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Best Proximity Point Theorems in Partially Ordered *b*-Quasi Metric Spaces

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Abstract: In this paper, we introduce the notion of an ordered rational proximal contraction in partially ordered b-quasi metric spaces. We shall then prove some best proximity point theorems in partially ordered b-quasi metric spaces.

Keywords: best proximity point; b-metric space; b_q-metric space; ordered rational proximal contraction; b_q-proximally order preserving mapping

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1. Introduction

Fixed point theory is one of the most useful techniques in nonlinear functional analysis. The Banach contraction principle [1], which is the simplest statement regarding the fixed points of nonlinear mappings states that every contraction (self-mapping) $T : X \to X$ on a complete metric space (X, d) has a unique fixed point. This principle has been generalized by many researchers in several directions (see [2–4]). On the other hand, the study of fixed points for non-self mappings is also very interesting. More precisely, for two given nonempty closed subsets A and B of a complete metric space (X, d), a non-self contraction $T : A \to B$ does not necessarily have a fixed point. In this case, it is quite natural to investigate an element $x \in X$ such that $d(x, Tx)$ is in some sense minimum; more precisely, a point $x \in A$ for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T.

Since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be a self mapping, the best proximity point theorems are natural generalizations of the Banach contraction principle.

In the year 1969, Kay Fan [5] presented a classical result for best approximation theorem, which is regarded as the starting point of the current theory:

Theorem 1. *[5]. If A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$.*

Afterwards, several authors have derived extensions of Fan's Theorem and the best approximation theorem; here, we just mention the works of Prolla [6], Sehgal and Singh [7,8].

Fixed point theory for partially ordered metric spaces was initiated by Nieto and Rodriguez-Lopez [9]. In 2013, Hemant Kumar Nashine, Poom Kumam and Calogero Vetro [10] introduced the concept of a rational proximal contraction mapping. Using this notion, they succeeded to establish some best proximity point theorems under mild conditions; indeed, their hypotheses were a combination of compactness and completeness conditions.

In 1989, Bakhtin [11] introduced the concept of a b-metric space, as a generalized metric space with non-Hausdorff topology. He proved the contraction mapping principle in b-metric spaces that generalizes the Banach contraction principle of metric spaces. For related results, we refer the reader to [12–16].

In this paper, we prove some best proximity point theorems for ordered rational proximal contractions of first and second kind in the setting of partially ordered b-quasi metric spaces.

2. Preliminaries

This section is devoted to some preliminaries that later on will be used.

Let (X, d) be a metric space, A, B be two nonempty subsets of X, and $T : A \to B$ be a mapping. A point $x \in A$ is called a best proximity point of T if

$$d(x, Tx) = d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

We denote the set of all best proximity points of T by $B_{est}(T)$.

We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \quad \text{for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \quad \text{for some } x \in A\}.$$

In [17], sufficient conditions are given to guarantee the non-emptiness of A_0 and B_0.

Definition 1. *A metric space (X, d) is boundedly compact, if all closed bounded subsets of X are compact.*

Definition 2. *[18]. Let (X, d) be a metric space and A, B be two nonempty subsets of X. Then, B is said to be approximatively compact with respect to A, if every sequence $\{y_n\}$ in B, satisfying the condition $d(x, y_n) \to d(x, B)$ for some $x \in A$, has a convergent subsequence.*

Definition 3. *[10] Let (X, d) be a metric space and let A and B be nonempty subsets of X. A mapping $T : A \to B$ is said to be a rational proximal contraction of the first kind if there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, such that for all $x_1, x_2, u_1, u_2, \in A$, the conditions*

$$\begin{cases} d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{cases}$$

imply that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \frac{\beta[1 + d(x_1, u_1)]d(x_2, u_2)}{1 + d(x_1, x_2)} + \gamma[d(x_1, u_1) + d(x_2, u_2)] + \delta[d(x_1, u_2) + d(x_2, u_1)].$$

Definition 4. *[10] Let (X, d) be a metric space and let A and B be nonempty subsets of X. A mapping $T : A \to B$ is said to be a rational proximal contraction of the second kind if there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, such that for all $x_1, x_2, u_1, u_2, \in A$, the conditions*

$$\begin{cases} d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{cases}$$

imply that

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) + \frac{\beta[1 + d(Tx_1, Tu_1)]d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} + \gamma[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] + \delta[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)].$$

Theorem 2. *([10], Theorem 3.1) Let (X, d) be a complete metric space and A and B be two nonempty, closed subsets of X such that B is approximatively compact with respect to A. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a non-self mapping such that:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is a rational proximal contraction mapping of the first kind.*

Then, there exists $x \in A$ such that $B_{est}(T) = \{x\}$. Furthermore, for any fixed $x_0 \in A_0$ the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x.

They also proved that, if, instead, A is approximatively compact with respect to B, and T is a continuous rational proximal contraction mapping of the second kind, then T has a best proximity point (see [10], Theorem 3.2). Furthermore, if T is a rational proximal contraction mapping of the first kind, as well as of the second kind, then T has a unique best proximity point in A (see [10], Theorem 3.3).

Definition 5. *[11] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping. Then, (X, d) is said to be a b-metric space if the following conditions are satisfied:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(iii) *There exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.*

Definition 6. *[19] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping. Then, (X, d) is said to be a b-dislocated quasi metric space if the following conditions are satisfied:*

(i) *if $d(x, y) = 0 = d(y, x)$ then $x = y$ for all $x, y \in X$;*
(ii) *There exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. In addition, if d satisfies in the following extra condition, then (X, d) is a b-quasi metric space;*
(iii) $d(x, x) = 0$ *for all* $x \in X$.

We simply write b_q-metric space for b-quasi metric space, and b_{dq}-metric space for b-dislocated metric space.

