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A Generalization of b-Metric Space and Some Fixed Point Theorems

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Abstract: In this paper, inspired by the concept of b-metric space, we introduce the concept of extended b-metric space. We also establish some fixed point theorems for self-mappings defined on such spaces. Our results extend/generalize many pre-existing results in literature.

Keywords: fixed point; b-metric

1. Introduction

The idea of b-metric was initiated from the works of Bourbaki [1] and Bakhtin [2]. Czerwik [3] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [4] discussed some kind of relaxation in triangular inequality and called this new distance measure as non-linear elastic mathing (NEM). Similar type of relaxed triangle inequality was also used for trade measure [5] and to measure ice floes [6]. All these applications intrigued and pushed us to introduce the concept of extended b-metric space. So that the results obtained for such rich spaces become more viable in different directions of applications.

Definition 1. *Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric (Bakhtin [2], Czrerwik [3]) if it satisfies the following properties for each $x, y, z \in X$.*

(b1): $d(x, y) = 0 \Leftrightarrow x = y$;
(b2): $d(x, y) = d(y, x)$;
(b3): $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Example 1. *1. Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d : X \times X \to \mathbb{R}^+$ as:*

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{1/p}$$

where $x = \{x_n\}, y = \{y_n\}$. Then d is a b-metric space [7–9] with coefficient $s = 2^{1/p}$.
2. Let $X := L_p[0, 1]$ be the space of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p < \infty$ with $0 < p < 1$. Define $d : X \times X \to \mathbb{R}^+$ as:

$$d(x, y) = \Big(\int_0^1 |x(t) - y(t)|^p dt\Big)^{1/p}$$

Then d is b-metric space [7–9] with coefficient $s = 2^{1/p}$.

The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When $s = 1$, the concept of b-metric space coincides with the concept of metric space. For some details on subject see [7–12].

Definition 2. *Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is said to be:*

(I) Cauchy [12] if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;
(II) Convergent [12] if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;
(III) The b-metric space (X, d) is complete [12] if every Cauchy sequence is convergent.

In the following we recollect the extension of Banach contraction principle in case of b-metric spaces.

Theorem 1. *Let (X, d) be a complete b-metric space with constant $s \geq 1$, such that b-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in [0, 1)$ such that $ks < 1$. Then T has a unique fixed point [13].*

2. Results

In this section, we introduce a new type of generalized metric space, which we call as an extended b-metric space. We also establish some fixed point theorems arising from this metric space.

Definition 3. *Let X be a non empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies:*

$(d_\theta 1)$ $d_\theta(x, y) = 0$ iff $x = y$;
$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;
$(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b-metric space.

Remark 1. *If $\theta(x, y) = s$ for $s \geq 1$ then we obtain the definition of a b-metric space.*

Example 2. *Let $X = \{1, 2, 3\}$. Define $\theta : X \times X \to \mathbb{R}^+$ and $d_\theta : X \times X \to \mathbb{R}^+$ as:*

$$\theta(x, y) = 1 + x + y$$

$$d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$$

$$d_\theta(1, 2) = d_\theta(2, 1) = 80, d_\theta(1, 3) = d_\theta(3, 1) = 1000, d_\theta(2, 3) = d_\theta(3, 2) = 600$$

Proof. $(d_\theta 1)$ and $(d_\theta 2)$ trivially hold. For $(d_\theta 3)$ we have:

$$d_\theta(1, 2) = 80,\ \theta(1, 2)\,[d_\theta(1, 3) + d_\theta(3, 2)] = 4(1000 + 600) = 6400$$

$$d_\theta(1, 3) = 1000,\ \theta(1, 3)\,[d_\theta(1, 2) + d_\theta(2, 3)] = 5(80 + 600) = 3400$$

Similar calculations hold for $d_\theta(2, 3)$. Hence for all $x, y, z \in X$

$$d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$$

Hence (X, d_θ) is an extended b-metric space. □

Example 3. *Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions define on $[a,b]$. Note that X is complete extended b-metric space by considering $d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$, with $\theta(x,y) = |x(t)| + |y(t)| + 2$, where $\theta : X \times X \to [1,\infty)$.*

The concepts of convergence, Cauchy sequence and completeness can easily be extended to the case of an extended b-metric space.

Definition 4. *Let (X, d_θ) be an extended b-metric space.*

(i) *A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.*
(ii) *A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.*

Definition 5. *An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.*

Note that, in general a b-metric is not a continuous functional and thus so is an extended b-metric.

