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Exponential Energy Decay of Solutions for a Transmission Problem With Viscoelastic Term and Delay

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Abstract: In our previous work (Journal of Nonlinear Science and Applications 9: 1202–1215, 2016), we studied the well-posedness and general decay rate for a transmission problem in a bounded domain with a viscoelastic term and a delay term. In this paper, we continue to study the similar problem but without the frictional damping term. The main difficulty arises since we have no frictional damping term to control the delay term in the estimate of the energy decay. By introducing suitable energy and Lyapunov functionals, we establish an exponential decay result for the energy.

Keywords: wave equation; transmission problem; exponential decay; viscoelastic; delay

MSC: AMS Subject Classification (2000): 35B37, 35L55, 93D15, 93D20

1. Introduction

In our previous work [1], we considered the following transmission system with a viscoelastic term and a delay term:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)\mathrm{d}s \\ \qquad\qquad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty) \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty) \end{cases} \tag{1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega = (0,L_1) \cup (L_2,L_3)$, a, b, μ_1, μ_2 are positive constants, and $\tau > 0$ is the delay. In that work, we first proved the well-posedness by using the Faedo–Galerkin approximations together with some energy estimates when $\mu_2 \leq \mu_1$. Then, a general decay rate result was established under the hypothesis that $\mu_2 < \mu_1$. As for the previous results and developments of transmission problems, and the research of wave equations with viscoelastic damping or time delay effects, we have stated and summarized in great detail in our previous work [1], thus we just omit it here. The readers, for a better understanding of present work, are strongly recommended to [1] and the reference therein (see [2–33]).

It is worth pointing out that, in our previous work, the assumption "$\mu_2 < \mu_1$" plays an important role in the proof of the above-mentioned general decay result. In this paper, we intend to investigate system (1) with $\mu_1 = 0$. That is, we study the exponential decay rate of the solutions for the following transmission system with a viscoelastic term and a delay term but without the frictional damping:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)\mathrm{d}s & \\ \qquad\qquad +\mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty) \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1,L_2) \times (0,+\infty) \end{cases} \tag{2}$$

under the boundary and transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, & \\ u(L_i,t) = v(L_i,t), & i = 1,2 \\ \left(a - \displaystyle\int_0^t g(s)\mathrm{d}s\right) u_x(L_i,t) = bv_x(L_i,t), & i = 1,2 \end{cases} \tag{3}$$

and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, \quad t \in [0,\tau] \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in (L_1,L_2) \end{cases} \tag{4}$$

where μ_2 is a real number, a, b are positive constants and $u_1(x) = f_0(x,0)$.

The main difficulty in dealing with this problem is that in the first equation of system (2), we have no frictional damping term to control the delay term in the estimate of the energy decay. To overcome this difficulty, our basic idea is to control the delay term by making use of the viscoelastic term. In order to achieve this goal, a restriction of the size between the parameter μ_2 and the relaxation function g and a suitable energy is needed. This is motivated by Dai and Yang's work [34], in which the viscoelastic wave equation with delay term but without a frictional damping term was studied and an exponential decay result was established. In the work here, we will establish an exponential decay rate result for the energy.

The remaining part of this paper is organized as follows. In Section 2, we give some notations and hypotheses needed for our work and state the main results. In Section 3, under some restrictions of μ_2 (see (35) below), we prove the exponential decay of the solutions for the relaxation function satisfying assumption (H_1) and (H_2).