Definition 7. *[20]*

- *A sequence $\{x_n\}$ in a b_{dq}-metric space (X, d), b-dislocated quasi converges (simply b_{dq}-converges) to $x \in X$, if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.*

  In this case, x is called a b_{dq}-limit of $\{x_n\}$, and we write $x_n \to x$.
- *A sequence $\{x_n\}$ in a b_{dq}-metric space (X, d) is called Cauchy if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

- *A b_{dq}-metric space (X, d) is complete if every Cauchy sequence in it is b_{dq}-convergent.*

Definition 8. *[21] Let $(X, d, \leq)$ be a partially ordered b_q-metric space and let A, B be two nonempty subsets of X. A mapping $T : A \to B$ is said to be b_q-proximally order preserving if for all $x_1, x_2, u_1, u_2 \in A$*

$$\begin{cases} x_1 \leq x_2, \\ d(u_1, Tx_1) = d(Tx_1, u_1) = d(A, B), \\ d(u_2, Tx_2) = d(Tx_2, u_2) = d(A, B), \end{cases}$$

imply that $u_1 \leq u_2$.

In the above definition, if $A = B$, then the b_q-proximally order preserving map T reduces to a nondecreasing map.

In addition, if $(X, d, \leq)$ is a partially ordered metric space, then the b_q-proximally order preserving map T reduces to a proximally order preserving map.

3. Main Results

In this section, we define the notion of an ordered rational proximal contraction mapping in partially ordered b_q-metric spaces. Then, we prove some best proximity point theorems for this mappings.

Definition 9. *Let $(X, d, \leq)$ be a partially ordered b-quasi metric space with $s \geq 1$ and let A, B be two nonempty subsets of X. A mapping $T : A \to B$ is said to be an ordered rational proximal contraction of the first kind, if there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $s\alpha + \beta + (s+1)\gamma + s(s+1)\delta < 1$ such that for all $x_1, x_2, u_1, u_2, \in A$,*

$$\begin{cases} x_1 \leq x_2, \\ d(u_1, Tx_1) = d(Tx_1, u_1) = d(A, B), \\ d(u_2, Tx_2) = d(Tx_2, u_2) = d(A, B), \end{cases}$$

$$\begin{aligned} \Longrightarrow d(u_1, u_2) \leq{}& \alpha d(x_1, x_2) + \beta \frac{[1 + d(x_1, u_1)] d(x_2, u_2)}{1 + d(x_1, x_2)} \\ &+ \gamma\, [d(x_1, u_1) + d(x_2, u_2) + \delta[d(x_1, u_2) + d(x_2, u_1)], \end{aligned}$$

and

$$\begin{aligned} d(u_2, u_1) \leq{}& \alpha d(x_2, x_1) + \beta \frac{[1 + d(u_1, x_1)] d(u_2, x_2)}{1 + d(x_2, x_1)} \\ &+ \gamma\, [d(u_1, x_1) + d(u_2, x_2) + \delta[d(u_1, x_2) + d(u_2, x_1)]. \end{aligned}$$

Clearly, if $s = 1$ and (X, d) is a metric space, then the above definition reduces to Definition 3.

Definition 10. *Let $(X, d, \leq)$ be a partially ordered b-quasi metric space with $s \geq 1$ and let A, B be two nonempty subsets of X. A mapping $T : A \to B$ is said to be an ordered rational proximal contraction of the second kind, if there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $s\alpha + \beta + (s+1)\gamma + s(s+1)\delta < 1$ such that for all $x_1, x_2, u_1, u_2, \in A$,*

$$\begin{cases} x_1 \leq x_2, \\ d(u_1, Tx_1) = d(Tx_1, u_1) = d(A, B), \\ d(u_2, Tx_2) = d(Tx_2, u_2) = d(A, B), \end{cases}$$

$$\begin{aligned} \Longrightarrow d(Tu_1, Tu_2) \leq{}& \alpha d(Tx_1, Tx_2) + \beta \frac{[1 + d(Tx_1, Tu_1)] d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\ &+ \gamma\, [d(Tx_1, Tu_1) + d(Tx_2, Tu_2) + \delta[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)], \end{aligned}$$

and

$$d(Tu_2, Tu_1) \leq \alpha d(Tx_2, Tx_1) + \beta \frac{[1 + d(Tu_1, Tx_1)]d(Tu_2, Tx_2)}{1 + d(Tx_2, Tx_1)}$$
$$+ \gamma\, [d(Tu_1, Tx_1) + d(Tu_2, Tx_2) + \delta[d(Tu_1, Tx_2) + d(Tu_2, Tx_1)].$$

Clearly, if $s = 1$ and (X, d) is a metric space, then the above definition reduces to Definition 4.

Theorem 3. *Let $(X, d, \leq)$ be a complete partially ordered b_q-metric space with $s \geq 1$, and $A,\ B$ be two nonempty subsets of X such that A is closed. Let $T : A \to B$ be a mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is a continuous ordered rational proximal contraction of the first kind,*
(iii) *T is b_q-proximally order preserving,*
(iv) *there exist points $x_0, x_1 \in A_0$ such that*

$$x_0 \leq x_1 \text{ and } d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B).$$

Then, there exists $x \in A$ such that $x \in B_{est}(T)$. In addition, if any two elements of $B_{est}(T)$ are comparable, then T has a unique best proximity point.

Proof. By our assumption, there exist points $x_0, x_1 \in A_0$ such that

$$x_0 \leq x_1 \text{ and } d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B).$$

Since $T(A_0) \subseteq B_0$ and $x_1 \in A_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(Tx_1, x_2) = d(A, B)$. Thus, we have

$$\begin{cases} x_0 \leq x_1, \\ d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B), \\ d(x_2, Tx_1) = d(Tx_1, x_2) = d(A, B). \end{cases}$$

Since T is b_{dq}-proximally order preserving, $x_1 \leq x_2$. Continuing this process, we obtain a nondecreasing sequence $\{x_n\}$ in A_0 such that for all $n \in \mathbb{N}$,

$$d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B).$$

Therefore, we have

$$\begin{cases} x_{n-1} \leq x_n, \\ d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B), \\ d(x_{n+1}, Tx_n) = d(Tx_n, x_{n+1}) = d(A, B). \end{cases} \tag{1}$$