Example 4. *Let $X = \mathbb{N} \bigcup \infty$ and let $d : X \times X \to \mathbb{R}$ be defined by [14]:*

$$d(x,y) = \begin{cases} 0 & \text{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \text{if } m, n \text{ are even or } mn = \infty \\ 5 & \text{if } m, n \text{ are odd and } m \neq n \\ 2 & \text{otherwise} \end{cases}$$

Then (X, d) is a b-metric with $s = 3$ but it is not continuous.

Lemma 1. *Let (X, d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.*

Our first theorem is an analogue of Banach contraction principle in the setting of extended b-metric space. Throughout this section, for the mapping $T : X \to X$ and $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, T^2x_0, T^3x_0, \cdots\}$ represents the orbit of x_0.

Theorem 2. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional. Let $T : X \to X$ satisfy:*

$$d_\theta(Tx, Ty) \leq k d_\theta(x,y) \quad \text{for all } x, y \in X \tag{1}$$

where $k \in [0,1)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1, 2, \cdots$. Then T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0, \; Tx_0 = x_1, \; x_2 = Tx_1 = T(Tx_0) = T^2(x_0) \ldots, x_n = T^n x_0 \ldots.$$

Then by successively applying inequality (1) we obtain:

$$d_\theta(x_n, x_{n+1}) \leq k^n d_\theta(x_0, x_1) \tag{2}$$

By triangular inequality and (2), for $m > n$ we have:

$$\begin{aligned}
d_\theta(x_n, x_m) &\leq \theta(x_n, x_m)k^n d_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)k^{n+1}d_\theta(x_0, x_1) + \cdots + \\
&\quad \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)...\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)k^{m-1}d_\theta(x_0, x_1) \\
&\leq d_\theta(x_0, x_1)\Big[\theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_{n-1}, x_m)\theta(x_n, x_m)k^n + \\
&\quad \theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\theta(x_{n+1}, x_m)k^{n+1} + \cdots + \\
&\quad \theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\theta(x_{n+1}, x_m)...\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)k^{m-1}\Big]
\end{aligned}$$

Since, $\lim_{n,m\to\infty}\theta(x_{n+1}, x_m)k < 1$ so that the series $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let:

$$S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x_m), \quad S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(x_i, x_m)$$

Thus for $m > n$ above inequality implies:

$$d_\theta(x_n, x_m) \leq d_\theta(x_0, x_1)\Big[S_{m-1} - S_n\Big]$$

Letting $n \to \infty$ we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete let $x_n \to \xi \in X$:

$$\begin{aligned}
d_\theta(T\xi, \xi) &\leq \theta(T\xi, \xi)[d_\theta(T\xi, x_n) + d_\theta(x_n, \xi)] \\
&\leq \theta(T\xi, \xi)[kd_\theta(\xi, x_{n-1}) + d_\theta(x_n, \xi)] \\
d_\theta(T\xi, \xi) &\leq 0 \quad as \quad n \to \infty \\
d_\theta(T\xi, \xi) &= 0
\end{aligned}$$

Hence ξ is a fixed point of T. Moreover uniqueness can easily be invoked by using inequality (1), since $k < 1$. □

In the following we include another variant which is analogue to fixed point theorem by Hicks and Rhoades [15]. We need the following definition.

Definition 6. *Let $T : X \to X$ and for some $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, fx_0, f^2x_0, \cdots\}$ be the orbit of x_0. A function G from X into the set of real numbers is said to be T-orbitally lower semi-continuous at $t \in X$ if $\{x_n\} \subset \mathcal{O}(x_0)$ and $x_n \to t$ implies $G(t) \leq \lim_{n\to\infty} \inf G(x_n)$.*

Theorem 3. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional. Let $T : X \to X$ and there exists $x_0 \in X$ such that:*

$$d_\theta(Ty, T^2y) \leq kd_\theta(y, Ty) \quad \text{for each } y \in \mathcal{O}(x_0) \tag{3}$$

where $k \in [0, 1)$ be such that for $x_0 \in X$, $\lim_{n,m\to\infty}\theta(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1, 2, \cdots$. Then $T^n x_0 \to \xi \in X$ (as $n \to \infty$). Furthermore ξ is a fixed point of T if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Proof. For $x_0 \in X$ we define the iterative sequence $\{x_n\}$ by:

$$x_0, \ Tx_0 = x_1, \ x_2 = Tx_1 = T(Tx_0) = T^2(x_0) \ldots, x_n = T^n x_0 \ldots.$$

Now for $y = Tx_0$ by successively applying inequality (3) we obtain:

$$d_\theta(T^n x_0, T^{n+1}x_0) = d_\theta(x_n, x_{n+1}) \leq k^n d_\theta(x_0, x_1) \tag{4}$$