2. Preliminaries and Main Results

In this section, we present some materials that shall be used in order to prove our main result. Let us first introduce the following notations:

$$(g * h)(t) := \int_0^t g(t-s)h(s)\mathrm{d}s$$
$$(g \diamond h)(t) := \int_0^t g(t-s)(h(t)-h(s))\mathrm{d}s$$
$$(g \Box h)(t) := \int_0^t g(t-s)|h(t)-h(s)|^2\mathrm{d}s$$

We easily see that the above operators satisfy

$$(g * h)(t) = \left(\int_0^t g(s)\mathrm{d}s\right) h(t) - (g \diamond h)(t)$$
$$|(g \diamond h)(t)|^2 \leq \left(\int_0^t |g(s)|\mathrm{d}s\right) (|g|\Box h)(t)$$

Lemma 1. *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g'\Box h - g(t)|h|^2 - \frac{d}{dt}\left\{g\Box h - \left(\int_0^t g(s)\mathrm{d}s\right)|h|^2\right\}$$

For the relaxation function g, we assume
(H1) g: $\mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad \beta_0 := a - \int_0^\infty g(s)\mathrm{d}s = a - \bar{g} > 0$$

(H2) There exists a positive constant ξ satisfying $\xi > \xi_0 > 0$ (ξ_0 defined by (39) below) and

$$g'(t) \leq -\xi g(t), \quad \forall t \geq 0$$

According to previous results in the literature (see [1]), we state the following well-posedness result, which can be proved by using the Faedo–Galerkin method.

Theorem 2. *Assume that (H1) and (H2) hold. Then, given $(u_0, v_0) \in \mathcal{V}$, $(u_1, v_1) \in \mathcal{L}^2$, and $f_0 \in L^2((0,1), H^1(\Omega))$, Equations (2)–(4) have a unique weak solution in the following class:*

$$(u, v) \in C(0, \infty; \mathcal{V}) \cap C^1(0, \infty; \mathcal{L}^2)$$

where

$$\mathcal{V} = \Big\{(u, v) \in H^1(\Omega) \cap H^1(L_1, L_2) : u(0, t) = u(L_3, 0) = 0, u(L_i, t) = v(L_i, t)$$
$$\left(a - \int_0^t g(s)\mathrm{d}s\right) u_x(L_i, t) = bv_x(L_i, t), i = 1, 2\Big\}$$

and

$$\mathcal{L}^2 = L^2(\Omega) \times L^2(L_1, L_2)$$

To state our decay result, we introduce the following energy functional:

$$\begin{aligned} E(t) =& \frac{1}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x + \frac{1}{2}\left(a - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{1}{2}\int_\Omega (g\Box u_x)\mathrm{d}x \\ &+ \frac{1}{2}\int_{L_1}^{L_2}\left[v_t^2(x,t) + bv_x^2(x,t)\right]\mathrm{d}x + \frac{\zeta}{2}\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} u_s^2((x,s)\mathrm{d}x\mathrm{d}s \end{aligned} \tag{5}$$

where σ and ζ are positive constants to be determined later.

Remark 1. We note that the energy functional defined here is different from that of [1] in the construction of the last term. This is motivated by the idea of [28], in which wave equations with time dependent delay was studied.

Our decay results read as follows:

Theorem 3. *Let (u, v) be the solution of Equations (2)–(4). Assume that $(H1)$, $(H2)$ and*

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0 \tag{6}$$

hold. Let a_0 be the constants defined by (35) below. If $|\mu_2| < a_0$, then there exists constants $\gamma_1, \gamma_2 > 0$ such that

$$E(t) \leq \gamma_2 e^{-\gamma_1 \xi(t-t_0)}, \quad t \geq t_0 \tag{7}$$

3. Proof of Theorem 3

For the proof of Theorem 3, we use the following lemmas.

Lemma 4. *Let (u, v) be the solution of Equations (2)–(4). Then, we have the inequality*

$$\begin{aligned} \frac{d}{dt}E(t) \leq& \frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x + \left(\frac{|\mu_2|}{2} + \frac{\zeta}{2}\right)\int_\Omega u_t^2(x,t)\mathrm{d}x - \left(\frac{\zeta}{2}e^{-\sigma\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x \\ &- \frac{1}{2}g(t)\int_\Omega u_x^2(x,t)\mathrm{d}x - \frac{\sigma\zeta}{2}\int_{t-\tau}^t \int_\Omega e^{-\sigma(t-s)}u_s^2((x,s)\mathrm{d}x\mathrm{d}s \end{aligned} \tag{8}$$