Since T is an ordered rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x_n, x_{n+1}) + d(x_{n+1}, x_n) \leq\ & \alpha\big[d(x_{n-1}, x_n) + d(x_n, x_{n-1})\big] \\ & +\beta\big[\frac{(1 + d(x_{n-1}, x_n))d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} + \frac{(1 + d(x_n, x_{n-1})d(x_{n+1}, x_n)}{1 + d(x_n, x_{n-1})}\big] \\ & +\gamma\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_{n-1}) + d(x_{n+1}, x_n)\big] \\ & +\delta\ \big[d(x_{n-1}, x_{n+1}) + d(x_{n+1}, x_{n-1})\big]. \end{aligned}$$

By using the following triangle inequalities

$$d(x_{n-1}, x_{n+1}) \leq s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})],$$
$$d(x_{n+1}, x_{n-1}) \leq s[d(x_{n+1}, x_n) + d(x_n, x_{n-1})],$$

we obtain

$$\begin{aligned}d(x_n, x_{n+1}) + d(x_{n+1}, x_n) \leq (\alpha + \gamma + s\delta)\big[d(x_{n-1}, x_n) + d(x_n, x_{n-1})\big]\\ + (\beta + \gamma + s\delta)\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_n)\big].\end{aligned}$$

If we set

$$t_n := d(x_n, x_{n+1}) + d(x_{n+1}, x_n),$$

then we have

$$t_n \leq \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta} t_{n-1}. \tag{2}$$

Since $\frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta} < \frac{1}{s} \leq 1$, it follows that $t_n \leq t_{n-1}$. Therefore, the sequence $\{t_n\}$ is decreasing. Thus, there exists some $t \geq 0$ such that $\lim_{n\to\infty} t_n = t$. By taking the limit as $n \to \infty$ in Equation (2), we have

$$t \leq \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta}\, t < t.$$

Thus, $t = 0$, that is,

$$\lim_{n\to\infty} t_n = 0 \;\Rightarrow\; \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3}$$

In addition, if we set $h := \frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta}$, then we have

$$t_n \leq h t_{n-1} \leq h^2 t_{n-2} \leq \cdots \leq h^n t_0.$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Let $n \leq m$, so we have

$$\begin{aligned}d(x_n, x_m) + d(x_m, x_n) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] + s[d(x_m, x_{n+1}) + d(x_{n+1}, x_n)]\\ &\leq st_n + s[d(x_{n+1}, x_m) + d(x_m, x_{n+1})]\\ &\leq st_n + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) + d(x_m, x_{n+2}) + d(x_{n+2}, x_{n+1})]\\ &\leq st_n + s^2 t_{n+1} + s^2[d(x_{n+2}, x_m) + d(x_m, x_{n+2})]\\ &\leq st_n + s^2 t_{n+1} + \cdots + s^{m-n+1} t_m\\ &\leq \big[sh^n + s^2 h^{n+1} + \cdots + s^{m-n+1} h^m\big] t_0\\ &\leq sh^n (1 + sh + \cdots + (sh)^{m-n}) t_0\\ &\leq sh^n (1 + sh + \cdots) t_0\\ &= sh^n \frac{1}{1 - sh} t_0.\end{aligned}$$

Since $sh < 1$ and $h < 1$, then

$$\lim_{n,m\to\infty} \big[d(x_n, x_m) + d(x_m, x_n)\big] = 0.$$

Thus, $\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in A. Since X is a complete b_q-metric space, and A is a closed subset of X, there exists x in A such that $x_n \to x$. Since T is continuous, it follows that $Tx_n \to Tx$. Thus, we have

$$d(x_{n+1}, Tx_n) \to d(x, Tx) \text{ and } d(Tx_n, x_{n+1}) \to d(Tx, x).$$

On the other hand, $d(x_{n+1}, Tx_n) = d(Tx_n, x_{n+1}) = d(A, B)$. Therefor $d(x, Tx) = d(Tx, x) = d(A, B)$. That is, $x \in B_{est}(T)$.

As for the uniqueness of best proximity point, we now assume that there exist $x, \acute{x} \in B_{est}(T)$ such that $x \leq \acute{x}$. Thus, we have

$$\begin{cases} x \leq \acute{x}, \\ d(x, Tx) = d(Tx, x) = d(A, B), \\ d(\acute{x}, T\acute{x}) = d(T\acute{x}, \acute{x}) = d(A, B). \end{cases}$$

Since T is an ordered rational proximal contraction of the first kind, after some manipulations, we get

$$d(x, \acute{x}) + d(\acute{x}, x) \leq \alpha\big[d(x, \acute{x}) + d(\acute{x}, x)\big] + 0 + 0 + 2\delta\big[d(x, \acute{x}) + d(\acute{x}, x)\big].$$

Thus,

$$d(x, \acute{x}) + d(\acute{x}, x) \leq (\alpha + 2\delta)\big[d(x, \acute{x}) + d(\acute{x}, x))\big].$$

Since $\alpha + 2\delta < 1$, it follows that

$$d(x, \acute{x}) + d(\acute{x}, x) = 0 \ \Rightarrow \ d(x, \acute{x}) = d(\acute{x}, x) = 0 \ \Rightarrow \ x = \acute{x}.$$

Therefore, the best proximity point of T is unique. □

Corollary 1. *Let (X, d) be a complete metric space, and $A, \ B$ be two nonempty subsets of X such that A is closed. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a non-self mapping such that:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is a continuous rational proximal contraction of the first kind.*

Then, there exists a unique $x \in A$ such that $x \in B_{est}(T)$.

