi \)-contractions of type I. Section 3 reformulates the above results for another type of proximal contractions, so-called type II. Both sections contain and discuss some illustrative examples.
2. Results and Discussion on Optimal Fuzzy Best Proximity Coincidence Points in Non-Archimedean Fuzzy Metric Spaces for Cyclic Fuzzy Order Preserving Proximal $\psi$-Contractions of Type I

This section is devoted to give a framework related to the existence of best proximity coincidence points in partially ordered non-Archimedean fuzzy metric spaces for cyclic fuzzy order preserving proximal $\psi$-contractions of type I. A set of necessary definitions are given to set and prove the results and a set of “ad hoc” discussed examples is also given.

Definition 5. Let $A$ be a non-empty subset of a non-Archimedean fuzzy metric space $(X, M, \ast)$. A self-mapping $f$ on $A$ is said to be:

1. a fuzzy isometry if $M(fx, fy, t) = M(x, y, t)$ for all $x, y \in A$ and $t \in \mathbb{R}_+$;
2. fuzzy non-contractive if for any $x, y \in A$ and $t \in \mathbb{R}_+$, we have $M(fx, fy, t) \leq M(x, y, t)$.

Definition 6. Let $(X, \prec)$ be a preordered set and let $X_i \subseteq X$ be non-empty sets; $\forall i \in \mathbb{P}$. A $p(\geq 2)$-cyclic mapping $T : \bigcup_{i \in \mathbb{P}} X_i \to \bigcup_{i \in \mathbb{P}} X_i$ is said to be non-decreasing, or order preserving with respect to a preorder relation $\prec$ on $\bigcup_{i \in \mathbb{P}} X_i(t)$, if:

(a) the binary preorder relation $\preceq$ on $\bigcup_{i \in \mathbb{P}} X_i$ is a partial order relation on each set $X_i(t); \forall i \in \mathbb{P}$;
(b) for any $x, y \in X_i$ and any $i \in \mathbb{P}$ if $x \preceq y$ then $Tx \preceq Ty$.

Remark 1. Note that Definition 6 could be restated under stronger conditions with the binary preorder relation $\preceq$ holding on $\bigcup_{i \in \mathbb{P}} X_i$ while being a partial order relation on each set $X_i; \forall i \in \mathbb{P}$. Note that $X_i \supseteq A_i(t); \forall i \in \mathbb{P}$ and $\bigcup_{i \in \mathbb{P}} X_i \supseteq \left(X_i \bigcup_{i \neq \emptyset} X_{i+1}(t)\right)$. In this context, Definition 6 can be applied to the partially order preserving non-Archimedean fuzzy metric space $(X, M, \ast, \preceq)$ (i.e., $(X, \prec)$ is a partially order preserving set and $(X, M, \ast)$ is a non-Archimedean fuzzy metric space, [38]) to a $p$-cyclic mapping $T : \bigcup_{i \in \mathbb{P}} X_i \to \bigcup_{i \in \mathbb{P}} X_i$ where $X_i \subseteq Xare non-empty sets; $\forall i \in \mathbb{P}$.

The concepts of order preserving, order reversing and monotone mappings $T : A \to B$ have been discussed in [11], where related results have been obtained. An “ad hoc” adaptation of the concept of order preserving for cyclic mappings is proposed in the subsequent definitions:

Definition 7. A $p(\geq 2)$-cyclic mapping $T : \bigcup_{i \in \mathbb{P}} X_i \to \bigcup_{i \in \mathbb{P}} X_i$ is said to be a (strong) $p$-cyclic fuzzy ordered proximal $\psi$-contraction of type I if for any $u, v \in X_{i0}(t), x, y \in X_i$ and any given $i \in \mathbb{P}$, the following condition holds:

$$
\begin{align*}
M(u, Tx, t) &= M(X_i, X_{i+1}, t) \\
M(v, Ty, t) &= M(X_i, X_{i+1}, t)
\end{align*}
\Rightarrow M(u, v, t) \geq \psi(M(x, y, t)),
$$

where $\psi \in \Psi$ for all $t \in \mathbb{R}_+$.

If the above conditions hold for any $u, v, x, y \in X_{i0}(t)$ and any given $i \in \mathbb{P}$ then $T : \bigcup_{i \in \mathbb{P}} X_i \to \bigcup_{i \in \mathbb{P}} X_i$ is said to be a weak $p$-cyclic fuzzy ordered proximal $\psi$-contraction of type I.

Definition 8. A $p(\geq 2)$-cyclic mapping $T : \bigcup_{i \in \mathbb{P}} X_i \to \bigcup_{i \in \mathbb{P}} X_i$ is said to be a (strong) $p$-cyclic fuzzy ordered proximal $\psi$-contraction of type II if for some $\alpha \in (0, 1)$, any $u, v \in X_{i0}(t), x, y \in X_i$ and any given $i \in \mathbb{P}$, the following condition holds:

$$
\begin{align*}
M(u, Tx, t) &= M(X_i, X_{i+1}, t) \\
M(v, Ty, t) &= M(X_i, X_{i+1}, t)
\end{align*}
\Rightarrow M(u, v, t) \geq \psi\left(M\left(x, y, \alpha^{-1}t\right)\right),
$$

where $\psi \in \Psi$ for all $t \in \mathbb{R}_+$.
If the above conditions hold for any \( u, v, x, y \in X_0(1) \) and any given \( i \in \mathcal{P} \) then \( T : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) is said to be a weak \( p \)-cyclic fuzzy ordered proximal \( \psi \)-contraction of type II.

**Definition 9.** A \( p(\geq 2) \)-cyclic mapping \( T : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) is said to be (strong) \( p \)-cyclic proximal fuzzy order preserving if for any \( u, v \in X_i(1), x, y, z \in X_i \) and any given \( i \in \mathcal{P} \) then \( z \preceq Txz \) and the following condition holds:

\[
\begin{align*}
M(u, Tx, t) &= M(X_i, X_{i+1}, t) \\
M(v, Ty, t) &= M(X_i, X_{i+1}, t)
\end{align*}
\]

If the above conditions hold for any \( u, v, x, y, z \in X_0(1) \) and any given \( i \in \mathcal{P} \) then \( T : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) is said to be a weak \( p \)-cyclic proximal fuzzy order preserving mapping.

**Definition 10 [20].** A point \( z \) in an abstract non-empty set \( X \) is said to be an optimal fuzzy best proximity coincidence point of the pair of mappings \((g, T)\), where \( g : A \to A \) is a self-mapping and \( T : A \to B \) is, in general, a non-self mapping, \( A \) and \( B \) are non-empty subsets of \( X \) if \( M(gz, Tz, t) = M(A, B, t) \).

**Remark 2.**

1. Note that Definition 10 is applicable to the case when the mapping \( T : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) is \( p \)-cyclic (so that \( T(X_i) \subseteq X_{i+1}; \forall i \in \mathcal{P} \)) and \( g : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) is p-cyclic proximal fuzzy order preserving (with respect to \( \preceq \)) order preserving of type I and let \( g : \cup_{i \in \mathcal{P}}X_i \to \cup_{i \in \mathcal{P}}X_i \) be surjective, fuzzy-contractive and inverse monotone mapping such that, for any \( x, y \in \cup_{i \in \mathcal{P}}X_i(1) \), \( gx \) and \( gy \) are comparable with respect to \( \preceq \) only if \( x, y \) are comparable. Suppose also that each pair of elements of \( \cup_{i \in \mathcal{P}}X_0(t) \) has a lower bound and an upper bound and that for any \( t \geq 0 \), \( X_0(t) \) is non-empty, \( T(X_0(t)) \subseteq X_{0+1}(t) \) and \( X_0(t) \subseteq g(X_0(t)) \); \( \forall i \in \mathcal{P} \). If for each given \( x_{0i} \in X_0(t) \) for each \( i \in \mathcal{P} \), there exists some element \( x_i \) in \( X_i(1) \) such that:

\[
M(gx_i, Tx_{i+1}, t) = M(X_i, X_{i+1}, t) \text{ with } x_{0i} \preceq x_i
\]

then there exists a unique element \( x^*_i \in clX_0(t) \) for each \( i \in \mathcal{P} \) that is an optimal fuzzy best proximity coincidence point of the pair \((g, T)\) in \( X_i \) such that \( x^*_i = Tx^*_i; \forall i \in \mathcal{P} \), and then:

\[
M(gx^*_i, x^*_{i+1}, t) = M(X_i, X_{i+1}, t); \forall i \in \mathcal{P}
\]

Furthermore, each of the subsequences \( \{z_{in}\} \subseteq \{z_n\} \cap X_0(t) \) for each \( i \in \mathcal{P} \) for \( n \in \mathbb{Z}_+ \), with \( z_0 = x_0 \in X_0(t) \), being defined by any given first element \( z_0 \in \cup_{i \in \mathcal{P}}X_0(t) \) so that the proximal constraint:

\[
M(gz_{n+1}, Tz_n, t) = M(X_i, X_{i+1}, t)
\]
holds for any given $n \in \mathbb{Z}_{0+}$, $\forall j \in \mathbb{N}$, is a Cauchy sequence that is convergent to the optimal fuzzy best proximity coincidence point $x_j^* \in clX_0(t)$ (the closure of $X_0(t)$) of the pair $(g, T)$ in $X_t$ provided that the two subsequent conditions hold:

1. $n_{j-1,k} + 1 \leq n \leq n_{j,k}$, with $n_{j,k} + 1 - n_{j-1,k} \geq 2$, for $j = i, i + 1, ..., i + p - 2$, and $n_{i+p-2,k} + 1 \leq n \leq n_{i+p-1,k} + 1$, with $n_{i+p-1,k} - n_{i+p-2,k} \geq 2$; $\forall i, j \in \mathbb{N}$, $\forall k \in \mathbb{N}_{0+}$;

2. the chosen arbitrary $i \in \mathbb{N}$ is such that the initial points $(z_0, z_1) \in X_0(t) \times X_0(t)$ and that the sets of positive integers $\{n_{j,k} - n_{j-1,k}\}$, for $j \in \mathbb{N}$ and $k \in \mathbb{N}_{0+}$ have $p$ positive upper-bounding integers $\pi_j \geq n_{j,k} - n_{j-1,k}$; $\forall k \in \mathbb{N}_{0+}$, $\forall j \in \mathbb{N}$ with $n_{i-1,0} = -1$ and $n_{i,0} \geq 1$.

