

Article

Amenability Modulo an Ideal of Second Duals of Semigroup Algebras

Hamidreza Rahimi ^{1,*} and Khalil Nabizadeh ²

¹ Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, P. O. Box 13185/768, Tehran, Iran

² Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran; functional1971@ymail.com

* Correspondence: rahimi@iauctb.ac.ir

Academic Editor: Hvedri Inassaridze

Received: 22 May 2016; Accepted: 2 September 2016; Published: 13 September 2016

Abstract: The aim of this paper is to investigate the amenability modulo, an ideal of Banach algebras with emphasis on applications to homological algebras. In doing so, we show that amenability modulo, an ideal of A^{**} implies amenability modulo, an ideal of A . Finally, for a large class of semigroups, we prove that $l^1(S)^{**}$ is amenable modulo I_σ^{**} if and only if an appropriate group homomorphic image of S is finite, where I_σ is the closed ideal induced by the least group congruence σ .

Keywords: amenability modulo an ideal; semigroup algebra; group congruence

1. Introduction

The concept of the amenability modulo, an ideal of Banach algebras, was initiated by the first author and Amini [1]. They stated a version of Johnson's theorem for a large class of semigroups, including inverse semigroups, E -inversive semigroup, and E -inversive E -semigroups; $l^1(S)$ is amenable modulo I_σ if and only if S is amenable. Some characterization of amenability modulo, an ideal of Banach algebras, and their hereditary properties were investigated in [2,3]. A characterization of the amenability of Banach algebras with applications to homological algebra were studied by P. C. Curtis and R. J. Loy [4]. Amenability of the second dual of a Banach algebra was considered by Gourdeau (1997) [5]; if a Banach algebra A^{**} is amenable, then A is amenable; a fact which is also proven (using a different proof) by Ghahramani, Loy, and Willis [6]. Then, it was shown that for a locally compact group G , if $L^1(G)^{**}$ is amenable, then G is finite [6]. Whereas, we are not aware of a similar result for semigroups, in general.

In section two of this paper, we give a brief review of amenability modulo an ideal of Banach algebras, the necessary preliminaries of homological algebra, and amenability modulo an ideal of Banach algebras in terms of short exact sequences. Thence, we show that for a Banach algebra A and a closed ideal I of A such that $I^2 = I$, if A^{**} is amenable modulo I^{**} , then A is amenable modulo I .

In section three, we show that amenability modulo an ideal of the second dual of a semigroup algebra implies amenability modulo an ideal of the semigroup algebra. We then prove that $l^1(S)^{**}$ is amenable modulo I_σ^{**} if and only if S/σ is finite, where I_σ^{**} is the induced closed ideal by the least group congruence σ .

2. Structure of Amenability Modulo an Ideal

In this section, we first express the concept of amenability modulo an ideal of Banach algebras and some of their characterizations in terms of virtuals and diagonals. To see details, the reader may refer to [1–3].

Let A be a Banach algebra and X be a Banach A -bimodule. A bounded linear mapping $D : A \rightarrow X$ is called a derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all $a, b \in A$. A derivation D is called inner if there exists $x \in X$ such that $D = ad_x$, where $ad_x : A \rightarrow X$ is defined by $ad_x(a) = a \cdot x - x \cdot a$. For any Banach A -bimodule X , its dual X^* is naturally equipped with a Banach A -bimodule structure via $\langle x, a.f \rangle = \langle xa, f \rangle, \langle x, f.a \rangle = \langle ax, f \rangle$, for all $a \in A, x \in X, f \in X^*$. The set of all bounded derivations from A into X is denoted by $Z^1(A; X)$, the set of all inner derivations from A to X is denoted by $N^1(A; X)$, and the quotient space $H^1(A; X) = \frac{Z^1(A; X)}{N^1(A; X)}$ is called the first cohomology group of A with coefficients in X . A Banach algebra A is called amenable if $H^1(A; X^*) = \{0\}$ for every Banach A -bimodule X [7].

Definition 1. ([1], Definition 1) Let I be a closed ideal of A . A Banach algebra A is amenable modulo I if for every Banach A -bimodule X such that $I \cdot X = X \cdot I = 0$, and every derivation D from A into X^* , there is $\phi \in X^*$ such that $D = ad_\phi$ on the set theoretic difference $A \setminus I := \{a \in A : a \notin I\}$.