Corollary 2. *Let $(X, d, \leq)$ be a complete partially ordered b_q-metric space with $s \geq 1$, and there exist points $x_0, x_1 \in X$ such that $x_0 \leq x_1$. Let $T : X \to X$ be a self-mapping having b_q-proximally order preserving property, and furthermore, assume that there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $s\alpha + \beta + (s+1)\gamma + s(s+1)\delta < 1$ such that for all $x_1, x_2 \in X$, where $x_1 \leq x_2$,*

$$\begin{aligned} d(Tx_1, Tx_2) \leq{}& \alpha d(x_1, x_2) + \beta\frac{[1 + d(x_1, Tx_1)]d(x_2, Tx_2)}{1 + d(x_1, x_2)}, \\ &+ \gamma\,[d(x_1, Tx_1) + d(x_2, Tx_2) + \delta[d(x_1, Tx_2) + d(x_2, Tx_1)], \end{aligned}$$

and

$$\begin{aligned} d(Tx_2, Tx_1) \leq{}& \alpha d(x_2, x_1) + \beta\frac{[1 + d(Tx_1, x_1)]d(Tx_2, x_2)}{1 + d(x_2, x_1)} \\ &+ \gamma\,[d(Tx_1, x_1) + d(Tx_2, x_2) + \delta[d(Tx_1, x_2) + d(Tx_2, x_1)]. \end{aligned}$$

Then, there exists $x \in A$, such that $x \in B_{est}(T)$.
In addition, if any two elements of $B_{est}(T)$ are comparable, then T has a unique best proximity point.

Theorem 4. *If in Theorem 3, rather than assuming T is continuous and A is a closed subset of X, we assume that*

(1) *A_0 is a closed subset of X,*
(2) *if $\{x_n\}$ is a nondecreasing sequence in A, such that $x_n \to x$ as $n \to \infty$, then $x_n \leq x$.*

Then, the conclusion of Theorem 3 holds.

Proof. As in the proof of Theorem 3, there exists a nondecreasing sequence $\{x_n\}$ in A_0 such that for all $n \in \mathbb{N}$, $d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B)$, and that $\{x_n\}$ is a Cauchy sequence in A_0. Since A_0

is closed, there exists an $x \in A_0$ such that $x_n \to x$, and by assumption, $x_n \leq x$. As $T(A_0) \subseteq B_0$, there exists $\acute{x} \in A_0$ such that

$$d(\acute{x}, Tx) = d(Tx, \acute{x}) = d(A, B). \tag{4}$$

Therefore, we have

$$\begin{cases} x_{n-1} \leq x \\ d(x_n, Tx_{n-1}) = d(x_n, Tx_{n-1}) = d(A, B) \\ d(\acute{x}, Tx) = d(Tx, \acute{x}) = d(A, B). \end{cases}$$

Because T is an ordered rational proximal contraction, it follows from the definition that

$$\begin{aligned} d(x_n, \acute{x}) + d(\acute{x}, x_n) \leq \alpha\big[d(x_{n-1}, x) + d(x, x_{n-1})\big] & \\ + \beta\Big[\frac{[1 + d(x_{n-1}, x_n)]d(x, \acute{x})}{1 + d(x_{n-1}, x)} + \frac{[1 + d(x_n, x_{n-1})]d(\acute{x}, x)}{1 + d(x, x_{n-1})}\Big] & \\ + \gamma\big[d(x_{n-1}, x_n) + d(x, \acute{x}) + d(x_n, x_{n-1}) + d(\acute{x}, x)\big] & \\ + \delta\big[d(x_{n-1}, \acute{x}) + d(x, x_n) + d(\acute{x}, x_{n-1}) + d(x_n, x)\big]. & \end{aligned} \tag{5}$$

By taking limit as $n \to \infty$ in (5), we obtain

$$d(x, \acute{x}) + d(\acute{x}, x) \leq (\beta + \gamma + \delta)\big[d(x, \acute{x}) + d(\acute{x}, x)\big].$$

Since $\beta + \gamma + \delta < 1$, it follows that

$$d(x, \acute{x}) + d(\acute{x}, x) = 0 \;\Rightarrow\; d(x, \acute{x} = d(\acute{x}, x) = 0 \;\Rightarrow\; x = \acute{x}.$$

Then from (4), $d(x, Tx) = d(Tx, x_{)} = d(A, B)$. The proof of uniqueness is similar to the one in Theorem 3. □

Theorem 5. *Let $(X, d, \leq)$ be a boundedly compact partially ordered b_q-metric space with $s \geq 1$, and A, B be two nonempty and closed subsets of X. Let $T : A \to B$ be a mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is a continuous ordered rational proximal contraction of the second kind,*
(iii) *T is b_q-proximally order preserving,*
(iv) *there exist points $x_0, x_1 \in A_0$, such that*

$$x_0 \leq x_1 \textit{ and } d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B).$$

Then, there exists $x \in A$ such that $x \in B_{est}(T)$. In addition, if any two elements of $B_{est}(T)$ are comparable, then for all $x, y \in B_{est}(T)$, $Tx = Ty$.

Proof. By our assumption, there exist points $x_0, x_1 \in A_0$, such that

$$x_0 \leq x_1 \text{ and } d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B).$$

Since $T(A_0) \subseteq B_0$ and $x_1 \in A_0$, there exists $x_2 \in A_0$, such that $d(x_2, Tx_1) = d(Tx_1, x_2) = d(A, B)$. Thus, we have

$$\begin{cases} x_0 \leq x_1, \\ d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B), \\ d(x_2, Tx_1) = d(Tx_1, x_2) = d(A, B). \end{cases}$$

Since T is b_q-proximally order preserving, it follows that $x_1 \leq x_2$. Continuing this process, we obtain a nondecreasing sequence $\{x_n\}$ in A_0, such that for all $n \in \mathbb{N}$,

$$d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B).$$

Therefore, we have

$$\begin{cases} x_{n-1} \le x_n, \\ d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B), \\ d(x_{n+1}, Tx_n) = d(Tx_n, x_{n+1}) = d(A, B). \end{cases} \tag{6}$$

Since T is an ordered rational proximal contraction of the second kind, we have

$$\begin{aligned} & d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_n) \\ & \le \alpha\big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\big] \\ & + \beta\Big[\frac{(1 + d(Tx_{n-1}, Tx_n))d(Tx_n, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_n)} + \frac{(1 + d(Tx_n, Tx_{n-1})d(Tx_{n+1}, Tx_n)}{1 + d(Tx_n, Tx_{n-1})}\Big] \\ & + \gamma\big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)\big] \\ & + \delta\big[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n-1})\big]. \end{aligned}$$