Proof. Differentiating (5) and using (2), we have

$$\begin{aligned} \frac{d}{dt}E(t) =& \int_\Omega \left[u_t u_{tt} + \left(a - \int_0^t g(s)\mathrm{d}s\right)u_x u_{xt} - \frac{1}{2}g(t)u_x^2\right]\mathrm{d}x + \int_{L_1}^{L_2}[v_t v_{tt} + b v_x v_{xt}]\,\mathrm{d}x \\ &+ \int_0^t g(t-s)\int_\Omega u_{xt}(u_x(t) - u_x(s))\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_\Omega g'\Box u_x \mathrm{d}x + \frac{\zeta}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x \\ &- \frac{\zeta}{2}\int_\Omega e^{-\sigma\tau}u_t^2(x,t-\tau)\mathrm{d}x - \frac{\sigma\zeta}{2}\int_{t-\tau}^t\int_\Omega e^{-\sigma(t-s)}u_s^2(x,s)\mathrm{d}x\mathrm{d}s \\ =& \frac{1}{2}\int_\Omega g'\Box u_x\mathrm{d}x - \frac{1}{2}g(t)\int_\Omega u_x^2\mathrm{d}x - |\mu_2|\int_\Omega u_t(t)u_t(t-\tau)\mathrm{d}x + \frac{\zeta}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x \\ &- \frac{\zeta}{2}\int_\Omega e^{-\sigma\tau}u_t^2(x,t-\tau)\mathrm{d}x - \frac{\sigma\zeta}{2}\int_{t-\tau}^t\int_\Omega e^{-\sigma(t-s)}u_s^2(x,s)\mathrm{d}x\mathrm{d}s \end{aligned} \tag{9}$$

By Cauchy inequalities, we get

$$\begin{aligned} \frac{d}{dt}E(t) \leq& \frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x + \left(\frac{|\mu_2|}{2} + \frac{\zeta}{2}\right)\int_\Omega u_t^2(x,t)\mathrm{d}x + \left(\frac{|\mu_2|}{2} - \frac{\zeta}{2}e^{-\sigma\tau}\right)\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x \\ &- \frac{1}{2}g(t)\int_\Omega u_x^2(x,t)\mathrm{d}x - \frac{\sigma\zeta}{2}\int_{t-\tau}^t\int_\Omega e^{-\sigma(t-s)}u_s^2((x,s)\mathrm{d}x\mathrm{d}s \end{aligned}$$

The proof is complete. □

Remark 2. In ([1] Lemma 4.1), we proved that the energy functional defined in [1] is non-increasing. However, since $\left(\frac{|\mu_2|}{2} + \frac{\zeta}{2}\right)\int_\Omega u_t^2\mathrm{d}x \geq 0$, $E(t)$ may not be non-increasing here.

Now, we define the functional $\mathscr{D}(t)$ as follows:

$$\mathscr{D}(t) = \int_\Omega u u_t \mathrm{d}x + \int_{L_1}^{L_2} v v_t \mathrm{d}x$$

Then, we have the following estimate:

Lemma 5. *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned} \frac{d}{dt}\mathscr{D}(t) \le \int_\Omega u_t^2 \mathrm{d}x + \int_{L_1}^{L_2} v_t^2 \mathrm{d}x + \left(\delta_1|\mu_2|L^2 + \delta_1 - \left(a - \int_0^t g(s)\mathrm{d}s\right)\right)\int_\Omega u_x^2 \mathrm{d}x \\ + \frac{1}{4\delta_1}\int_0^t g(s)\mathrm{d}s \int_\Omega (g\Box u_x)\mathrm{d}x + \frac{|\mu_2|}{4\delta_1}\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x - \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x \end{aligned} \quad (10)$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (2), we have