Theorem 1. ([1], Theorem 1) Let I be a closed ideal of A .

- (i) If A/I is amenable and $I^2 = I$, then A is amenable modulo I .
- (ii) If A is amenable modulo I , then A/I is amenable.
- (iii) If A is amenable modulo I and I is amenable, then A is amenable.

First, we investigate the structure of amenability modulo an ideal of Banach algebras in terms of virtual diagonals and approximate diagonals. Some results are given in [2], but here we revise and improve them. Let A be a Banach algebra and I be a closed ideal of A . Then $\frac{A}{I}$ with module actions $a.\bar{b} := \overline{ab}$ and $\bar{b}.a = \overline{ba}$ is a Banach A -bimodule where \bar{a} is the image of a in $\frac{A}{I}$. In addition, there is a canonical A -bimodule structure on $\frac{A}{I} \hat{\otimes} A$ defined by the linear extension of $a.(\bar{b} \otimes c) := \overline{ab} \otimes c$, and $(\bar{b} \otimes c).a := \overline{ba} \otimes c$, ($a, b, c \in A$).

By the diagonal operator we mean the bounded linear operator defined by the linear extension of $\pi : (\frac{A}{I} \hat{\otimes} A) \rightarrow \frac{A}{I}$ by $\pi(\bar{b} \otimes c) = \overline{bc}$. Clearly, π is an A -bimodule homomorphism.

Definition 2. (i) By a virtual diagonal modulo I , we mean an element $M \in (\frac{A}{I} \hat{\otimes} A)^{**}$ such that;

$$a \cdot \pi^{**}(M) - \bar{a} = 0 \ (a \in A), \text{ and } a \cdot M - M \cdot a = 0, \ (a \in A \setminus I),$$

(ii) by an approximate diagonal modulo I , we mean a bounded net $(m_\alpha)_\alpha \subset (\frac{A}{I} \hat{\otimes} A)$ such that;

$$a.\pi(m_\alpha) - \bar{a} \rightarrow 0 \ (a \in A), \text{ and } a.m_\alpha - m_\alpha.a \rightarrow 0, \ (a \in A \setminus I),$$

(iii) by a diagonal modulo I , we mean an element $m \in (\frac{A}{I} \hat{\otimes} A)$ such that;

$$a.\pi(m) - \bar{a} = 0 \ (a \in A), \text{ and } a.m - m.a = 0, \ (a \in A \setminus I).$$

We recall that a bounded net $\{u_\alpha\}_\alpha \subseteq A$ is called approximate identity modulo I if $\lim_\alpha u_\alpha \cdot a = \lim_\alpha a \cdot u_\alpha = a$, ($a \in A \setminus I$). If A is amenable modulo I then A has an approximate identity modulo I ([2], Theorem 4). Using the classical method [2,7], with appropriate modifications, we have the following result:

Theorem 2. The following conditions are equivalent;

- (i) A is amenable modulo I ,
- (ii) There is an approximate diagonal modulo I ,
- (iii) There is a virtual diagonal modulo I .

Let A be a Banach algebra and X, Y, Z be A -bimodules. A sequence $\Sigma : 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is called exact if f is one-to-one, $Im(g) = Z$, and $Im(f) = Ker(g)$. Additionally, the exact sequence Σ

is called admissible if there is a bounded linear map $F : Y \rightarrow X$ such that $Ff = I$ on X and the exact sequence Σ splits if there is a Banach A -module $F : Y \rightarrow X$ such that $Ff = I$ on X .

Let A be a Banach algebra and I be a closed ideal of A . Suppose that $\pi : \frac{A}{I} \hat{\otimes} A \rightarrow \frac{A}{I}$ is the diagonal operator, $i : K = \ker \pi \rightarrow \frac{A}{I} \hat{\otimes} A$ is the inclusion map, then $\Pi : 0 \rightarrow K \xrightarrow{i} \frac{A}{I} \hat{\otimes} A \xrightarrow{\pi} \frac{A}{I} \rightarrow 0$ is a sequence of A -bimodules. In addition, $(\frac{A}{I})^*$ and $(\frac{A}{I} \hat{\otimes} A)^*$ are clearly Banach A -bimodules where the module actions are defined by;