By using the below triangle inequalities:

$$\begin{aligned} d(Tx_{n-1}, Tx_{n+1}) &\le s[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})], \\ d(Tx_{n+1}, Tx_{n-1}) &\le s[d(Tx_{n+1}, Tx_n)] + d(Tx_n, Tx_{n-1})], \end{aligned}$$

we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_n) \le (\alpha + \gamma + s\delta)\big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\big] \\ + (\beta + \gamma + s\delta)\big[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_n)\big]. \end{aligned}$$

If we set

$$t_n := d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_n),$$

then we have

$$t_n \le \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta} t_{n-1}. \tag{7}$$

Since $\frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta} < \frac{1}{s} \le 1$, it follows that $t_n \le t_{n-1}$. Therefore, the sequence $\{t_n\}$ is decreasing. Thus, there exists some $t \ge 0$ such that $\lim_{n\to\infty} t_n = t$. By taking the limit as $n \to \infty$ in Equation (7), we have

$$t \le \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta}\, t < t.$$

Thus, $t = 0$, that is,

$$\lim_{n\to\infty} t_n = 0 \;\Rightarrow\; \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = \lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0. \tag{8}$$

In addition, if we set $h := \frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta}$, then we have

$$t_n \le h t_{n-1} \le h^2 t_{n-2} \le \cdots \le h^n t_0.$$

Now, we prove that $\{Tx_n\}$ is a Cauchy sequence in B_0. Let $n \le m$, so we have

$$\begin{aligned}
&d(Tx_n, Tx_m) + d(Tx_m, Tx_n) \\
&\leq s[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_m)] + s[d(Tx_m, Tx_{n+1}) + d(Tx_{n+1}, Tx_n)] \\
&\leq st_n + s[d(Tx_{n+1}, Tx_m) + d(Tx_m, Tx_{n+1})] \\
&\leq st_n + s^2[d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+2}, Tx_m) + d(Tx_m, Tx_{n+2}) + d(Tx_{n+2}, Tx_{n+1})] \\
&\leq st_n + s^2 t_{n+1} + s^2[d(Tx_{n+2}, Tx_m) + d(Tx_m, Tx_{n+2})] \\
&\leq st_n + s^2 t_{n+1} + \cdots + s^{m-n+1} t_m \\
&\leq \left[sh^n + s^2 h^{n+1} + \cdots + s^{m-n+1} h^m\right] t_0 \\
&\leq sh^n (1 + sh + \cdots + (sh)^{m-n}) t_0 \\
&\leq sh^n (1 + sh + \cdots) t_0 \\
&= sh^n \frac{1}{1-sh} t_0.
\end{aligned}$$

Since $sh < 1$ and $h < 1$, then

$$\lim_{n,m\to\infty} \left[d(Tx_n, Tx_m) + d(Tx_m, Tx_n)\right] = 0.$$

Thus, $\lim_{n,m\to\infty} d(Tx_n, Tx_m) = \lim_{n,m\to\infty} d(Tx_m, Tx_n) = 0$. Therefore, $\{Tx_n\}$ is a Cauchy sequence in B_0. So $\{Tx_n\}$ is a bounded sequence. Since X is a boundedly compact b_q-metric space and B is closed, there exist a subsequence $\{Tx_{n_r}\}$ of $\{Tx_n\}$ and a $y \in B$ such that $\lim_{r\to\infty} Tx_{n_r} = y$.

In addition, we have

$$\begin{aligned}
d(A, y) &\leq d(x_{n_r}, y) \leq s\left[d(x_{n_r}, Tx_{n_{r-1}}) + d(Tx_{n_{r-1}}, y)\right] \leq sd(A, B) + sM, \\
d(y, A) &\leq d(y, x_{n_r}) \leq s\left[d(y, Tx_{n_{r-1}}) + Tx_{n_{r-1}}, x_{n_r})\right] \leq sM + sd(A, B),
\end{aligned}$$

where $M = \sup\{d(Tx_{n_{r-1}}, y), d(y, Tx_{n_{r-1}}) : \ n \in \mathbb{N}\}$.

Therefore, $\{x_{n_r}\}$ is a bounded sequence. Since X is a boundedly compact b_q-metric space and A is closed, there exist a subsequence $\{x_{n_{r_i}}\}$ of $\{x_{n_r}\}$ and an $x \in A$, such that $\lim_{i\to\infty} x_{n_{r_i}} = x$, and by assumption, $x_{n_{r_i}} \leq x$. Since T is continuous, we conclude that

$$\begin{aligned}
d(x, Tx) &= \lim_{i\to\infty} d(x_{n_{r_i}}, Tx_{n_{r_{i-1}}}) = d(A, B), \\
d(Tx, x) &= \lim_{i\to\infty} d(Tx_{n_{r_{i-1}}}, x_{n_{r_i}}) = d(A, B).
\end{aligned}$$

Therefore, $d(x, Tx) = d(Tx, x) = d(A, B)$. That is, $x \in B_{est}(T)$. Now, assume that there exist $x, \acute{x} \in B_{est}(T)$ such that $x \leq \acute{x}$. Thus, we have

$$\begin{cases}
x \leq \acute{x}, \\
d(x, Tx) = d(Tx, x) = d(A, B), \\
d(\acute{x}, T\acute{x}) = d(T\acute{x}, \acute{x}) = d(A, B).
\end{cases}$$

Since T is an ordered θ-η-rational proximal contraction of the second kind, after some manipulations, we get

$$d(Tx, T\acute{x}) + d(T\acute{x}, Tx) \leq \alpha\left[d(Tx, T\acute{x}) + d(T\acute{x}, Tx)\right] + 0 + 0 + 2\delta\left[d(Tx, T\acute{x}) + d(T\acute{x}, Tx)\right].$$

Thus,

$$d(Tx, T\acute{x}) + d(T\acute{x}, Tx) \leq (\alpha + 2\delta)\left[d(Tx, T\acute{x}) + d(T\acute{x}, Tx))\right].$$

Since $\alpha + 2\delta < 1$, it follows that

$$d(Tx, T\acute{x}) + d(T\acute{x}, Tx) = 0 \;\Rightarrow\; d(Tx, T\acute{x}) = d(T\acute{x}, Tx) = 0 \;\Rightarrow\; Tx = T\acute{x}.$$

□

Corollary 3. *Let (X, d) be a complete metric space, and A, B be two nonempty subsets of X such that A is closed. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a non-self mapping, such that:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is a continuous rational proximal contraction of the second kind.*

Then, there exists an $x \in A$, such that $x \in B_{est}(T)$ and for all $x, y \in B_{est}(T)$, $Tx = Ty$.