**Proof.** Denote the restricted mappings $T : \cup_{i \in \mathbb{N}} X_i | X_i \to \cup_{i \in \mathbb{N}} X_i | X_{i+1}$ and $g : \cup_{i \in \mathbb{N}} X_i | X_i \to \cup_{i \in \mathbb{N}} X_i | X_i$; $\forall i \in \mathbb{N}$ of the functions $T : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ and $g : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$, respectively, by $T_i : X_i \to X_{i+1}$ and $g_i : X_i \to X_{i+1}$; $\forall i \in \mathbb{N}$. Let $z_0 = x_0$ arbitrary in $X_0(t)$ for the given $i \in \mathbb{N}$ and some $z_1 = x_{i,0} \in X_0(t)$ for any given arbitrary $i \in \mathbb{N}$ be such that $z_0 \geq z_1$ and $M(gz_1, Tz_0, t) = M(X_1, X_1, t, l)$. Since $Tz_0 \in T(X_0(t)) \subseteq X_0(t)$ and $X_0(t) \subseteq g(X_0(t), t)$, $z_2 \in X_0(t)$ exists such that $M(gz_2, Tz_1, t) = M(X_{i,1}, X_{i+1}, t, l)$. Since $T : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ is weak $p$-cyclic proximal fuzzy order preserving and $z_0 \leq z_2$ then $gz_1 \leq g\bar{z}_2$ and, since $g : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ is surjective and $X_0(t) \subseteq g(X_0(t), t)$ it then follows that $\tau_3 \leq z_2$. Assume that this is not the case and proceed by contradiction. Since $g\bar{z}_1$ and $g\bar{z}_2$ are comparable, which holds, by hypothesis, only if $z_1$ and $z_2$ are comparable, then $\tau_3 \geq z_2$ since $z_1 \leq z_2$ is assumed false. However, then $g\bar{z}_1 \geq g\bar{z}_2$, which contradicts $g\bar{z}_1 \leq g\bar{z}_2$. Then, $z_1 \leq \tau_3$ and, as a result, $z_0 \leq \tau_3 \leq z_2$. Proceding in the same way, we can build a sequence $\{z_n\}$ with $z_n \in X_0(t)$ for $1 \leq n \in \mathbb{Z}_{0+} \leq n_0$, where $1 \leq n_0 \in \mathbb{Z}_{0+} \leq n_i - 1$, such that $M(gz_{n+1}, Tz_{n}, t) = M(X_{i,n+1}, t, l); z_{n+1} \leq \tau_{n+1}; \forall n \in \mathbb{Z}_{0+}$. Then, take $z_{n_0+1} = Tz_{n_0} \in X_{n_0+1}(t)$ such that $z_{n_0+1} \leq z_{n_0+2}$ since $T : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ is a weak $p$-cyclic proximal fuzzy order preserving, $z_{n_0} \in X_0(t)$ and $z_{n_0+1} = Tz_{n_0} \in X_{n_0+1}(t)$. As a result, $z_{n+1} \leq z_{n+2}$; $\forall n \in \mathbb{Z}_{0+} \leq n_0 + 1$. Again, for such a $z_{n_0+1} \in X_{n_0+1}(t)$, there is some $z_{n_0+1} \in X_{n_0+1}(t)$ such that $z_{n_0+1} \leq z_{n_0+2}$ and for $1 \leq n_0+1,0 \in \mathbb{Z}_{0+} \leq n_i - 1$. Then, the elements of $\{z_n\}$ from $n = 0$ to $n = n_0 + 1$ are ordered, with the order preserved with respect to the preorder relation $\leq$ and $z_1 \leq z_2 \leq z_3 \leq \ldots \leq z_{n_0+1} \leq z_{n_0+2} \leq \ldots \leq z_{n_0+1}$ for $m = 1, 2$. By keeping $k = 0$, we proceed in the same way by running $j$ from $i$ to $i + p - 2$ and prove that the finite subsequence of $\{z_n\}$ from $n = 0$ to $n = n_{i+p-2} + 2$ is also totally ordered with respect to $\leq$ since $n_{j-1,0} + 1 \leq n \leq n_{j,0}$ for $j = i, i + 1, ..., i + p - 2$; $\forall j \in \mathbb{N}$ for the given $i \in \mathbb{N}$. Now, the same reasoning is used for $j = i + 1$ and $p - 1$ and for $k = 1$ and $n_{i+p-2,1} + 1 \leq n \leq n_{i+p-1,1}$ to conclude that the elements of $\{z_n\}$ from $n = 0$ to $n = n_{i+p-1} + 2$ are ordered with respect to the relation $\leq$. Proceeding recursively for each $j = i, i + 1, ..., i + p - 2$ and each integer $k \in \mathbb{Z}_{+}$, it is proved that, $z_n \in X_0(t)$ for $n \leq n_{j-1,k} \leq n_{j,k}$, for $i = i, i + 1, ..., i + p - 2$; $\forall k \in \mathbb{N}_{0+}$ and $z_{n_0} \in X_0(t)$, and that the sequence $\{z_n\}$ is totally ordered with respect to $\leq$. Define the strictly ordered set of positive integers $N^* = \{n_{j-1,k}\}$; $j = i, i + 1, ..., i + p - 1$, $\forall k \in \mathbb{N}_{0+}$ with $n_{j,k} < n_{j+1,k}$ and $n_{k,j} < n_{k+1,j}$, $\forall j \in \mathbb{N}$, $\forall k \in \mathbb{N}_{0+}$. Since $T : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ is a weak $p$-cyclic proximal fuzzy order preserving (with respect to $\leq$) proximal $\psi$-contraction of type I and $g : \cup_{i \in \mathbb{N}} X_i \to \cup_{i \in \mathbb{N}} X_i$ is fuzzy non-contractive, one has for $z_n \in \{z_n\}$ since $\psi : [0, 1] \to [0, 1]$ is in the set $\Psi$:

$$\tau_n(t) = M(z_n, z_{n+1}, t) \geq M(gz_n, g\bar{z}_{n+1}, t) \geq \psi(M(z_{n-1}, z_n, t)) > \tau_{n-1}(t) = M(z_{n-1}, z_n, t)$$ (4)

if $n, (n + 1) \notin N^*$ and:

$$\tau_n(t) = M(z_n, z_{n+1}, t) \geq M(gz_n, g\bar{z}_{n+1}, t) \geq \psi(M(z_{n-1}, z_n, t)) > \tau_{n-1}(t) = M(z_{n-1}, z_n, t)$$ (5)

if $(n + 1), (n + 2) \in N^*$. Note that if $z_{n-1}$ and $z_n$ are in $X_0(t)$ then $z_n$ and $z_{n+1}$ are in $X_0(t)$, for some $f \in \mathbb{N}$, if $n, (n + 1) \notin N^*$ and in $X_0(t)$ $(n + 1), (n + 2) \in N^*$ since, by hypothesis, $n_{j-1,k} + 1 \leq$
It is now proven that for each \( i, i + 1, \ldots, i + \ell - 2 \) and \( i + \ell - 1 - k \geq 2 \), such that \( m_{jk} \) is a non-negative integer number that depends on the initial \( z_0 \in X_{0j}(t) \), such that \( z_{m_{jk}} \in X_{0j}(t) \) and \( z_{m_{jk} + 1} \) are in \( X_{0j}(t) \cap \{z_n\} \) for all \( k \in Z_{0+} \) and each \( j \in \overline{p} \).

Also, for the subsequence \( \{z_{m_{jk} + \ell}\} \subseteq \{z_n\} \) for \( \ell = 0, 1, 2 \), and any given \( j \in \overline{p} \), \( \tau_{m_{jk} + \ell}(t) \) and \( \ell = 0, 1, 2 \) are strictly increasing sequences with \( n \in Z_{0+} \) for each given \( j \in \overline{p} \), so that \( \tau_{m_{jk} + \ell}(t) \to \infty \) as \( n \to \infty \) for \( \ell = 0, 1, 2 \), and convergent to a limit \( \tau_j(t) \) in \( (0, 1) \) for each \( t \in R_+ \) and \( j \in \overline{p} \) since \( \psi : [0, 1] \to [0, 1] \) is continuous and non-decreasing and Equations (4) and (5) hold with:

\[
1 \geq \tau_{m_{jk} + \ell}(t) > \psi(\tau_{m_{jk} - 1 + \ell}(t)) > \tau_{m_{jk} + \ell}(t); \forall n \in Z_{0+}
\]

\[
1 \geq \tau_j(t) = \lim_{n \to \infty} \tau_{m_{jk} + \ell}(t) \geq \lim_{n \to \infty} \tau_{m_{jk} + \ell}(t); \forall n \in Z_{0+}
\]

For each \( j \in \overline{p} \). Assume that there is \( t_0 \in R_+ \) such that \( 1 > \tau_j(t_0) \) for some \( j \in \overline{p} \). Then, the subsequent contradiction follows:

\[
\tau_j(t_0) = \lim_{n \to \infty} \tau_{m_{jk} + \ell}(t_0) \geq \psi(\tau_j(t_0)) > \tau_j(t_0)
\]

for each \( j \in \overline{p} \). Thus, \( \tau_j(t) = 1 \) for all \( t \in R_+ \) and all \( j \in \overline{p} \). It is now proven that the subsequences \( \{z_{m_{jk}}\} \subseteq X_{0j}(t) \) are Cauchy sequences in \( X_{0j}(t) \) for each given \( j \in \overline{p} \). Suppose that there is a sequence \( \{z_{m_{jk}}\} \) that is not Cauchy for some \( j \in \overline{p} \). Then, there exists \( \varepsilon \in (0, 1) \) and \( t_0 \in R_+ \) such that for all \( k \in Z_{0+} \), there are \( n_{jk} \) \( m_{jk} \) \( n_k \) \( Z_{0+} \) such that \( M(z_{n_{jk}}, z_{m_{jk}}, t_0) \leq 1 - \varepsilon \), for some \( j \in \overline{p} \). Assume that \( m_{jk} \) is the least integer exceeding \( n_{jk} \) and satisfying the above inequality so that:

\[
M(z_{m_{jk} + \ell}, z_{m_{jk}} , t_0) > 1 - \varepsilon
\]