$$\langle \bar{a}, b.f \rangle := \langle \overline{ab}, f \rangle, \text{ and } \langle \bar{a}, f.b \rangle := \langle \overline{ba}, f \rangle, \quad (a, b \in A, f \in (\frac{A}{I})^*),$$

$$\langle \bar{b} \otimes c, a.\phi \rangle := \langle \bar{b} \otimes ca, \phi \rangle, \text{ and } \langle \bar{b} \otimes c, \phi.a \rangle := \langle \overline{ab} \otimes c, \phi \rangle, \quad (a, b, c \in A, \phi \in (\frac{A}{I} \hat{\otimes} A)^*),$$

so the dual sequence $\Pi^* : 0 \rightarrow (\frac{A}{I})^* \xrightarrow{\pi^*} (\frac{A}{I} \hat{\otimes} A)^* \xrightarrow{i^*} K^* \rightarrow 0$ is a sequence of A -bimodules.

Lemma 1. *If A has an identity, the sequence Π is admissible. If A has a bounded left or right approximate identity, Π^* is admissible.*

Proof. Let A have an identity, then $\phi : \frac{A}{I} \rightarrow \frac{A}{I} \hat{\otimes} A$ by $\phi(\bar{a}) = \bar{a} \otimes 1$ is the required right inverse for π , and ϕ^* is the required left inverse for π^* . Let now A have a bounded left approximate identity (e_j) and $u \in (\frac{A}{I} \otimes A)^{**}$ be a w^* -limit point of $(\bar{e}_j \otimes e_j)$, so by passing to a sub-net we may assume $\langle h, u \rangle = \lim_j \langle \bar{e}_j \otimes e_j, h \rangle$ ($h \in (\frac{A}{I} \otimes A)^*$). Set $\psi : (\frac{A}{I} \otimes A)^* \rightarrow (\frac{A}{I})^*$ by $\langle \bar{a}, \psi(h) \rangle = \langle \bar{a}.h, u \rangle$ ($\bar{a} \in \frac{A}{I}, h \in (\frac{A}{I} \hat{\otimes} A)^*$). It is not far to see that ψ is well-defined. Now for every $f \in (\frac{A}{I} \hat{\otimes} A)^*, \bar{a} \in \frac{A}{I}$ we have;

$$\begin{aligned} \langle \bar{a}, \psi \pi^* f \rangle &= \langle a.\pi^*(f), u \rangle = \lim_\alpha \langle \bar{e}_\alpha \otimes e_\alpha, a.\pi^*(f) \rangle \\ &= \lim_\alpha \langle \pi(\bar{e}_\alpha \otimes e_\alpha.a), f \rangle \\ &= \lim_\alpha \langle \bar{e}_\alpha^2.\bar{a}, f \rangle \\ &= \langle \bar{a}, f \rangle, \end{aligned}$$

hence $\psi \pi^* = id_{(\frac{A}{I})^*}$ as required. \square

Theorem 3. *Let A be a Banach algebra and I be a closed ideal of A . Then the following assertions hold.*

- (i) *If A is amenable modulo I , then the exact sequence Π^* splits.*
- (ii) *If A has a bounded approximate identity and the exact sequence Π^* is split, then A is amenable modulo I .*

Proof. (i) Suppose that A is amenable modulo I and M is a virtual diagonal modulo I for A . Set $\theta^* : (\frac{A}{I} \hat{\otimes} A)^* \rightarrow (\frac{A}{I})^*$ by $\langle \bar{a}, \theta^*(f) \rangle = \langle f.\bar{a}, M \rangle$ ($f \in (\frac{A}{I} \hat{\otimes} A)^*, \bar{a} \in \frac{A}{I}$). Let $b \in A \setminus I$, then;

$$\begin{aligned} \langle \bar{a}, \theta^*(b.f) \rangle &= \langle (b.f).\bar{a}, M \rangle = \langle b.(f.\bar{a}), M \rangle \\ &= \langle f.\bar{a}, M.b \rangle = \langle f.\bar{a}, b.M \rangle \\ &= \langle f.(b.\bar{a}), M \rangle = \langle (\bar{a}.b), \theta^*(f) \rangle \\ &= \langle \bar{a}, b.\theta^*(f) \rangle, \end{aligned}$$

so θ^* is an A -bimodule morphism. To see $\theta^* \pi^* = I$, let $f \in (\frac{A}{I})^*$, then

$$\begin{aligned} \langle \bar{a}, \theta^* \pi^* f \rangle &= \langle (\pi^* f).\bar{a}, M \rangle = \langle \pi^*(f.\bar{a}), M \rangle \\ &= \langle f.\bar{a}, \pi^{**} M \rangle = \langle f, \bar{a}.\pi^{**} M \rangle \\ &= \langle f, a.\pi^{**} M \rangle = \langle f, \bar{a} \rangle \end{aligned}$$