Theorem 6. *Let $(X, d, \leq)$ be a complete partially ordered b_q-metric space with $s \geq 1$, and A, B be two nonempty and closed subsets of X. Let $T : A \to B$ be a mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$,
(ii) *T is an ordered rational proximal contraction of the first and second kind,*
(iii) *T is b_q-proximally order preserving,*
(iv) *there exist points $x_0, x_1 \in A_0$, such that*

$$x_0 \leq x_1 \text{ and } d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B).$$

(v) *if $\{x_n\}$ is a nondecreasing sequence in A, such that $x_n \to x$ as $n \to \infty$, then $x_n \leq x$.*

Then, there exists $x \in A$ such that $x \in B_{est}(T)$. In addition, if any two elements of $B_{est}(T)$ are comparable, then T has a unique best proximity point.

Proof. As in the proof of Theorem 3, there exists a nondecreasing sequence $\{x_n\}$ in A_0, such that for all $n \in \mathbb{N}$

$$\begin{cases} x_{n-1} \leq x_n, \\ d(x_n, Tx_{n-1}) = d(Tx_{n-1}, x_n) = d(A, B), \end{cases}$$

and that $\{x_n\}$ is a Cauchy sequence in A.

In addition, similar to the proof of Theorem 5, $\{Tx_n\}$ is a Cauchy sequence in B_0. Since A and B are closed, there exist $x \in A$ and $y \in B$, such that $x_n \to x$ and $Tx_n \to y$ and by assumption, $x_n \leq x$. Now, we have

$$d(x, y) = \lim_{n\to\infty} d(x_n, Tx_{n-1}) = d(A, B).$$

Therefore, $x \in A_0$. As $T(A_0) \subseteq B_0$, there exists $\acute{x} \in A_0$, such that

$$d(\acute{x}, Tx) = d(Tx, \acute{x}) = d(A, B). \tag{9}$$

Therefore, we have

$$\begin{cases} x_{n-1} \leq x, \\ d(x_n, Tx_{n-1}) = d(x_n, Tx_{n-1}) = d(A, B), \\ d(\acute{x}, Tx) = d(Tx, \acute{x}) = d(A, B). \end{cases}$$

Because T is an ordered rational proximal contraction, it follows from the definition that

$$\begin{aligned} d(x_n, \acute{x}) + d(\acute{x}, x_n) \leq{}& \alpha\big[d(x_{n-1}, x) + d(x, x_{n-1})\big] \\ &+ \beta\Big[\frac{[1 + d(x_{n-1}, x_n)]d(x, \acute{x})}{1 + d(x_{n-1}, x)} + \frac{[1 + d(x_n, x_{n-1})]d(\acute{x}, x)}{1 + d(x, x_{n-1})}\Big] \\ &+ \gamma\big[d(x_{n-1}, x_n) + d(x, \acute{x}) + d(x_n, x_{n-1}) + d(\acute{x}, x)\big] \\ &+ \delta\big[d(x_{n-1}, \acute{x}) + d(x, x_n) + d(\acute{x}, x_{n-1}) + d(x_n, x)\big]. \end{aligned} \tag{10}$$

By taking the limit as $n \to \infty$ in (10), we obtain

$$d(x, \acute{x}) + d(\acute{x}, x) \leq (\beta + \gamma + \delta)\big[d(x, \acute{x}) + d(\acute{x}, x)\big].$$

Since $\beta + \gamma + \delta < 1$, it follows that

$$d(x, \acute{x}) + d(\acute{x}, x) = 0 \;\Rightarrow\; d(x, \acute{x} = d(\acute{x}, x) = 0 \;\Rightarrow\; x = \acute{x}.$$

Then, from Equation (9),

$$d(x, Tx) = d(Tx, x) = d(A, B).$$

The proof of uniqueness is similar to that of Theorem 3. $\square$

4. Examples

In this section, we provide some examples, which accordingly show the applications of our results.

Example 1. *Let $X = [0, 1]$ and let $\leq$ be the usual ordering of X. Let $d : X \times X \to [0, \infty)$ be defined by*

$$d(x, y) = \begin{cases} |x - y| & 0, \leq x, y \leq \frac{1}{2}, \\ 0 & x = y, \\ 2 & x = 0, y = \frac{2}{3}, \\ \frac{1}{2} & else. \end{cases}$$

Then, $(X, d, \leq)$ is a partially ordered b_q-metric space with $s = 2$.