Then, one obtains for all \( k \in Z_{0+} \):

\[
1 - \varepsilon \geq M(z_{m_{jk}}, z_{m_{jk}}, t_0) \geq M(z_{m_{jk}}, z_{m_{jk} - 1}, t_0) \geq M(z_{m_{jk} - 1}, z_{m_{jk}}, t_0) > \tau_{m_{jk} - 1}(t_0) * (1 - \varepsilon)
\]

and, since \( \lim_{k \to \infty} \tau_{m_{jk} - 1}(t_0) = \tau_j(t_0) = 1 \) for all \( j \in \overline{p} \), \( \tau_j(t) = 1 \) is a continuous \( t \)-norm and \( \varepsilon > 0 \), one gets by taking the limits in Equation (10) as \( k \to \infty \) the following contradiction:

\[
1 - \varepsilon > 1 * (1 - \varepsilon) = 1 - \varepsilon
\]

As a result, \( \{z_{m_{jk}}\} \) is a Cauchy sequence for all \( j \in \overline{p} \). Since \( (X, M, *, \leq) \) is complete, there exists \( x^*_j \in clX_{0j}(t) \) such that \( \{z_{m_{jk}}\} \to x^*_j \); \( \forall j \in \overline{p} \) since \( m_{jk} \to \infty \) as \( n \to \infty \) and \( \lim M(z_{m_{jk}}, x^*_j, t) = 1; \forall j \in \overline{p}, \forall t \in R_+ \). Since \( T : \cup_{j \in \overline{p}} X_j \to \cup_{j \in \overline{p}} X_j \) is continuous, this also implies that:

\[
M(gx^*_j, T(x^*_j), t) = \lim_{n \to \infty} M(Tz_{m_{jk}}, gzm_{jk + 1}, t) = \lim_{n \to \infty} M(Tx^*_j, gzm_{jk + 1}, t) = \lim_{n \to \infty} M(gx^*_j, Tz_{m_{jk}}, t) = M(X_j, X_{j + 1}, t)
\]

so that \( x^*_j \in clX_{0j}(t) \) is the common best proximity point of the pair \( (g, T) \) in \( X_{0j}(t) \) for each \( j \in \overline{p} \). It is now proven that \( x^*_j \in clX_{0j}(t) \) is unique for each \( j \in \overline{p} \). This is equivalent to proving that, for any fixed element \( x_0 \), \( x_0 \in X_{0j}(t) \), the subsequence \( \{x_{m_{jk}}\} \subseteq X_{0j}(t) \) of \( \{x_{m_{jk}}\} \subseteq \cup_{j \in \overline{p}} X_{0j}(t) \) converges to the same \( x^*_j \); \( \forall j \in \overline{p} \). Since \( T : \cup_{j \in \overline{p}} X_j \to \cup_{j \in \overline{p}} X_j \) is a weak \( p \)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type I and \( g : \cup_{j \in \overline{p}} X_j \to \cup_{j \in \overline{p}} X_j \) is fuzzy non-contractive, one gets the subsequent contradiction under the assumption that \( x^*_j \neq x_j^* \) for some \( j \in \overline{p} \) such that, if
where \( m \neq 0 \) and \( x^j \) and \( \tau^j \) are comparable, the convergent subsequences \( \{ x_{m^n} \} \subset X_{0j}(t) \cap \{ x_n \} \rightarrow x^j \),
\( \{ x_{\tau^n} \} \subset X_{0j}(t) \cap \{ \tau_n \} \rightarrow \tau^j \neq x^j \), such that the sequences of nonnegative integers \( \{ m^n \} \) and
\( \{ \tau^n \} \) with \( m^n, \tau^n \rightarrow \infty \) as \( n \rightarrow \infty \); \( \forall j \in \mathbb{P} \), defined by:
\[
M(\{ g\tau^n_{m^n+1}, T\tau^n_{m^n+1}, t \}) = M(\{ gx^n_{m^n+1}, Tx^n_{m^n+1}, t \}) = M(\{ X_j, X_{j+1}, t \}), \forall n \in \mathbb{Z}_0^+
\]
satisfy the set of inequalities:
\[
M(\{ x^j, \tau^j, t \}) \geq M(\{ gx^j, g\tau^j, t \}) \geq M(\{ T\tau^j, T\tau^j, t \}) = M(\{ \tau^j, \tau^j, t \})
\]
so that \( x^j = \tau^j \); \( \forall j \in \mathbb{P} \). Now, assume that the corresponding elements of the sequences \( \{ x_n \} \) and
\( \{ \tau_n \} \), with distinct initial values \( x_0 \) and \( \tau_0 \), are not all pair-wise comparable. By the hypothesis of the
mapping \( T \) being a fuzzy order preserving proximal \( \psi \)-contraction of type I and the hypothesis of the
mapping \( g \) being inverse monotone, both sequences are lower-bounded and upper-bounded by sequences \( \{ u_n \} \) and
\( \{ \pi_n \} \), with the corresponding subsequences \( \{ u_{m^n} \} \) and \( \{ \pi_{m^n} \} \), respectively, in
any of the sets \( X_0(t) \), which are constructed from:
\[
M(\{ gu_{m^n+1}, Tu_{m^n+1}, t \}) = M(\{ g\pi_{m^n+1}, T\pi_{m^n+1}, t \}) = M(\{ X_j, X_{j+1}, t \}), \forall j \in \mathbb{P}
\]
provided that \( m^n, m^n + 1 \in \{ m^n \} \subset \mathbb{Z}_0^+ \), \( \tau^n, \tau^n + 1 \in \{ \tau^n \} \subset \mathbb{Z}_0^+ ; \forall j \in \mathbb{P} \) and have order
comparison properties of the form \( u_{m^n} \leq x_{m^n} \leq \pi_{m^n} \) and \( u_{m^n} \leq \tau_{m^n} \leq \pi_{m^n} \); \( \forall n \in \mathbb{Z}_0^+ \), \( \forall j \in \mathbb{P} \) since
the sequences \( \{ x_{m^n} \} \) and \( \{ \tau_{m^n} \} \); \( \forall j \in \mathbb{P} \), are totally ordered, and also both lower-bounded and
upper-bounded by the pair-wise comparable sequences \( \{ u_{m^n} \} \) and \( \{ \pi_{m^n} \} \), respectively, since they
are convergent. Then, \( \lim u_{m^n} = \lim \pi_{m^n} = \lim \tau_{m^n} = \lim x_{m^n} = x^j ; \forall j \in \mathbb{P} \).

It is now proved that \( x^j_{j+1} = Tx^j_j; \forall j \in \mathbb{P} \). Since \( x^j_{j+1}, Tx^j_j \in X_{0j+1}(t) \), then:
\[
M(\{ gx^j_j, T(Tx^j_j), t \}) = \lim_{n \rightarrow \infty} M(\{ Tz^j_{m^n+1}, g\tau^j_{m^n+1}, t \}) = M(\{ Tx^j_{j+1}, g\tau^j_{j+1}, t \}) \quad (14)
\]
where \( \{ z^j_{m^n+1} \} \subset X_{0j+1}(t) \) is a Cauchy sequence of initial value \( Tx^j_j \) that is convergent to some
\( z^j_{j+1} \in c\{ X_{0j+1}(t) \} \), while another subsequence in \( X_{0j}(t) \) of a sequence with initial consecutive values
\( x_0, x_1, \in X_{0j}(t) \) converges to \( x^j_{j+1} = z^j_{j+1} \in X_{0j+1}(t) \). However, since \( x^j_j \in X_{0j}(t) \) is unique and
\( Tx^j_j \in X_{0j+1}(t) \) for all \( j \in \mathbb{P} \), then:
\[
M(\{ X_{j+1}, X_{j+2}, t \}) = M(\{ Tx^j_{j+1}, g\tau^j_{j+1}, t \}) = M(\{ X_{j+1}, X_{j+2}, t \}) \quad (15)
\]
since \( z^j_{j+1} \) is a unique limit of Cauchy sequences in \( X_{0j+1}(t) \) then \( z^j_{j+1} = Tx^j_j; \forall j \in \mathbb{P} \). So the unique
limit of all Cauchy subsequences in \( X_{0j+1}(t) \) is \( x^j_j = z^j_{j+1} = Tx^j_j; \forall j \in \mathbb{P} \). \( \square \)

Remark 3. Theorem 1 guarantees the existence of Cauchy sequences that are constructed from the proximal
constraints and their convergence to unique optimal fuzzy best proximity coincidence points of the pair \( (g, T) \) in
Sketch of Proof. Note that the proximal constraint (16) may be rewritten as:

\[ x_\ell \] and convergent to a unique \( z_{n} \) such that the nonnegative integers \( z_n \) are located at \( \text{cl}X_0(t) \) for any \( n \in \mathbb{Z}_+ \) if \( z_n \in X_0(t) \) for any \( i \neq j \) in \( \mathcal{P} \) and some fixed \( j \in \mathcal{P} \) (i.e., the proximal subsequence from each subset \( X_i \) of \( X \) to each next adjacent subset is only computed eventually at the subset \( X_0(t) \), while at the remaining subsets only the cyclic self-mapping \( T \) is involved). (2) the proximal constraint Equation (3), subject to its subsequent constraints, is replaced at the subset \( X_j \) of \( X \) by the subsequent one:

\[
M\left(g_z n_{j,k+1} + 1, T z_{n_{j,k+1} + 1 - n_j} n_{j,k}, t\right) = M\left(X_j, X_{j+1}, t\right); \forall k \geq k^* \in \mathbb{Z}_{0^+}, i \in \mathcal{P}
\]  

(16)

for some given \( j \) in \( \mathcal{P} \) and some subsequence \( \{z_{n_i}\} \subseteq \{z_n\} \cap X_0(t) \), for \( n \in \mathbb{Z}_+ \) with \( z_0 \in X_i \) for some \( i \in \mathcal{P} \) satisfying \( z_{n_{j,k+1}} = T^{n_{j,k+1} - n_j} z_{n_{j,k}} \) with \( z_{n_{j,k+1}} \in X_0(t) \), for some set of bounded positive integers \( m_j(k) \); \( \forall k \geq k^* \in \mathbb{Z}_{0^+} \), for the given \( j \in \mathcal{P} \) and some sequence of positive integers \( \{n_{j,k}\} \) being strictly increasing with \( k \geq k^* \) in \( \mathbb{Z}_{0^+} \) (i.e., the proximal subsequence at the subset \( X_0(t) \) is not necessary computed at each \( k \)-th cycle on the whole cyclic disposal of the \( p \) subsets \( X_0(t) \subseteq X_i \subseteq X \) for all \( i \in \mathcal{P} \) since \( m_j(k) \) can exceed the value \( p \) for some values \( k \geq k^* \) in \( \mathbb{Z}_{0^+} \).