(ii) Let θ^* be an A -bimodule morphism such that $\theta^* \pi^* = I$ and (e_α) be the approximate identity of A . By passing to a sub-net, we can suppose that $E \in (\frac{A}{I} \hat{\otimes} A)^{**}$ is w^* -accumulation point of $(\bar{e}_\alpha \hat{\otimes} e_\alpha)$. We show that $M = \theta^{**} \pi^{**} E$ is a virtual diagonal modulo I for A . Let $a \in A \setminus I, f \in (\frac{A}{I} \otimes A)^*$, then;

$$\begin{aligned} \langle f, a.M \rangle &= \langle f, a.\theta^{**} \pi^{**} E \rangle = \langle f.a, \theta^{**} \pi^{**} E \rangle = \langle \pi^* \theta^* f.a, E \rangle \\ &= \lim \langle \theta^* f, a.e_\alpha^2 \rangle = \lim \langle \theta^* f, e_\alpha^2.a \rangle \\ &= \dots = \langle f, M.a \rangle \end{aligned}$$

Additionally,

$$\begin{aligned} \langle f, a.\pi^{**} M \rangle &= \langle f, a.\pi^{**} \theta^{**} \pi^{**} E \rangle = \langle \theta^* \pi^* (f.a), \pi^{**} (E) \rangle \\ &= \langle (f.a), \pi^{**} (E) \rangle = \lim \langle f, a.\bar{e}_\alpha^2 \rangle \\ &= \lim \langle f, \bar{a}.\bar{e}_\alpha^2 \rangle = \langle f, \bar{a} \rangle \end{aligned}$$

Hence M is a virtual diagonal modulo I for A , so A is amenable modulo I [2]. \square

Suppose that X, Z are A -bimodules, then the space of all bounded linear operators $T : Z \rightarrow Y$ which is denoted by $B(X; Y)$ is an A -bimodule where the module actions are defined by $\langle z, a.T \rangle = a.\langle z, T \rangle$, and $\langle z, T.a \rangle = \langle a.z, T \rangle$ ($T \in B(Z; X), z \in Z, a \in A$). In addition, $Z \hat{\otimes} X$ is an A -bimodule with the module actions $a.(z \otimes x) = (a.z) \otimes x, (z \otimes x).a = z \otimes (x.a)$. Now, the map $T : (Z \hat{\otimes} X)^* \rightarrow B(Z; X^*)$ given by $T : \phi \mapsto T_\phi$, where $\langle x, T_\phi(z) \rangle = \langle z \otimes x, \phi \rangle$ is an isometric A -module morphism and $B(Z; X^*) \simeq (Z \hat{\otimes} X)^*$ [4].

Definition 3. Let A be a Banach algebra, I be a closed ideal of A , and X, Y, Z be Banach A -bimodules. An exact (admissible) sequence $\Sigma : 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is an exact (admissible) sequence modulo I if $X \cdot I = I \cdot X = Y \cdot I = I \cdot Y = Z \cdot I = I \cdot Z = 0$.

Theorem 4. Let A be a Banach algebra and $\Sigma : 0 \rightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be an admissible sequence modulo I . If A is amenable modulo I , then Σ splits.