Let $A = [0, \frac{\pi}{8}]$ and $B = [0, \frac{1}{4}]$ be subsets of X. Clearly, $d(A, B) = 0$. Now, we define $T : A \to B$ by $Tx = \frac{1}{4}\sin x$. Thus, for all $x_1, x_2, u_1, u_2, \in A$, if

$$\begin{cases} x_1 \leq x_2, \\ d(u_1, Tx_1) = d(Tx_1, u_1) = d(A, B) = 0, \\ d(u_2, Tx_2) = d(Tx_2, u_2) = d(A, B) = 0, \end{cases}$$

then $u_1 = Tx_1 = \frac{1}{4}\sin x_1,\ u_2 = Tx_2 = \frac{1}{4}\sin x_2$ and $0 \leq u_1, u_2 \leq \frac{1}{4}\sin\frac{\pi}{8} \leq \frac{1}{4} \leq \frac{1}{2}$. Thus,

$$d(u_1, u_2) = |u_1 - u_2| = |\frac{1}{4}\sin x_1 - x_2| \leq \frac{1}{4}|x_1 - x_2|.$$

Because $0 \leq x_1, x_2 \leq \frac{\pi}{8} \leq \frac{1}{2}$, $d(x_1, x_2) = |x_1 - x_2|$, we have

$$d(u_1, u_2) \leq \frac{1}{4}d(x_1, x_2).$$

If we consider $\alpha = \frac{1}{4}, \beta = \gamma = \delta = 0$, then $s\alpha + \beta + (s+1)\gamma + s(s+1)\delta = 2\frac{1}{4} = \frac{1}{2} < 1$. Thus, T is an ordered rational proximal contraction of the first kind.

The next example shows that an ordered rational proximal contraction of the second kind is not necessarily an ordered rational proximal contraction of the first kind.

Example 2. *Let $X = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and let $\leq$ be the usual ordering of X.*

Let $d : X \times X \to [0, \infty)$ be defined by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} & \begin{cases} x_1 = y_1, x_2 = y_2, \\ or \\ x_1 = 0, y_1 = \frac{1}{2}, x_2 = y_2, \\ or \\ x_1 = y_1, x_2 = 0, y_2 = \frac{1}{2}, \end{cases} \\ 0 & (x_1, y_1) = (x_2, y_2), \\ 2 & x_1 = 0, y_1 = \frac{1}{3}, x_2 = \frac{1}{3}, y_2 = 0, \\ \frac{1}{2} & else. \end{cases}$$

Then, $(X, d, \leq)$ is a partially ordered b_q-metric space with $s = 2$. Define

$$A := \{(x, x) : 0 \leq x \leq \frac{1}{2}\} \cup \{(0, \frac{1}{2})\}$$

and $B := \{(0, 0), (0, \frac{1}{2})\}$. Then, $d(A, B) = 0$. Now, we define $T : A \to B$ by

$$T(x, y) = \begin{cases} (0, 0) & x = y, \text{ and } x \text{ is rational}, \\ (0, \frac{1}{2}) & x = y, \text{ and } x \text{ is not rational}, \\ (0, 0) & x = 0, y = \frac{1}{2}. \end{cases}$$

For every $(x_1, y_1), (x_2, y_2), (u_1, \acute{u_1}), (u_2, \acute{u_2})$ in A, we have

$$d\big((u_1, \acute{u_1}), T(x_1, y_1)\big) = d\big(T(x_1, y_1), (u_1, \acute{u_1})\big) = d(A, B) = 0,$$

from which it follows that $(u_1, \acute{u_1}) = T(x_1, y_1) = (0, 0)$ or $(0, \frac{1}{2})$. This implies that $T(u_1, \acute{u_1}) = (0, 0)$.
Similarly,

$$d\big((u_2, \acute{u_2}), T(x_2, y_2)\big) = d\big(T(x_2, y_2), (u_2, \acute{u_2})\big) = d(A, B) = 0,$$

from which it follows that $(u_2, \acute{u_2}) = T(x_2, y_2) = (0, 0)$ or $(0, \frac{1}{2})$. Hence, $T(u_2, \acute{u_2}) = (0, 0)$. Therefore, for all $\alpha, \beta, \gamma, \delta$ with $2\alpha + \beta + 3\gamma + 6\delta < 1$, we have

$$\begin{aligned} d\big(T(u_1, \acute{u_1}), T(u_2, \acute{u_2})\big) &= 0 \\ &\leq \alpha\, d\big(T(x_1, y_1), T(x_2, y_2)\big) + \\ &\quad \beta \frac{\big[1 + d\big(T(x_1, y_1), T(u_1, \acute{u_1})\big)\big] d\big(T(x_2, y_2), T(u_2, \acute{u_2})\big)}{1 + d\big(T(x_1, y_1), T(x_2, y_2)\big)} + \\ &\quad \gamma\, \big[d\big(T(x_1, y_1), T(u_1, \acute{u_1})\big) + d\big(T(x_2, y_2), T(u_2, \acute{u_2})\big)\big] + \\ &\quad \delta\, \big[d\big(T(x_1, y_1), T(u_2, \acute{u_2})\big) + d\big(T(x_2, y_2), T(u_1, \acute{u_1})\big)\big]. \end{aligned}$$

Similarly,

$$\begin{aligned} d\big(T(u_2, \acute{u_2}), T(u_1, \acute{u_1})\big) &= 0 \\ &\leq \alpha\, d\big(T(x_2, y_2), T(x_1, y_1)\big) + \\ &\quad \beta \frac{\big[1 + d\big(T(u_1, \acute{u_1}), T(x_1, y_1)\big)\big] d\big(T(u_2, \acute{u_2}), T(x_2, y_2)\big)}{1 + d\big(T(x_2, y_2), T(x_1, y_1)\big)} + \\ &\quad \gamma\, \big[d\big(T(u_1, \acute{u_1}), T(x_1, y_1)\big) + d\big(T(u_2, \acute{u_2}), T(x_2, y_2)\big)\big] + \\ &\quad \delta\, \big[d\big(T(u_2, \acute{u_2}), T(x_1, y_1)\big) + d\big(T(u_1, \acute{u_1}), T(x_2, y_2)\big)\big]. \end{aligned}$$

Thus, T is an ordered rational proximal contraction of the second kind.