$$\begin{aligned} \frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2 \mathrm{d}x - \int_\Omega (au_x - g * u_x)u_x \mathrm{d}x - |\mu_2|\int_\Omega u_t(x,t-\tau)u\mathrm{d}x + \int_{L_1}^{L_2} v_t^2\mathrm{d}x - \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x \\ &= \int_\Omega u_t^2 \mathrm{d}x - \left(a - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega u_x^2\mathrm{d}x - \int_\Omega (g \diamond u_x)u_x\mathrm{d}x - |\mu_2|\int_\Omega u_t(x,t-\tau)u\mathrm{d}x \\ &+ \int_{L_1}^{L_2} v_t^2\mathrm{d}x - \int_{L_1}^{L_2} b v_x^2\mathrm{d}x \end{aligned} \quad (11)$$

By the boundary condition (3), we have

$$u^2(x,t) = \left(\int_0^x u_x(x,t)\mathrm{d}x\right)^2 \le L_1\int_0^{L_1} u_x^2(x,t)\mathrm{d}x, \quad x \in [0, L_1]$$

$$u^2(x,t) \le (L_3 - L_2)\int_{L_2}^{L_3} u_x^2(x,t)\mathrm{d}x, \quad x \in [L_2, L_3]$$

which implies

$$\int_\Omega u^2(x,t)\mathrm{d}x \le L^2 \int_\Omega u_x^2 \mathrm{d}x, \quad x \in \Omega \quad (12)$$

where $L = \max\{L_1, L_3 - L_2\}$. By exploiting Young's inequality and (12), we get for any $\delta_1 > 0$

$$-|\mu_2|\int_\Omega u_t(x,t-\tau)u\mathrm{d}x \le \frac{|\mu_2|}{4\delta_1}\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x + \delta_1|\mu_2|L^2\int_\Omega u_x\mathrm{d}x \quad (13)$$

Young's inequality implies that

$$\int_\Omega (g\diamond u_x)u_x\mathrm{d}x \le \delta_1\int_\Omega u_x^2\mathrm{d}x + \frac{1}{4\delta_1}\int_0^t g(s)\mathrm{d}s\int_\Omega (g\Box u_x)\mathrm{d}x \quad (14)$$

Inserting the estimates (13) and (14) into (11), then (10) is fulfilled. The proof is complete. □

Now, as in Lemma 4.5 of [24], we introduce the function

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x \in [0, L_1] \\ \dfrac{L_1}{2} - \dfrac{L_1 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1, L_2) \\ x - \dfrac{L_2 + L_3}{2}, & x \in [L_2, L_3] \end{cases} \quad (15)$$

It is easy to see that $q(x)$ is bounded, that is, $|q(x)| \le M$, where $M = \max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}$ is a positive constant. In addition, we define the functionals:

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t(au_x - g * u_x)\mathrm{d}x, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_x v_t\mathrm{d}x \quad (16)$$

Then, we have the following estimates.

Lemma 6. *The functionals $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy*

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_1(t) \leq & \left[-\frac{q(x)}{2}(au_x - g * u_x)^2\right]_{\partial\Omega} - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} + \left[\frac{a}{2} + \frac{M^2}{4\delta_2}\right]\int_\Omega u_t^2 \mathrm{d}x \\
& + \left[2a^2 + \delta_2 M^2 a^2|\mu_2| + g^2(0)\delta_2 + (4 + \delta_2|\mu_2|)\left(\int_0^t g(s)\mathrm{d}s\right)^2\right]\int_\Omega u_x^2 \mathrm{d}x \\
& + \left[\delta_2|\mu_2|M^2 + \frac{|\mu_2|}{4\delta_2} + \frac{|\mu_2|M^2}{4\delta_2}\right]\int_\Omega u_t^2(x, t-\tau)\mathrm{d}x \\
& + \left(4 + \frac{|\mu_2|}{4\delta_2}\right)\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega (g\Box u_x)\mathrm{d}x - g(0)\delta_2\int_\Omega (g'\Box u_x)\mathrm{d}x
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) \leq & -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2 \mathrm{d}x + \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x\right) + \frac{L_1}{4}v_t^2(L_1) + \frac{L_3 - L_2}{4}v_t^2(L_2) \\
& + \frac{b}{4}\Big((L_3 - L_2)v_x^2(L_2, t) + L_1 v_x^2(L_1, t)\Big)
\end{aligned}
\tag{18}
$$