Proof. Since Σ is an admissible modulo I , there exists $G : Z \rightarrow Y$ such that $gG = I$. Put $D : A \rightarrow B(Z; Y)$ by $D(a) = a.G - G.a$. Then D is a non-zero bounded derivation on A and $g(D(a)(z)) = 0$ so $D(A) \subseteq B(Z; \ker g) = B(Z; \text{Im } f)$. Thus, $f^{-1}D : A \rightarrow B(Z; X^*) \simeq (Z \hat{\otimes} X)^*$ is a bounded derivation. Clearly $(Z \hat{\otimes} X)^* \cdot I = I \cdot (Z \hat{\otimes} X)^* = 0$. Since A is amenable modulo I , there exists $\psi \in B(Z; X^*)$ such that $f^{-1}D(a) = a.\psi - \psi.a$ ($a \in A \setminus I$). Let $\hat{G} = G - f\psi$, then \hat{G} is an A -module morphism and $g\hat{G} = I$ on Z . \square

At the end of this section, we would like to discuss the connection between amenability modulo an ideal of Banach algebras and their closed ideals. Following Curtis and Loy ([4], Theorem 3.7), if A is an amenable Banach algebra, $J \neq 0$ is a closed left, right, or two-sided ideal in A , then J has a bounded right, left, or two-sided approximate identity if and only if J^* is complemented in A^* . In the following, we state the same result for amenable modulo an ideal Banach algebras.

Lemma 2. Let A be a Banach algebra and I, J be closed ideals of A such that $J \supseteq I$. Let A be amenable modulo I , then $\frac{I}{J}$ has a bounded approximate identity if and only if $(\frac{I}{J})^\perp$ is complemented in $(\frac{A}{I})^*$.

Proof. Since A is amenable modulo I , $\frac{A}{I}$ is amenable (by Theorem 1). It is clear that $\frac{I}{J}$ is a closed ideal of $\frac{A}{I}$, so $\frac{I}{J}$ has a bounded approximate identity if and only if $(\frac{I}{J})^\perp$ is complemented in $(\frac{A}{I})^*$ (by [4], Theorem 3.7). \square

Theorem 5. Let A be a Banach algebra, I be a closed ideal of A with $I^2 = I$, and $J \supseteq I$ be a closed ideal of A . If A is amenable modulo I and $\frac{J}{I}$ has a bounded approximate identity, then J is amenable modulo I .

Proof. Since A is amenable modulo I , $\frac{A}{I}$ is amenable (by Theorem 1). Since $\frac{J}{I}$ has a bounded approximate identity, $\frac{J}{I}$ is amenable [8]. Thus, J is amenable modulo I (by Theorem 1 again). \square

3. Amenability Modulo an Ideal of $l^1(S)^{**}$

Following Arens [9], for a Banach algebra A there are two algebra multiplications on the second dual of A which extend multiplication on A . It is shown that A^{**} is a Banach algebra under two Arens products [10]. Amenability of second conjugate Banach algebras are considered in [6]; for a Banach algebra A , amenability of A^{**} necessitates amenability of A . In the following, we present a similar result for amenable modulo an ideal Banach algebras.

Theorem 6. Let A be a Banach algebra, I be a closed ideal of A such that $I^2 = I$. If A^{**} is amenable modulo I^{**} , then A is amenable modulo I .

Proof. Since A^{**} is amenable modulo I^{**} , $\frac{A^{**}}{I^{**}} \simeq (\frac{A}{I})^{**}$ is amenable (by Theorem 1). Thus $\frac{A}{I}$ is amenable (by [6], Theorem 1.8). Since $I^2 = I$, amenability of $\frac{A}{I}$ implies that A is amenable modulo I (by Theorem 1). \square

We apply the obtained results to characterize amenability of second dual of semigroup algebras. First of all, we have to recall some basic properties of semigroup theory to which we shall refer; for full details, see [11,12]. A semigroup S is called an *E-semigroup* if the set of all idempotents $E(S)$ forms a sub-semigroup of S ; S is called *E-inversive* if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$. A semigroup S is called *regular* if $V(a) = \{x \in S : a = axa, x = xax\} \neq \emptyset$ for every $a \in S$; S is called an *inverse semigroup* if the inverse of each element is unique; S is called a *semilattice* if is a commutative and idempotent semigroup; and S is called *eventually inverse* if every element of S has some power that is regular and $E(S)$ is a semilattice.

A congruence ρ on semigroup S is called a group congruence if S/ρ is group. The existence of the least group congruence on semigroups have also been investigated by various authors, see [13–15]. If the least group congruence on semigroup S exists, we denote it by σ . In general, we know that there is no the least group congruence on semigroups. For example, consider the semigroup of positive integers $(\mathbb{N}, +)$ (with respect to addition). It is not hard to see that every group congruence on \mathbb{N} is of the form $\{(p, q) \in \mathbb{N} \times \mathbb{N} : kn = p - q, \text{ for some } k \in \mathbb{Z}\}$, but \mathbb{N} does not have the least group congruence. Let ρ be a group congruence on semigroup S ; by the induced ideal I_ρ , we mean an ideal in the semigroup algebra $l^1(S)$ generated by the set $\{\delta_s - \delta_t : s, t \in S \text{ with } (s, t) \in \rho\}$. We recall the following Lemma of [1].