Then, there exists a unique element \( x_i^* \in \text{cl}X_0(t) \) for each \( i \in \mathcal{P} \), which is an optimal fuzzy best proximity coincidence point of the pair \( (g, T) \) in \( X_i \), such that \( x_i^* \in T x_i^* \); \( \forall i \in \mathcal{P} \), and then \( M(g x_i^* x_i^* + 1, t) = M(X_i, X_{i+1}, t); \forall i \in \mathcal{P} \). Furthermore, each of the subsequences \( \{x_{n_{i}}\} \subseteq \{z_{n_{i}}\} \cap X_0(t) \) for \( i \in \mathcal{P} \) being defined by any given first element \( z_0 \in \cup_{i \in \mathcal{P}} X_0(t) \), so that the proximal constraint \( M(g x_i^* x_i^* + 1, t) = M(X_i, X_{i+1}, t); \forall n \in \mathbb{Z}_{0^+} \), is a Cauchy sequence which is convergent to the optimal fuzzy best proximity coincidence point \( x_i^* \in \text{cl}X_0(t) \) of the pair \( (g, T) \) in \( X_i \).

Sketch of Proof. Note that the proximal constraint (16) may be rewritten as:

\[
M\left(g z n_{j,k+1} + 1, T z_{n_{j,k+1} + 1 - n_j} n_{j,k}, t\right) = M\left(X_j, X_{j+1}, t\right)
\]

by defining \( z_{n_{j,k+1} + 1} = T^{n_{j,k+1} - n_j} z_{n_{j,k}} \). Thus, we can define a strictly increasing sequence of nonnegative integers \( \{\ell_n\} \) satisfying:

\[
\ell_0 = n_{j,k^*} \geq 0, \ldots, \ell_n = n_{j,k_1} + m_j(k_n) \geq 0
\]

such that the nonnegative integers \( m_j(k_n) \geq n_{j,k_1} - k_n + m_j(k_{n-1}) \) for all \( n \in \mathbb{Z}_+ \) and \( m_j(k_0) \geq 0 \), \( k_0 = k^* \in \mathbb{Z}_{0^+} \), \( n_{j,k_1} \in \mathbb{Z}_{0^+} \) with \( n_{j,k_1} \geq 0 \) such that \( z_{n_{j,k_0}} \in X_0(t) \). Then, the subsequence \( \{z_{\ell_n}\} \subseteq X_0(t) \) and satisfies the proximal condition (16). Then, according to Theorem 1, such a subsequence is Cauchy and convergent to a unique \( x_i^* \in \text{cl}X_0(t) \), which is a unique optimal fuzzy best proximity coincidence points of the pair \( (g, T) \) in \( X_i \). Since this sequence is convergent, all subsequences \( \{T x_{{\ell_n}}\} \rightarrow x_{i+j}^* \); \( j = i, i + 1, \ldots, i + p - 1 \) and \( x_{i+j}^* \) is the unique limit point and also the unique optimal fuzzy best proximity coincidence point of the pair \( (g, T) \) in \( X_{i+j} \). \( \Box \)
Theorem 1 and Corollary 1 also hold if \( T : \cup_{i \in \mathcal{F}} X_i \to \cup_{i \in \mathcal{F}} X_i \) is a \( p \)-cyclic fuzzy order preserving (strong) proximal \( \psi \)-contraction of type I such that the convergence of the constructed subsequences in each \( X_i \) for \( i \in \mathcal{F} \) converge asymptotically to be proximal subsequences converging to a unique optimal fuzzy best proximity coincidence point of \((g, T)\) in each \( X_i \) so that \( M(z_{n-1}, z_n, t) \to M(X_i, X_{i+1}, t) \) as \( n \to \infty \). In particular, Equation (1) is replaced with Equation (17) below. A related result is as follows:

**Corollary 2.** Theorem 1 and Corollary 1 also hold "mutatis–mutandis" if \( T : \cup_{i \in \mathcal{F}} X_i \to \cup_{i \in \mathcal{F}} X_i \) is a continuous \( p \)-cyclic fuzzy order preserving (with respect to \( \leq \) defined on \( \cup_{i \in \mathcal{F}} X_i \)) proximal \( \psi \)-contraction of type I provided that for each given for each \( i \in \mathcal{F} \) and some existing element \( x_{1i} \) in \( X_i \):

\[
M(gx_{1i}, Tx_{0i}, t) \leq M(X_i, X_{i+1}, t) \text{ with } x_{0i} \preceq x_{1i}
\] (17)

**Sketch of Proof.** Now take some arbitrary \( z_0 = x_{0i} \) in \( X_i \) and any arbitrary given \( i \in \mathcal{F} \) and some \( z_1 = x_{1i} \in X_i \) for any given arbitrary \( i \in \mathcal{F} \) such that (17) holds (note that equality in Equation (17) holds if and only if \( x_{0i}, x_{1i} \in X_i(1) \)). Since \( T : \cup_{i \in \mathcal{F}} X_i \to \cup_{i \in \mathcal{F}} X_i \) is a \( p \)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type I and \( g : \cup_{i \in \mathcal{F}} X_i \to \cup_{i \in \mathcal{F}} X_i \) is fuzzy non-contractive, one has for any built subsequence \( z_n \in \{ z_i \} \subset X_i \) of the whole iteration, since \( \psi : [0, 1] \to [0, 1] \) is in the set \( \mathcal{F} \), and following Equations (4), (5) and (17):

\[
M(X_i, X_{i+1}, t) \geq M(z_n, z_{n+1}, t) \geq M(gz_n, gz_{n+1}, t) \geq \psi (M(z_{n-1}, z_n, t)) \geq M(z_{n-1}, z_n, t)
\] (18)

if \( n, (n + 1) \notin N^* \), and:

\[
M(z_{n+1}, z_{n+2}, t) = M(Tz_n, Tz_{n+1}, t) \geq \psi (M(z_n, z_{n+1}, t)) \geq M(z_{n-1}, z_n, t)
\] (19)

if \( (n + 1), (n + 2) \in N^* \). One can conclude from the steps of the proof of Theorem 1 and from the sketch of the proof of Corollary 1 that \( M(z_{n-1}, z_n, t) \to M(X_i, X_{i+1}, t) \) as \( n \to \infty \) so that the subsequence \( z_n \) converges to a best proximity point in \( cl( X_{0i}(t) ) \), which is a unique optimal fuzzy best proximity coincidence point of \((g, T)\) in \( X_i \).

**Example 1.** Let \( X \subset R^2_{+} \) be defined by \( X = X_1 \cup X_2 \), where \( X_1 = \{(0, 1 + x) : x \in R_{0+}\} \) and \( X_2 = \{(1 + x, 0) : x \in R_{0+}\} \). Note that \( X_1 \cap X_2 = \emptyset \). Consider the complete ordered fuzzy metric space \((X, M, *, \preceq)\) under \( M(x, y, t) = \frac{t}{t + d(x, y)} \) \( \forall x, y \in X, \forall t \in R_{+} \) and \( M(X_1, X_2, t) = \frac{t}{t + 1} \), where \( d(x, y) = |x_1 - y_1| + |x_2 - y_2| \) for any \( x = (x_1, x_2), y = (y_1, y_2) \in X, \) and \( \preceq \) is a coordinate-wise ordering for all \( x, y, z, w \in X, \) for \( i = 1, 2 \) defined by:

(a) \((x, y) \preceq (z, w) \) if and only if \( x \leq z, y \leq w \);
(b) \((x, y) \preceq T(y, x) \) for any \((x, y) \in X, \) i.e., if \((x, y) \in X_i, T(x, y) \in X_j \) for \( i, j (\neq i) = 1, 2 \).

The proximal subsets are \( X_{01}(t) = \{(0, 1)\}, X_{02}(t) = \{(1, 0)\} \).

Define the continuous2—cyclic mapping \( T_1(X_1) = (1 + \lambda x, 0) \) and \( T_2(X_2) = (0, 1 + \lambda x) \) for some given real constant \( \lambda \in [0, 1] \). Note that \( T_1(X_0(1)) = X_{02}(1) \) and \( T_1(X_{02}(1)) = X_{01}(1) \); \( \forall \lambda \in R_{0+} \). If \( \lambda = 0 \) then \( T_0(X_0(1)) = X_{02}(2) = \{(1, 0)\} \) and \( T_1(X_{02}(1)) = X_{01}(1) = \{(0, 1)\} \).

Define \( g : X \to X \) as \( g(x) = \rho (0, x) = (1 + \rho x, 0) \) for any given \( x \in R_{0+} \) and some given real \( \rho \geq 1 \). Note that \( g_\rho : X_{01}(1) = X_{03}(1), g_\rho (X_{02}(2)) = X_{02}(2) \) and that \( g_\rho : X \to X \) is surjective and inverse monotone.