Now, we prove T is not an ordered rational proximal contraction of the first kind. It is clear that diam $A = \sqrt{2}/2$. If

$$(x_1, y_1) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ (u_1, \acute{u}_1) = (0, \frac{1}{2}),\ (x_2, y_2) = (u_2, \acute{u}_2) = (0, 0),$$

then, $d\big((u_1, \acute{u}_1), (u_2, \acute{u}_2)\big) = \frac{1}{2}$ *and*

$$\begin{aligned} d\big((u_1, \acute{u}_1), T(x_1, y_1)\big) &= d\big(T(x_1, y_1), (u_1, \acute{u}_1)\big) = d(A, B) = 0, \\ d\big((u_2, \acute{u}_2), T(x_2, y_2)\big) &= d\big(T(x_2, y_2), (u_2, \acute{u}_2)\big) = d(A, B) = 0. \end{aligned}$$

For all nonnegative real numbers $\alpha, \beta, \gamma, \delta$ with $2\alpha + \beta + 3\gamma + 6\delta < 1$, we have $\alpha + \gamma + 2\delta < \frac{1}{2}$. Therefore, from $d\big((x_2, y_2), (u_2, \acute{u}_2)\big) = 0$, we have

$$\begin{aligned} & \alpha\, d\big((x_1, y_1), (x_2, y_2)\big) + \\ & \beta \frac{\big[1 + d\big((x_1, y_1), (u_1, \acute{u}_1)\big)\big] d\big((x_2, y_2), (u_2, \acute{u}_2)\big)}{1 + d\big((x_1, y_1), (x_2, y_2)\big)} + \\ & \gamma \left[d\big((x_1, y_1), (u_1, \acute{u}_1)\big) + d\big((x_2, y_2), (u_2, \acute{u}_2)\big)\right] + \\ & \delta \left[d\big((x_1, y_1), (u_2, \acute{u}_2)\big) + d\big((x_2, y_2), (u_1, \acute{u}_1)\big)\right] \\ \leq\ & (\alpha + \gamma + 2\delta)\ diam A < \frac{1}{2}\frac{\sqrt{2}}{2} < \frac{1}{2} =\ d\big((u_1, \acute{u}_1), (u_2, \acute{u}_2)\big). \end{aligned}$$

Thus, T is not an ordered rational proximal contraction of the first kind.

The above example shows that every ordered rational proximal contraction is not necessarily continuous. The next example reveals that in Theorems 3 and 6, the closedness condition on A is necessary. In addition, in Theorem 4, the closedness condition on A_0 is necessary.

Example 3. *Let $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|$, and let $\leq$ be the usual ordering of X, then $(X, d, \leq)$ is a complete partially ordered metric space. Thus, X is a partially b_q-metric space with $s = 1$. Let $A = [\frac{1}{2}, 1)$ and $B = [0, 1]$ be subsets of X. Obviously, $d(A, B) = 0$ and $A_0 = B_0 = A$. Suppose $T : A \to B$ is defined by $Tx = \sqrt{x}$. Therefore, T is continuous and $T(A_0) \subseteq B_0$. Since T is nondecreasing, T is b_q-proximally order preserving. Thus, for all $x_1, x_2, u_1, u_2, \in A$, if*

$$d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) = 0,$$

then

$$u_1 = Tx_1 = \sqrt{x_1},\ u_2 = Tx_2 = \sqrt{x_2},$$

and

$$Tu_1 = {x_1}^{\frac{1}{4}}, Tu_2 = {x_2}^{\frac{1}{4}}.$$

From the mean value theorem, there exist $c_1, c_2 \in [x_1, x_2]$ such that

$$d(u_1, u_2) = |\sqrt{x_1} - \sqrt{x_2}| \leq \frac{1}{2\sqrt{c_1}}|x_1 - x_2| \leq \frac{1}{2\sqrt{\frac{1}{2}}}|x_1 - x_2| \leq \frac{\sqrt{2}}{2} d(x_1, x_2).$$

If we consider $\alpha = \frac{\sqrt{2}}{2}, \beta = \gamma = \delta = 0$, then $\alpha + \beta + 2\gamma + 2\delta = \frac{\sqrt{2}}{2} < 1$. Thus, T is an ordered rational proximal contraction of the first kind. Moreover,

$$d(Tu_1, Tu_2) = |x_1{}^{\frac{1}{4}} - x_2{}^{\frac{1}{4}}| \leq \frac{1}{4c_2{}^{\frac{3}{4}}}|x_1 - x_2| \leq \frac{1}{4(\frac{1}{2})^{\frac{3}{4}}}|x_1 - x_2|$$
$$\frac{1}{4(\frac{1}{2})^{\frac{3}{4}}}|\sqrt{x}_1 - \sqrt{x}_2||\sqrt{x}_1 + \sqrt{x}_2| \leq \frac{1}{4(\frac{1}{2})^{\frac{3}{4}}}2|\sqrt{x}_1 - \sqrt{x}_2|$$
$$\leq (\frac{1}{2})^{\frac{1}{4}}|\sqrt{x}_1 - \sqrt{x}_2| \leq (\frac{1}{2})^{\frac{1}{4}}d(Tx_1, Tx_2).$$

If we consider $\alpha = (\frac{1}{2})^{\frac{1}{4}}, \beta = \gamma = \delta = 0$, *then* $\alpha + \beta + 2\gamma + 2\delta = (\frac{1}{2})^{\frac{1}{4}} < 1$. *Thus,* T *is an ordered rational proximal contraction of the second kind.*

If $x_0 = \frac{1}{2}$ *and* $x_1 = \sqrt{\frac{1}{2}}$, *then* $x_0 \leq x_1$ *and* $d(x_1, Tx_0) = d(Tx_0, x_1) = d(A, B) = 0$. *It is clear that all the conditions of Theorems 3, 4 and 6, except the condition of closedness of* A *and* A_0, *hold. On the other hand, we have*

$$x \in A \text{ and } x \in B_{est}(T) \Rightarrow d(x, Tx) = 0 \Rightarrow x = \sqrt{x} \Rightarrow x = 0 \text{ or } 1$$

which is a contradiction.

5. Conclusions

In this paper we introduced the notion of an ordered rational proximal contraction of the first type, as well as of the second type. We then proved the existence and uniqueness of best proximity point theorems for these mappings in the setting of b-quasi metric spaces; in this way, we generalized a number of existing results to this new setting. Finally, in the preceding section, we discussed some applications of our new results.

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