Lemma 3. (i) ([1], Lemma 1) Let S be a semigroup and ρ be a group congruence on S , then $l^1(S/\rho) \simeq l^1(S)/I_\rho$, where I_ρ is a closed ideal of $l^1(S)$.

(ii) ([1], Lemma 2) Let S be an *E-inversive* semigroup with commuting idempotents and σ be the least group congruence on S , then $l^1(S/\sigma) \simeq l^1(S)/I_\sigma$ where I_σ is a closed ideal of $l^1(S)$ and $I_\sigma^2 = I_\sigma$.

Theorem 7. Let S be a semigroup, and ρ be a group congruence on S such that I_ρ has an approximate identity. If $l^1(S)^{**}$ is amenable modulo I_ρ^{**} , then $l^1(S)$ is amenable modulo I_ρ .

Proof. It is obvious that for a group congruence ρ on semigroup S , I_ρ is a closed ideal of $l^1(S)$, so I_ρ^{**} is a closed ideal of $l^1(S)^{**}$. Since I_ρ has an approximate identity, $I_\rho^2 = I_\rho$. Using Theorem 6, $l^1(S)$ is amenable modulo I_ρ . \square

If S is an *E-inversive E-semigroup* such that $E(S)$ is commutative, then the relation $\sigma^* = \{(a, b) \in S \times S \mid ea = fb \text{ for some } e, f \in E_S\}$ is the least group congruence on S [13,15].

Theorem 8. Let S be an E -inversive E -semigroup with commuting idempotents and σ^* be the least group congruence on S . If $l^1(S)^{**}$ is amenable modulo $I_{\sigma^*}^{**}$, then $l^1(S)$ is amenable modulo I_{σ^*} .

Proof. Since σ^* is the least group congruence, $l^1(S/\sigma^*) \simeq l^1(S)/I_{\sigma^*}$ and $I_{\sigma^*}^2 = I_{\sigma^*}$ (by Lemma 3). Now if $l^1(S)^{**}$ is amenable modulo $I_{\sigma^*}^{**}$, then $\frac{l^1(S)^{**}}{I_{\sigma^*}^{**}} \simeq (\frac{l^1(S)}{I_{\sigma^*}})^{**}$ is amenable (by Theorem 1). Thus $\frac{l^1(S)}{I_{\sigma^*}}$ is amenable. Since $I_{\sigma^*}^2 = I_{\sigma^*}$, $l^1(S)$ is amenable modulo I_{σ^*} . \square

It is shown that if S is an eventually semigroup, then the relation $\sigma' = \{(s, t) : es = et, \text{ for some } e \in E(S)\}$ is the least group congruence on S [13,14].

Corollary 1. If S be an eventually inverse semigroup such that $l^1(S)^{**}$ is amenable modulo $I_{\sigma'}^{**}$, then $l^1(S)$ is amenable modulo $I_{\sigma'}$.

Following Ghahramani, Loy, and Willis, ([6], Theorem 1.3), if G is a locally compact group or a discrete weakly cancellative semigroup, then $l^1(G)^{**}$ is amenable if and only if G is finite. In the next theorem, we show that the same remains true for a large class of semigroups.

Theorem 9. If S satisfies one of the following statements:

- (i) S is an E -inversive semigroup with commuting idempotents;
- (ii) S be an eventually inverse semigroup;

and σ is the least group congruence on S , then $(l^1(S))^{**}$ is amenable modulo I_{σ}^{**} if and only if S/σ is finite.