Now, we build a sequence \( \{ x_n \} \subset X_1 \cup X_2 \) by taking an initial point \( x_0 = (0, y_0) \in X_1 \) for some real \( y_0 = 1 + x \geq 1 \) for some given fixed real \( x \in R_{0+} \). Now, \( T_1 x_0 = T_1(0, y_0) = (1 + \lambda y_0, 0) = (1 + \lambda (y_0 - 1), 0) \) for some given real \( \lambda \in [0, 1] \). Take \( \psi (t) = \sqrt{t} \). Next, take some \( y_1 \in R_{+} \) such that \( x_1 = (0, y_1) \in X_{01}(1) \) and \( x_{1} \preceq x_0 \) (i.e., \( y_1 \leq y_0 \)), \( g_\rho (x_1) = \rho x_1 = \rho (0, y_1) \) for some given real \( \rho \in [1, \infty) \).
In general, \( x_0 \in X_1, T_0 \in X_2, \{ x_n \} \subset X_1, \{ T_\lambda x_n \} \subset X_2 \), with \( T_\lambda x_n = (1 + \lambda(y_n - 1), 0) \), so that \( x_{n+1} \preceq x_n \) for \( n \in Z_+ \) and:

\[
M(g_\rho x_{n+1}, T_\lambda x_n, t) = M(\rho(0, y_{n+1}), T_\lambda(0, y_n), t) = M((0, \rho y_{n+1}), (1 + \lambda(y_n - 1), 0), t) = \frac{1}{t+|\rho y_{n+1}+\lambda(1-y_n)-1|}.
\]

Note that if \( \rho = 2 \) and \( \lambda \in [0,1] \) then \( (\lambda/\rho) \in [0,1/2] \) and \( \{ y_n \} \) is strictly decreasing and \( \{ y_n \} \rightarrow 1 \). Thus, for all \( n \in Z_+ \), one has \( M(g_\rho x_n, T_\lambda x_n, t) \geq M(g_\rho x_n, T_\lambda x_{n-1}, t) \), which converges to \( M(X_1, \{x_n\}) = \frac{1}{t+1} \) as \( n \rightarrow \infty \), \( \forall t \in R_+ \), and:

\[
M(x_{n+1}, x_n, t) \geq M(g_\rho x_{n+1}, g_\rho x_n, t) = M((0, \rho y_{n+1}), (0, \rho y_n), t) = \frac{1}{t+|\rho y_{n+1}-y_n|} \geq \frac{1}{t+|y_{n+1} - y_n|}
\]

\( \forall n \in Z_0^+, \forall t \in R_+ \), the last inequality being strict if \( y_{n+1} \neq y_n \), with \( \psi(0) = 0 \) while \( \psi(1) = 1 \) if and only if \( y_{n+1} = y_n \). Note that the sequence \( \{ M(x_{n+1}, x_n, t) \} \) is strictly increasing for all \( t \in R_+ \) and \( M(x_{n+1}, x_n, t) \rightarrow 1 \) as \( n \rightarrow \infty \), \( \forall t \in R_+ \) so that \( \{ y_{n+1} - y_n \} \rightarrow 0 \), \( x_{n+1} \preceq x_n \), \( \forall n \in Z_0^+ \), and:

\[
\{ x_n \} \subset X_0(t) \rightarrow (0, 1), \quad \{ T_\lambda x_n \} \subset X_0(t) \rightarrow (1, 0)
\]

Note that this example extends the validity of Corollary 1—via Corollary 2—to the construction of sequences in the whole \( X_1 \) (instead of on just the proximal subset \( X_0(t) \subset X_1 \)), which converge to the unique proximal point limit \( (0, 1) \in X_0(t) \) to \( X_2 \).

**Example 2.** If, in Example 1, we take the initial points either in the proximal set \( X_0(t) = \{ 0, 1 \} \) or in the proximal set \( X_0(t) = \{ 1, 0 \} \), i.e., \( x_0 \in \{ 0, 1 \} \) \( \{ 1, 0 \} \), then, according to Corollary 1, \( T_\lambda(0, 1) = (1, 0) \) and \( T_\lambda(1, 0) = (0, 1) \) since \( T_\lambda(X_1) = (1 + \lambda x, 0) \) and \( T_\lambda(X_2) = (0, 1 + \lambda x) \) for \( x \geq 0 \).

**Example 3.** Note that, in Example 1, we have taken the initial conditions in \( X_1 \) and the proximal sequences to \( X_2 \) are always constructed within \( X_1 \) and converge to the unique best proximity point of the proximal set \( X_0(t) = \{ 0, 1 \} \subset X_1 \) to \( X_2 \) so that the best proximity point in \( X_2 \) is focused as a limit point via the cyclic mapping since \( T_\lambda(X_0(t)) = X_0(t) = \{ 1, 0 \} \subset X_2 \). In Example 2, the points of the proximal built sequences are taken directly on both proximal sets. Now, we can construct converging proximal sequences with elements in both \( X_1 \) and \( X_2 \) that converge to the unique optimal fuzzy best proximity coincidence points of the pair \( (g, T) \) in both corresponding proximal sets. Take a point \( x_0 = (1 + \lambda x, 0) \in X_2 \). We now proceed by constructing a sequence with two consecutive elements in \( X_2 \), then the next one is in \( X_1 \) and again two consecutive proximal elements to \( X_2 \) in \( X_1 \) and so on so that the cyclic mapping \( T \) is also relevant to alternate elements with two or more elements of the sequences of interest in both subsets in the cyclic disposal. Thus, for instance, the sequence:

\[
\{(1 + \lambda x, 0), \left( \frac{1 + \lambda^2 x}{\rho^2}, 0 \right), \left( \frac{1 + \lambda^2 x}{\rho^2}, 0 \right), \left( \frac{1 + \lambda^4 x}{\rho^4}, 0 \right), \left( \frac{1 + \lambda^4 x}{\rho^4}, 0 \right), \ldots \}
\]

consists of two subsequences that converge to the best proximity points \((1,0)\) and \((0,1)\) for any given \( x \in R_0^+ \), which are the unique optimal fuzzy best proximity coincidence points of \((g, T)\).
3. Results and Discussion of Optimal Fuzzy Best Proximity Coincidence Points in Partially Ordered Non-Archimedean Fuzzy Metric Spaces for Cyclic Fuzzy Order Proximal $\psi$-Contractions of Type II

This section is devoted to a framework related to the existence of best proximity coincidence points in partially ordered non-Archimedean fuzzy metric spaces for cyclic fuzzy order preserving proximal $\psi$-contractions of type II. Three definitions are given; we then state and prove the results and two examples are also given.

The following definitions are used in the main results of this section:

**Definition 11** [8]. A sequence $\{t_n\}$ of positive real numbers is said to be $s$-increasing if there exists $n_0 \in \mathbb{Z}_{0+}$ such that $t_{n+1} = t_n + 1$ for all $n \geq n_0$.

**Definition 12** [7,8]. A fuzzy metric space $(X, M, *)$ is said to have the property $T$ if for any $s$-increasing sequence $\{t_n\}$ and any given real constant $\epsilon \in (0, 1)$, there exists $n_0 = n_0(\epsilon) \in \mathbb{Z}_{0+}$ such that $M(x, y, t_n) \geq 1 - \epsilon$ for all $n \geq n_0$ and all $x, y \in X$.

An alternative definition under weaker conditions follows:

**Definition 13.** A fuzzy metric space $(X, M, *)$ is said to have the property $T_0$ if for any $s$-increasing sequence $\{t_n\}$ and any given real constant $\epsilon \in (0, 1)$, there exist some $n_0 = n_0(\epsilon) \in \mathbb{Z}_{0+}$ and some $N \in \mathbb{Z}_+$ such that:

$$n_0 + N - 1 \leq t_n \leq n_0 + N$$

and for all $n \geq n_0$:

$$M(x, y, t_n) \geq 1 - \epsilon$$

The following result relates the properties $T_0$ and $T$ if $\lim_{t \to \infty} M(x, y, t) = 1; \forall x, y \in X$.

**Lemma 1.** Let $(X, M, \bullet)$ be a fuzzy metric space endowed with the product $t$-norm $\bullet : [0, 1]^2 \to [0, 1]$ such that its associate fuzzy metric fulfills $\lim_{t \to \infty} M(x, y, t) = 1; \forall x, y \in X$. Then, the fuzzy metric space $(X, M, \bullet)$, endowed with a $t$-norm $\ast : [0, 1]^2 \to [0, 1]$ has the properties $T_0$ and $T$ for any $t$-norm $\ast : [0, 1]^2 \to [0, 1]$.

**Proof.** If $\lim_{t \to \infty} M(x, y, t) = 1$ for any $s$-increasing sequence $\{t_n\}$, one has:

$$\lim_{n \to \infty} M(x, y, t_n) = \lim_{n \to \infty} M(x, y, t_{n+1}) = \lim_{n, m \to \infty} M(x, y, t_{n+m}) = 1; \forall x, y \in X, \forall \ell \in \mathbb{Z}_{0+}$$

and for any given real constant $\epsilon_1 \in (0, 1)$, there exists $n_0 = n_0(\epsilon_1, \ast) \in \mathbb{Z}_{0+}$ such that $M(x, y, t_n) \geq 1 - \epsilon_1$ for any $x, y \in X$ and $n \geq n_0$ since $M(x, y, \cdot)$ is a non-decreasing function on $(0, \infty)$. Also, for any given real constants $\epsilon \in (0, 1)$ and $\epsilon_1 \in (0, 1)$, and some integer $N \in \mathbb{Z}_+$, where $\bar{\epsilon}_1 = \bar{\epsilon}_1(\epsilon, N) < 1 - (1 - \epsilon)^{1/N+1}$ is in $(0, 1)$, one gets for the fuzzy metric space $(X, M, \bullet)$:

$$n_0 + N - 1 \leq t_n \leq n_0 + N$$

and $\bar{\epsilon}_1 = \bar{\epsilon}_1(\epsilon, \infty) \in (0, 1); \forall x, y \in X$ so that $(X, M, \bullet)$ possesses the property $T_0$.