Following the same procedure as in the proof of Theorem 2 we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete then $x_n = T^n x_0 \to \xi \in X$. Assume that G is orbitally lower semi continuous at $\xi \in X$, then:

$$\begin{aligned} d_\theta(\xi, T\xi) &\leq \liminf_{n\to\infty} d_\theta(T^n x_0, T^{n+1}x_0) && (5)\\ &\leq \liminf_{n\to\infty} k^n d_\theta(x_0, x_1) = 0 && (6)\end{aligned}$$

Conversely, let $\xi = T\xi$ and $x_n \in O(x)$ with $x_n \to \xi$. Then:

$$G(\xi) = d(\xi, T\xi) = 0 \leq \liminf_{n\to\infty} G(x_n) = d(T^n x_0, T^{n+1}x_0) \tag{7}$$

□

Remark 2. *When $\theta(x,y) = 1$ a constant function then Theorem 3 reduces to main result of Hicks and Rhoades ([15] (Theorem 1)). Hence Theorem 3 extends/generalizes ([15] (Theorem 1)).*

Example 5. *Let $X = [0, \infty)$. Define $d_\theta(x,y) : X \times X \to \mathbb{R}^+$ and $\theta : X \times X \to [1, \infty)$ as:*

$$d_\theta(x,y) = (x-y)^2, \quad \theta(x,y) = x + y + 2$$

Then d_θ is a complete extended b-metric on X. Define $T : X \to X$ by $Tx = \frac{x}{2}$. We have:

$$d_\theta(Tx, Ty) = \left(\frac{x}{2} - \frac{y}{2}\right)^2 \leq \frac{1}{3}(x-y)^2 = \; kd_\theta(x,y)$$

Note that for each $x \in X$, $T^n x = \frac{x}{2^n}$. Thus we obtain:

$$\lim_{m,n\to\infty} \theta(T^m x, T^n x) = \lim_{m,n\to\infty} \left(\frac{x}{2^m} + \frac{x}{2^n} + 2\right) < 3$$

Therefore, all conditions of Theorem 3 are satisfied hence T has a unique fixed point.

Example 6. *Let $X = [0, \frac{1}{4}]$. Define $d_\theta(x,y) : X \times X \to \mathbb{R}^+$ and $\theta : X \times X \to [1, \infty)$ as:*

$$d_\theta(x,y) = (x-y)^2, \quad \theta(x,y) = x + y + 2$$

Then d_θ is a complete extended b-metric on X. Define $T : X \to X$ by $Tx = x^2$. We have:

$$d_\theta(Tx, Ty) \leq \frac{1}{4} d_\theta(x,y)$$

Note that for each $x \in X$, $T^n x = x^{2n}$. Thus we obtain:

$$\lim_{m,n\to\infty} \theta(T^m x, T^n x) < 4$$

Therefore, all conditions of Theorem 3 are satisfied hence T has a unique fixed point.

3. Application

In this section, we give existence theorem for Fredholm integral equation. Let $X = C([a,b], \mathbb{R})$ be the space of all continuous real valued functions define on $[a,b]$. Note that X is complete extended b-metric space by considering $d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$, with $\theta(x,y) = |x(t)| + |y(t)| + 2$, where $\theta : X \times X \to [1, \infty)$. Consider the Fredholm integral equation as:

$$x(t) = \int_a^b M(t, s, x(s))ds + g(t), \quad t, s \in [a, b] \tag{8}$$

where $g\colon [a, b] \to \mathbb{R}$ and $M\colon [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $T\colon X \to X$ the operator given by:

$$Tx(t) = \int_a^b M(t, s, x(s))ds + g(t) \text{ for } t, s \in [a, b]$$

where, the function $g\colon [a, b] \to \mathbb{R}$ and $M\colon [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Further, assume that the following condition hold:

$$|M(t, s, x(s)) - M(t, s, Tx(s))| \leq \frac{1}{2}|x(s) - Tx(s)| \text{ for each } t, s \in [a, b] \text{ and } x \in X$$

Then the integral Equation (8) has a solution.

We have to show that the operator T satisfies all the conditions of Theorem 3. For any $x \in X$ we have:

$$\begin{aligned} |Tx(t) - T(Tx(t))|^2 &\leq \left(\int_a^b |M(t, s, x(s)) - M(t, s, Tx(s))| ds \right)^2 \\ &\leq \frac{1}{4} d_\theta(x, Tx) \end{aligned}$$

All conditions of Theorem 3 follows by the hypothesis. Therefore, the operator T has a fixed point, that is, the Fredholm integral Equation (8) has a solution.

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