Proof. $l^1(S)^{**}$ is amenable modulo I_{σ}^{**} if and only if $\frac{(l^1(S))^{**}}{(I_{\sigma})^{**}} \simeq (\frac{l^1(S)}{I_{\sigma}})^{**} \simeq l^1(S/\sigma)^{**}$ is amenable, if and only if S/σ is finite (by [6], Theorem 1.3 and we note that S/σ is a group). \square

Example 1. (1) Let $S = \{p^m q^n : m, n \geq 0\}$. It is known that S with multiplication operation $(p^{n_1} q^{m_1})(p^{m_2} q^{n_2}) = p^{m_1 - n_1 + \max\{n_1, m_2\}} q^{m_2 - n_2 + \max\{n_1, m_2\}}$, and $(p^n q^m)^* = (p^m q^n)$ is an E -unitary semigroup with $E(S) = \{p^n q^n : n = 0, 1, 2, \dots\}$ [11,12]. Set $x\sigma y$ if and only if $ex = ey$ for some $e \in E(S)$. Then σ is the least group congruence on S and $S/\sigma = \mathbb{Z}$. We know that $l^1(S)$ is amenable modulo I_{σ} whereas $l^1(S)$ is not amenable [1]. Additionally, since $S/\sigma = \mathbb{Z}$ is infinite, $(l^1(S))^{**}$ is not amenable modulo I_{σ}^{**} .

(2) Let $S = \mathbb{N}$ be the semigroup of positive integers with operation $n.m = \max\{n, m\}$ ($n, m \in \mathbb{N}$). Set $n\sigma m$ if and only if $pn = pm$ for some $p \in E(S)$. Clearly S/σ is the trivial maximum group image of \mathbb{N} . Since S/σ is finite, $l^1(S)^{**}$ is amenable modulo I_{σ}^{**} . We note that \mathbb{N} is an infinite weakly cancellative semigroup, so $l^1(S)^{**}$ is not amenable ([6], Theorem 1.3).

Acknowledgments: The authors sincerely thank the referee(s) for their valuable comments and suggestions, which were very useful to improve the paper significantly.

Author Contributions: The main ideas were developed by the first author. All authors formulated and proved the main technical claims. All authors wrote the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Amini, M.; Rahimi, H. Amenability of semigroups and their algebras modulo a group congruence. *Acta Math. Hung.* **2014**, *144*, 407–415.
2. Rahimi, H.; Tahmasebi, E. Amenability and Contractibility modulo an ideal of Banach algebras. *Abstr. Appl. Anal.* **2014**, *2014*, 514761.
3. Rahimi, H.; Tahmasebi, E. Hereditary properties of amenability modulo an ideal of Banach algebras. *J. Linear Topol. Algebra* **2014**, *3*, 107–114.
4. Curtis, P.C.; Loy, R.J. The structure of amenable Banach algebras. *J. Lond. Math. Soc.* **1989**, *40*, 89–104.
5. Gourdeau, F. Amenability and the second dual of a Banach algebras. *Stud. Math.* **1997**, *125*, 75–80.

6. Ghahramani, F.; Loy, R.J.; Willis, G.A. Amenability and weak amenability of second conjugate Banach algebras. *Proc. Am. Math. Soc.* **1996**, *124*, 1489–1497.
7. Runde, V. *Lectures on Amenability, Volume 1774 of Lecture Notes in Mathematics*; Springer: Berlin, Germany, 2002.
8. Lau, A.T.M.; Loy, R.J. Amenability of convolution algebras. *Math. Scand.* **1996**, *79*, 283–296.
9. Arens, R.E. The adjoint of a bilinear operation. *Am. Math. Soc.* **1951**, *2*, 839–848.
10. Duncan, J.; Hosseiniun, S.A.R. The second dual of a Banach algebra. *Proc. R. Soc. Edinb. Sect. A* **1979**, *19*, 309–325.
11. Clifford, A.H.; Preston, J.B. *The Algebraic Theory of Semigroups I*; American Mathematical Society, Surveys: Providence, RI, USA, 1961; Volume 7.
12. Howie, J.M. *Fundamentals of Semigroup Theory*; Clarendon Press: Oxford, UK, 1995.
13. Gigon, R.S. Congruences and group congruences on a semigroup. *Semigroup Forum* **2013**, *86*, 431–450.
14. Gomes, G.M.S. Group congruences on eventually inverse semigroups. *Port. Math.* **1992**, *49*, 417–428.
15. Siripitukdet, M.; Sattayaporn, S. The least group congruence on E-inversive semigroups and E-inversive E-semigroups. *Thai J. Math.* **2005**, *3*, 163–169.



© 2016 by the authors; licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).