Since $\lim_{n \to \infty} M(x, y, t_n) = 1; \forall x, y \in X$ then $\lim_{n \to \infty} M(x, y, t_m) = \lim_{n, s \to \infty} M(x, y, t_m) = 1; \forall x, y \in X$. Thus, since $(X, M, \bullet)$ has the property $T_0$ then for any $s$-increasing sequence $\{t_n\}$ and any
given real constant $\varepsilon \in (0, 1)$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{Z}_+$ such that
\[
\left. n_0 + 1 \right. M(x, y, t_n) \geq 1 - \varepsilon \quad \text{for all integer } n \geq n_0; \forall x, y \in X.\]
As a result, $(X, M, \bullet)$ has the property $T$ since:
\[
\lim_{m \to \infty} \left( \frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_m) \right) \geq 1 - \varepsilon \bullet 1 = 1 - \varepsilon.
\]

If the metric space is $(X, M, \Delta)$, with $\Delta : [0, 1]^2 \to [0, 1]$ being the triangular $\Delta$-norm, then one gets for any arbitrary $\varepsilon = \varepsilon_1 \in (0, 1)$ that
\[
\frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_n) \geq \frac{n_0 + N - 1}{m=m+\sum_{m=m}^{n+n}} [M(x, y, t_n)] \geq 1 - \varepsilon_1 = 1 - \varepsilon.
\]

Since the triangular $\Delta$-norm exceeds any other $\Delta$-norm, all of which are strictly larger than the drastic $\Delta$-norm $\rightarrow [0, 1]^2 \to [0, 1], \varepsilon_1 \in (0, 1)$ and $\varepsilon_1 \in (0, \varepsilon_2)$ with $\varepsilon_2 = \min(\varepsilon_1, 1 - 1 - (1 - \varepsilon_1)^{1/N})$ is in $(0, 1)$, that:
\[
\frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_n) \geq \frac{n_0 + N - 1}{m=m+\sum_{m=m}^{n+n}} [M(x, y, t_n)] \geq \frac{n_0 + N - 1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_n) \geq 0
\]

and $(X, M, \ast)$ has the property $T_0$ for any $\ast : [0, 1]^2 \to [0, 1]$. Also, $\frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_n) \geq 1 - \varepsilon$ for all integer $n \geq n_0; \forall x, y \in X, \lim_{n \to \infty} \frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_n) = \lim_{n \to \infty} \frac{n+N-1}{m=m+\sum_{m=m}^{n+n}} M(x, y, t_m) \geq 1 - \varepsilon; \forall x, y \in X$ for any $\ast : [0, 1]^2 \to [0, 1]$ so that $(X, M, \ast)$ has also the property $T$.

Lemma 1 concludes that property $T$ is less restrictive than property $T_0$ and holds under a standard property in probabilistic spaces $\lim_{\rightarrow \infty} M(x, y, t) = 1; \forall x, y \in X$. The subsequent result is close to Theorem 1 for the case when $T : \cup_{i \in X} X_i \to \cup_{i \in X} X_i$ is a weak $p$-cyclic fuzzy order preserving proximal $\varepsilon$-contraction of type II. The main result of this section follows below:

**Theorem 2.** Let $(X, M, \ast, \preceq)$ be a complete partially ordered non-Archimedean fuzzy metric space satisfying the property $T$ and let $X_i \subseteq X$ be non-empty sets; $\forall i \in X$ with $\preceq$ being a partial order defined on $\cup_{i \in X} X_i(t)$. Let a $p(\geq 2)$ -cyclic mapping $T : \cup_{i \in X} X_i \to \cup_{i \in X} X_i$ be continuous and weak $p$ -cyclic fuzzy order preserving (with respect to $\preceq$) proximal $\varepsilon$-contraction of type II and let $g : \cup_{i \in X} X_i \to \cup_{i \in X} X_i$ be surjective, fuzzy non-contractive and inverse monotone mapping such that, for any $x, y \in \cup_{i \in X} X_i(t), g x$ and $g y$ are comparable with respect to $\preceq$ only if $x, y$ are comparable. Suppose also that each pair of elements of $\cup_{i \in X} X_i(t)$ has a lower bound and an upper bound, and that for any $t \neq T, X_0(t) = X_0(t) \subseteq X_0(t)$ and $X_0(t) \subseteq g(X_i(t)); \forall i \in X$. If for each given $x_i \in X_i(t)$ for each $i \in X$, there exist some element $x_i$ in $\cap_{i \in X} X_i(t)$ such that:

\[
M(gx_i, T x_i) = M(x_i, x_i+1, t) \text{ with } x_{i+1} \in X_i(t)
\]

Then, there exists a unique element $x_i^* \in \cap_{i \in X} X_i(t)$ for each $i \in X$, which is an optimal fuzzy best proximity coincidence point of the pair $(g, T)$ in $X_i$ such that $x_i^* = T x_i^*; \forall i \in X$, and then $M(gx_i^*, T x_i^*) = M(x_i, x_i+1, t); \forall i \in X$. Furthermore, each of the subsequences $\{x_{ni}\} \subseteq X_0(t)$ for each $i \in X$ being defined by any given first element $x_0 \in \cup_{i \in X} X_i(t)$ so that the proximal constraint:

\[
M(gx_{ni}^*, T x_{ni}^*) = M(x_i, x_i+1, t)
\]
holds for any given \( n \in \mathbb{Z}_{0+}, \forall j \in \mathbb{P} \), is a Cauchy sequence which is convergent to the optimal fuzzy best proximity coincidence point \( x_j^* \in X_0(t) \) of the pair \((g, T)\) in \( X_1 \) provided that the two subsequent conditions hold:

1. \( n_{j-1,k} + 1 \leq n \leq n_{j,k} \), with \( n_{j,k} + 1 - n_{j-1,k} \geq 2 \), for \( j = i, i + 1, ..., i + p - 2 \), and \( n_{i+p-2,k} + 1 \leq n \leq n_{i+p-1,k+1} \), with \( n_{i+p-1,k+1} - n_{i+p-2,k} \geq 2 \), \( \forall i, j \in \mathbb{P}, \forall k \in \mathbb{Z}_{0+} \);
2. the chosen arbitrary \( i \in \mathbb{P} \) is such that the initial point \( z_0 \in X_0(t) \) and that the sets of positive integers \( \{n_{j,k} - n_{j,k-1}\} \) for \( j \in \mathbb{P} \) and \( k \in \mathbb{Z}_{0+} \) have \( p \) positive upper-bounding integers \( \pi_j \geq n_{j,k} - n_{j,k-1} \), \( \forall k \in \mathbb{Z}_{0+}, \forall j \in \mathbb{P} \) with \( n_{i-1,0} = -1 \) and \( n_{i,0} \geq 1 \).

**Proof.** Denote the restricted mappings \( T : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j | X_{i+1} \) and \( g : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j | X_{i+1} \), respectively, by \( T_j : X_j \rightarrow X_{i+1} \) and \( g_j : X_j \rightarrow X_{i+1} \); \( \forall j \in \mathbb{P} \). Let \( z_0 = x_{0i} \) arbitrary in \( X_0(t) \) and some \( z_1 = x_{1i} \in X_{0i} \) for any given arbitrary \( i \in \mathbb{P} \) be such that \( z_0 \preceq z_1 \) \( \forall n \in \mathbb{Z}_{0+} \) and \( M(gz_1, Tz_0, t) = M(X_i, X_{i+1}, t) \). Since \( Tz_1 \in T(X_0(t)) \subseteq X_{0,i+1}(t) \) and \( X_0(t) \subseteq g(X_0,i+1(t)) \), one gets that \( z_2 \in X_0(t) \) exists such that \( M(gz_2, Tz_1, t) = M(X_i, X_{i+1}, t) \). Since \( T : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j | X_{i+1} \) is a weak \( p \)-cyclic fuzzy order preserving proximal mapping and \( z_0 \preceq z_1 \) then \( gz_1 \preceq gz_2 \) and, since \( g : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j \) is surjective and \( X_0(t) \subseteq g(X_0,i+1(t)) \) then it follows that \( z_1 \preceq z_2 \). Assume that this is not the case and proceed by contradiction. Since \( gz_1 \) and \( gz_2 \) are comparable which holds, by hypothesis, only if \( z_1 \geq z_2 \) since \( z_1 \preceq z_2 \) is assumed false. However, \( gz_1 > gz_2 \) contradicts \( gz_1 \preceq gz_2 \). Then, \( z_1 \preceq z_2 \) and, as a result, \( z_0 \preceq z_1 \preceq z_2 \). Proceeding in the same way, we can build a sequence \( \{z_n\} \) with \( z_n \in X_0(t) \) for \( 0 \leq n \leq n_0 \), where \( 1 \leq n \leq n_0 \leq \mathbb{Z}_{0+} \), \( \forall n \in \mathbb{Z}_{0+} \) such that \( M(gz_n, Tz_n, t) = M(X_i, X_{i+1}, t) \) and \( z_n \preceq z_0 \). Then, take \( z_{n_0+1} = Tz_{n_0} \), such that \( z_{n_0+1} \preceq z_{n_0+2} \) since \( T : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j | X_{i+1} \) is a weak \( p \)-cyclic fuzzy order preserving proximal mapping, \( z_{n_0} \in X_0(t) \) and \( z_{n_0+1} = Tz_{n_0} \in X_0,i+1(t) \). As a result, \( z_{n_0+1} \preceq z_{n_0+2} \), \( \forall n \in \mathbb{Z}_{0+} \). Again, for such a \( z_{n_0+1} \in X_0,i+1(t) \), there is some \( z_{n_0+1} \in X_0,i+1(t) \) such that \( z_{n_0+1} \preceq z_{n_0+2} \) and for \( 0 \leq n \leq n_0 \leq \mathbb{Z}_{0+} \), \( \forall n \in \mathbb{Z}_{0+} \). The finite subsequence of \( \{z_n\} \) from \( n = 0 \) to \( n = n_0 \) is also ordered with respect to \( \preceq \), where \( \forall j \in \mathbb{P} \) for \( i = j, i + 1, ..., i + p \) and \( \forall k \in \mathbb{P} \) for \( i + p - 2 \leq n \leq n_0 \) and \( \forall k \in \mathbb{P} \) for \( i + p - 1 \leq n \leq n_0 \). Since \( T : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j | X_{i+1} \) is a weak \( p \)-cyclic fuzzy order preserving (with respect to \( \preceq \)) proximal \( \psi \)-contraction of type II and \( g : \bigcup_{j \in \mathbb{P}} X_j | X_i \rightarrow \bigcup_{j \in \mathbb{P}} X_j \) is fuzzy non-contractive, one has for \( z_n \in \{z_n\} \) and some real constant \( a \in (0, 1) \), since \( \psi : [0, 1] \rightarrow [0, 1] \) belongs to the set \( \Psi \):

\[
M(z_n, z_{n+1}, t) \geq M(gz_n, gz_{n+1}, t) \geq \psi \left( M(z_{n-1}, z_n, a^{-1}) \right) > M(z_{n-1}, z_n, a^{-1}t) \geq \dots \geq M(z_0, z_1, a^{-n}t)
\]
if \( n \), \((n + 1) \notin N^* \), \( \forall t \in R_+ \), and:
\[
M(z_{n+1}, z_{n+2}, t) = M(Tz_n, Tz_{n+1}, t) \\
\geq \psi \{ M(z_{n-1}, z_n, a^{-1}t) \} > M(z_{n-1}, z_n, a^{-1}t) \\
\geq \ldots \geq M(z_0, z_1, a^{-n}t)
\]

(23)

if \((n + 1), (n + 2) \in N^* \); \( \forall t \in R_+ \). Thus, one gets for any given \( \varepsilon \in R_+ \), some \( n_0 = n_0(\varepsilon) \in Z_+ \), \( \forall t \in R_+ \) and all \( m (\geq n) \), \( n(\geq n_0) \in Z_+ \) that:
\[
M(z_n, z_m, t) \geq \psi \{ M(z_n, z_{n+1}, t) \} \star M(z_{n+1}, z_{n+2}, t) \star \ldots \star M(z_{m-1}, z_m, t) \\
> M(z_0, z_1, a^{-n}t) \star M(z_1, a^{-n+1}t) \star \ldots \star M(z_0, z_1, a^{-m}t) \\
= \frac{m+1}{n \geq n_0} M(x, y, a^{-nt}) \\
\geq 1 - \varepsilon
\]

from the property \( T \) with \( t_n = a^{-nt} \); \( \forall n \in Z_+ \) being a \( \{t_n\} \) \( s \)-increasing sequence for \( n \geq n_0 \) such that \( t_n = a^{-nt} \geq (a^{-1} - 1) \) implies that \( t_{n+1} = a^{-n+1}t \geq t_n + 1 = a^{-nt} + 1 \) for all \( n \geq n_0 \) and all \( t \geq t_n = \frac{a^n}{a^{-1} - 1} \) (note that \( t_n \rightarrow 0 \) as \( n \rightarrow \infty \)). Any other \( s \)-increasing sequence \( \{t_n\} \) can be accommodated to satisfy \( t_{n_k} \leq t_{n-1} + 1 \leq t_n \leq t_{n_k} + 1 \). Thus, \( \lim_{k \geq k_0(n_k \geq n_0)} M(x, y, t_{n_k}) \geq 1 - \varepsilon \) by the non-Archimedean property. Then, \( (X, M, *, \preceq) \) satisfies the property \( T \) since Equation (24) holds for the \( s \)-increasing sequence \( \{a^{-nt}\}_{n \geq n_0} \) and it is extendable to any other \( s \)-increasing sequence. Thus, one has from Equation (22) that:
\[
\lim_{k \rightarrow \infty} M(z_{n_k}, z_{n_k+1}, t) = \lim_{k \rightarrow \infty} M(gz_{n_k}, gz_{n_k+1}, t) = \lim_{k \rightarrow \infty} M(z_1, z_0, a^{-nt}) = 1; \forall t \in R_+
\]

(25)

for a subsequence \( \{z_{n_k}\} \subseteq \{X_0(t)\} \) being defined, provided that arbitrary points \( z_0(\leq z_1) \in X_0(t), z_1 \in X_0(t) \) for some given arbitrary \( i \in \mathbb{P} \) and such that \( n_k, (n_k + 1) \notin N^* (N_k^* \subseteq N^* \) being the set where the iteration switches from \( X_i \) to \( X_{i+1} \), \( \forall t \in R_+ \). Thus, \( \{z_{n_k}\} \) is Cauchy and convergent to some \( x^*_j \in clX_0(t) \) since \( (X, M, *, \preceq) \) is complete. For the given initial \( z_0 \in X_0(t), z_1 \in X_0(t) \), we can always find two points \( \exists z_0, z_1 \in X_0(t) \) \( \exists z_{n_k} \subseteq \{z_n\} \cap X_0(t) \), which are initial adjacent points of a subsequence \( \{z_{n_k}\} \) for any arbitrary \( j \neq i \in \mathbb{P} \). It is found that \( \{z_{n_k}\} \) is Cauchy and convergent to some \( x^*_j \in clX_0(t) \). The set of \( p \) limit best proximity points \( x^*_j \in clX_0(t) \); \( \forall j \in \mathbb{P} \) is unique since all the above subsequences are also totally ordered with lower-bounding and an upper-bounding ordered sequences (see Theorem 1 for similar reasoning), which converge to a unique best proximity point of a set \( X_j \) to its adjacent set \( X_{j+1} \) for each \( j \in \mathbb{P} \). Then, for any constructed sequence \( \{x_n\} \in \cup_{j \in \mathbb{P}} X_0(t) \), we have convergent subsequences to the same limit within each subset \( X_n \), which is a unique best proximity point so that:
\[
\lim_{k \rightarrow \infty} M(x_{n_k}, x_{n_k+1}, t) = \lim_{k \rightarrow \infty} M(gx_{n_k}, gx_{n_k+1}, t) \\
= \lim_{k \rightarrow \infty} M(x^*_j, t) = \lim_{k \rightarrow \infty} M(gx^*_j, gx^*_j, t) = 1; \forall j \in \mathbb{P}
\]

(26)
so that \( \{ x_{n_k} \} \rightarrow x_j^*; \forall j \in \mathcal{P} \). It is now proven that each \( x_j^* \in clX_{0_j}(t) \) is an optimal fuzzy best proximity coincidence points of the pair \((g, T)\) in \( X_j \) for each \( j \in \mathcal{P} \). One gets from the properties of the \( t \)-norms, Equation (26), and the continuity of \( T: \cup_{i \in \mathcal{P}} X_i \rightarrow \cup_{i \in \mathcal{P}} X_i \):

\[
M(gx_j^*, Tx_j^*, t) \geq (M(gx_j^*, gx_{n_k}, t) * M(Tx_{n_k}, Tx_j^*, t)) * M(Tx_{n_k}, Tx_j^*, t) = (M(gx_j^*, gx_{n_k}, t) * M(Tx_{n_k}, Tx_j^*, t)) * M(Tx_{n_k}, Tx_j^*, t) \quad \forall k \in \mathbb{Z}_0^+, \forall j \in \mathcal{P}
\]

and:

\[
M(gx_j^*, Tx_j^*, t) \geq \liminf_{k \to \infty} (M(gx_j^*, gx_{n_k}, t) * M(Tx_{n_k}, Tx_j^*, t)) * M(Tx_{n_k}, Tx_j^*, t) = \liminf_{k \to \infty} M(gx_j^*, gx_{n_k}, t) = \lim_{k \to \infty} M(gx_j^*, tx_j^*, t) = M(X_j, X_{j+1}, t); \forall j \in \mathcal{P}
\]

However, we can interchange the locations of \( M(gx_{n_k}, Tx_{n_k}, t) \) and \( M(gx_j^*, Tx_j^*, t) \) in Equations (27) and (28) for obtaining corresponding sets of inequalities to conclude that:

\[
\lim_{k \to \infty} M(gx_{n_k}, Tx_{n_k}, t) = M(gx_j^*, Tx_j^*, t) = M(X_j, X_{j+1}, t) \quad \forall j \in \mathcal{P}
\]

Thus, \( x_j^* = Tx_{j-1}^* \in clX_{0_j}(t) \) is a unique best proximity point to \( X_{j+1} \) in \( X_j \) of \( T: \cup_{i \in \mathcal{P}} X_i \rightarrow \cup_{i \in \mathcal{P}} X_i \) and it is also an optimal fuzzy best proximity coincidence point of the pair \((g, T)\) in \( X_j \).

In the same way as Corollary 1 and under a close proof (see also Remark 3), we can get the subsequent Corollary to Theorem 2:

**Corollary 3.** Assume that the hypothesis of Theorem 2 holds except that, for any given initial point \( z_0 \in \cup_{i \in \mathcal{P}} X_{0_i}(t) \), the sequences \( \{ z_n \} \) are built so that:

1. \( z_{n+1} = Tz_n \) for any \( n \in \mathbb{Z}_+ \) if \( z_n \in X_{0_i}(t) \) for any \( i(\neq j) \in \mathcal{P} \) and some fixed \( j \in \mathcal{P} \) (i.e., the proximal subsequence from each subset \( X_j \) of \( X \) to each next adjacent subset is only computed eventually at the subset \( X_{0_i}(t) \), while at the remaining subsets only the cyclic self-mapping \( T \) is involved);
2. the proximal constraint Equation (3), subject to its subsequent constraints, is replaced at the subset \( X_j \) of \( X \) by the subsequent one:

\[
M(gz_{n_k}, Tx_{n_k}, t) = M(X_j, X_{j+1}, t); \forall k(\geq k^*) \in \mathbb{Z}_0^+, i \in \mathcal{P}
\]

for some given \( j \in \mathcal{P} \) and some subsequence \( \{ z_{n_k} \} \subseteq X_{0_j}(t) \), for \( n \in \mathbb{Z}_+ \) with \( z_0 \in X_i \) for some satisfying \( z_{n_k} = T^n z_{n_k} \) with \( z_{n_k} \) being strictly increasing with \( k(\geq k^*) \in \mathbb{Z}_0^+ \) (i.e., the proximal subsequence at the subset \( X_{0_j}(t) \) is not necessary computed at each \( k \)-th cycle on the whole cyclic disposal of the \( p \) subsets \( X_{0_i}(t) \subseteq X_i \subseteq X \) for all \( i \in \mathcal{P} \) since \( m_j(k) \) can exceed the value \( p \) for some values \( k(\geq k^*) \in \mathbb{Z}_0^+ \).
Results similar to those of Theorem 2 and Corollary 3 can be obtained by replacing the continuity assumption on \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) by the condition that \( X_{i+1} \) is fuzzy approximatively compact with respect to \( X_i; \forall i \in \mathbb{P} \), that is, each sequence \( \{T x_i\} \subset X_{i+1} \) such that \( \{M(x_i, T x_i, t)\} \rightarrow M(x_i, T x_{i+1}, t) \) for some \( x_i \in X_i \) has a convergent subsequence.

**Corollary 4.** Theorem 2 holds “mutatis-mutandis” if \( X_{i+1} \) is fuzzy and approximatively compact with respect to \( X_i; \forall i \in \mathbb{P} \) even if \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is not everywhere continuous.

**Proof.** The hypothesis of Theorem 2 still holds except that \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is not necessarily everywhere continuous while \( X_{i+1} \) is fuzzy and approximatively compact with respect to \( X_i; \forall i \in \mathbb{P} \). Then, the first part of the proof of Theorem 2 is still applicable while one concludes from Equation (28) that \( \{x_{n_i}\} \rightarrow x^*_j, \{g x_{n_i}\} \rightarrow g x^*_j \) as \( n_i \to \infty \forall j \in \mathbb{P} \) so that there is a convergent subsequence \( \{T x_{n_i}\} \subset clX_{0,j+1}(t) \) \( \rightarrow z^*_j + 1 \) for some \( z^*_j + 1 \in clX_{0,j+1}(t) \) for each \( j \in \mathbb{P} \) since:

\[
M\left(g x^*_j, T x^*_j, t\right) \geq \liminf_{k \to \infty} M\left(g x_{n_k}, T x_{n_k}, t\right)
\]

\[
= \lim_{k \to \infty} M\left(g x_{n_k}, z^*_j + 1, t\right)
\]

\[
= \lim_{k \to \infty} M\left(g x^*_j, T x_{n_k}, t\right)
\]

\[
= \lim_{k \to \infty} M\left(g x^*_j, z^*_j + 1, t\right)
\]

\[
= M(X_j, X_{j+1}, t); \forall j \in \mathbb{P}
\]

leads to:

\[
\left(M\left(g x_{n_i}, T x_{n_i}, t\right) - M\left(g x^*_j, T x^*_j, t\right)\right) \rightarrow 0,
\]

\[
M\left(g x^*_j, T x^*_j, t\right) \rightarrow M\left(g x^*_j, X_{j+1}, t\right) = M(X_j, X_{j+1}, t); \forall j \in \mathbb{P},
\]

\[
M\left(g x^*_j, z^*_j + 1, t\right) = M(X_j, X_{j+1}, t)
\]

and \( T x^*_j = z^*_j + 1 = x^*_j + 1; \forall j \in \mathbb{P} \). Assume that this is not true, i.e., \( z^*_j + 1 \neq x^*_j + 1 \) for some \( j \in \mathbb{P} \), so that:

\[
1 > M\left(z^*_j + 1, x^*_j + 1, t\right) \geq \left(\lim_{k \to \infty} M\left(z^*_j + 1, T x_{n_k}, t\right) * \lim_{n_k \to \infty} M\left(T x_{n_k}, T x_{n_k}, t\right)\right) * \lim_{n_k \to \infty} M\left(T x_{n_k}, T z^*_j, t\right)
\]

Taking limits in both sides of the above inequality as \( k, n_k, n_{jk} \to \infty \), and since \( \{x_{n_k}\}, \{x_{n_k}\} \to x^*_j \) and \( \{T x_{n_k}\} \to z^*_j + 1 \), yields the contradiction \( 1 > (1 * 1) * 1 = 1 \), so that \( T x^*_j = z^*_j + 1 = x^*_j + 1; \forall j \in \mathbb{P} \). Then, \( z^*_j + 1 = x^*_j + 1; \forall j \in \mathbb{P} \). □

**Corollary 5.** Corollary 3 holds “mutatis-mutandis” if \( X_{i+1} \) is fuzzy and approximatively compact with respect to \( X_i; \forall i \in \mathbb{P} \) even if \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is not everywhere continuous.

Theorem 2 and Corollaries 3–5 can be directly extended to a \((g, T)\) pair where the \( p(\geq 2)\)-cyclic mapping \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is a continuous \( p\)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type II, the partial order being defined with respect to \( \prec \), which is now defined on the whole union of subsets of the cyclic disposal \( \cup_{i \in \mathbb{P}} X_i \). This means that the constructed sequences possessing Cauchy subsequences within each subset converge to the best proximity points. Such points are simultaneously unique optimal fuzzy coincidence points of the pair \((g, T)\) that can be constructed on the whole subset \( X_i \) but converge to the corresponding proximal subsets.

**Remark 4.** Note from Definition 7 and from Definition 8 that if \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is a strong (respectively, weak) \( p\)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type II for some \( \alpha \in (0, 1) \) and \( M(X_i, X_{i+1}, \cdot) \) is non-decreasing on \((0, \infty)\) then \( T : \cup_{i \in \mathbb{P}} X_i \rightarrow \cup_{i \in \mathbb{P}} X_i \) is also a strong (respectively, weak) \( p\)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type I. This becomes obvious from \( M(X_i, X_{i+1}, \alpha^{-1} t) \geq \)
Example 4. Assume that \( T : \bigcup_{i \in \mathcal{I}} X_i \to \bigcup_{i \in \mathcal{I}} X_i \) is a \( p \)-cyclic fuzzy order preserving proximal \( \psi \)-contraction with \( M(x, y, t) = \frac{\lambda t}{M_d(x, y)} \); \( \forall x, y \in \bigcup_{i \in \mathcal{I}} X_i \) (respectively, \( \forall x, y \in \bigcup_{i \in \mathcal{I}} X_0(t) \)), \( \forall t \in R^+ \), for some real \( \lambda \in R^+ \) where \( d : X \times X \to R^+ \) is a metric, then \( \frac{dM(x, y)}{d(x, y)} + \lambda t \geq 0, \forall x, y \in \bigcup_{i \in \mathcal{I}} X_i \) (respectively, \( \forall x, y \in \bigcup_{i \in \mathcal{I}} X_0(t) \)), \( \forall t \in R^+ \) with the above inequality being strict if and only if \( x \neq y \). Thus, \( M(x, y, a^{-1}t) > M(x, y, t) \) for any \( a \in (0, 1) \) if and only if \( x \neq y \) for all \( t \in R^+ \). Then, \( T : \bigcup_{i \in \mathcal{I}} X_i \to \bigcup_{i \in \mathcal{I}} X_i \) is also a strong (respectively, weak) \( p \)-cyclic fuzzy order preserving proximal \( \psi \)-contraction of type I (see also Remark 4).

Example 5. Consider the mappings \( T \) and \( g \) defined as in Examples 1–3. From Definition 8, Equations (22) and (23) in the proof of Theorem 2 and Example 4 with \( \lambda = 1 \), it follows that \( \lim_{n \to \infty} M(z_n, z_{n+1}, t) \geq \lim_{n \to \infty} M(z_0, z_1, a^{-n}t) = 1 \) for any sequence \( \{z_n\} \subset X_1 \cup X_2 \) for \( i = 1, 2 \) constructed as in the proof of Theorem 2. Thus, it follows that the \( p \) subsequences \( \{z_{i_n}\} \subset \{z_n\} \cap X_i \) (or, respectively, in \( \{z_{i_n}\} \subset \{z_n\} \cap X_0(t) \) for \( i = 1, 2 \) for the weak proximal contraction case) for each \( i = 1, 2 \) converge to unique best proximity points, which are also the unique optimal fuzzy best proximity coincidence points of the pair \((g, T)\) of \( X_i \) at each \( X_0(t) \); \( i = 1, 2 \).

Remark 5. Consider the mappings \( T \) and \( g \) under the conditions of Theorem 3. From Equations (22) and (23) in the proof of Theorem 2 and Example 4 with \( \lambda = 1 \), it follows that \( \lim_{n \to \infty} M(z_n, z_{n+1}, t) \geq \lim_{n \to \infty} M(z_0, z_1, a^{-n}t) = M(z_0, z_1, \infty) = 1 \), even if the constraint \( M(z_{n+2}, z_{n+1}, t) \geq \psi (M(z_n, z_{n+1}, a^{-1}t)) \) in Definition 8 holds for all \( n \in Z_0 \) and only for some \( t \in R^+ \), for any sequence \( \{z_n\} \subset \bigcup_{i \in \mathcal{I}} X_i \) constructed as in the proof of Theorem 2. Thus, it follows that the \( p \) subsequences \( \{z_{i_n}\} \) of \( \{z_n\} \) fulfilling \( \{z_{i_n}\} \subset \{z_n\} \cap X_i \cup \bigcup_{i \in \mathcal{I}} X_i \) (or, respectively, in \( \{z_{i_n}\} \subset \{z_n\} \cap X_0(t) \) for the weak proximal contraction case); \( \forall i \in \mathcal{I} \), converge to unique best proximity points at each \( X_i \), which are also the unique optimal fuzzy best proximity coincidence points of the pair \((g, T)\) at each \( X_i \); \( \forall i \in \mathcal{I} \). Thus, Theorem 2 can be weakened by “ad hoc weakening” the implied part of the inequalities in Definition 8 so as to be fulfilled only for some (and not for all) \( t \in R^+ \).

4. Conclusions

This paper has dealt with some properties of interest in two types of fuzzy ordered proximal contractions of cyclic self-mappings \( T \), which is integrated in a pair \((g, T)\) of mappings that construct the relevant proximal sequences of interest. In particular, \( g \) is supposed to be a non-contractive fuzzy self-mapping in a non-Archimedean ordered fuzzy complete metric space \((X, M, *, \leq)\), endowed with a partial order \( \leq \) and a triangular norm \(*\), while \( T \) is a \( p \)-cyclic proximal contraction. The fuzzy set \( M \) on \( X \times X \times [0, \infty) \) is a fuzzy metric on the non-Archimedean fuzzy metric space \((X, M, \ast)\). Two types of such contractions (so-called type I and of type II) are considered. The main results obtained rely on the existence, uniqueness and limit properties for sequences to existing optimal fuzzy best proximity coincidence points for such pairs of mappings.

Acknowledgments: The first author is grateful to the Spanish Government and European Fund of Regional Development FEDER for Grant Nos. DPI2012-30651 and DPI2015-64766-R. The authors are also grateful to the referees for their useful suggestions.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.
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