Proof. Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (2), we get

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_1(t) =& -\int_\Omega q(x)u_{tt}(au_x - g * u_x)\mathrm{d}x - \int_\Omega q(x)u_t\left(au_{xt} - g(t)u_x(t) + (g' \diamond u_x)(t)\right)\mathrm{d}x \\
=& \left[-\frac{q(x)}{2}(au_x - g * u_x)^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega q'(x)(au_x - g * u_x)^2 \mathrm{d}x - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} \\
& + \frac{a}{2}\int_\Omega q'(x)u_t^2 \mathrm{d}x - \int_\Omega q(x)|\mu_2|u_t(x, t-\tau)(g * u_x)\mathrm{d}x \\
& + \int_\Omega q(x)au_x|\mu_2|u_t(x, t-\tau)\mathrm{d}x - \int_\Omega q(x)u_t[(g' \diamond u_x)(t) - g(t)u_x]\mathrm{d}x
\end{aligned}
\tag{19}
$$

We note that

$$
\begin{aligned}
& \frac{1}{2}\int_\Omega q'(x)(au_x - g * u_x)^2 \mathrm{d}x \\
\leq & 2\int_\Omega a^2 u_x^2 \mathrm{d}x + 2\int_\Omega (g * u_x)^2 \mathrm{d}x \\
\leq & 2\int_\Omega a^2 u_x^2 \mathrm{d}x + 2\int_\Omega \left(\int_0^t g(t-s)(u_x(s) - u_x(t) + u_x(t))\mathrm{d}s\right)^2 \mathrm{d}x \\
\leq & 2a^2\int_\Omega u_x^2 \mathrm{d}x + 4\left(\int_0^t g(s)\mathrm{d}s\right)^2\int_\Omega u_x^2 \mathrm{d}x + 4\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega (g\Box u_x)\mathrm{d}x
\end{aligned}
\tag{20}
$$

Young's inequality gives us for any $\delta_2 > 0$,

$$
\int_\Omega q(x)au_x|\mu_2|u_t(x, t-\tau)\mathrm{d}x \leq \delta_2 M^2 a^2|\mu_2|\int_\Omega u_x^2 \mathrm{d}x + \frac{|\mu_2|}{4\delta_2}\int_\Omega u_t^2(x, t-\tau)\mathrm{d}x \tag{21}
$$

$$
\begin{aligned}
& \int_\Omega q(x)|\mu_2|u_t(x, t-\tau)(g * u_x)\mathrm{d}x \\
=& |\mu_2|\int_\Omega (g \diamond u_x)q(x)u_t(x, t-\tau)\mathrm{d}x + |\mu_2|\int_0^t g(s)\mathrm{d}s\int_\Omega q(x)u_t(x, t-\tau)u_x \mathrm{d}x \\
\leq & \delta_2 M^2|\mu_2|\int_\Omega u_t^2(x, t-\tau)\mathrm{d}x + \frac{|\mu_2|}{4\delta_2}\int_0^t g(s)\mathrm{d}s\int_\Omega (g\Box u_x)\mathrm{d}x + \delta_2|\mu_2|\left(\int_0^t g(s)\mathrm{d}s\right)^2\int_\Omega u_x^2 \mathrm{d}x \\
& + \frac{|\mu_2|M^2}{4\delta_2}\int_\Omega u_t^2(x, t-\tau)\mathrm{d}x
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
&-\int_\Omega q(x)u_t[(g'\diamond u_x)(t)-g(t)u_x]\mathrm{d}x\\
\le&\frac{M^2}{4\delta_2}\int_\Omega u_t^2\mathrm{d}x+g^2(0)\delta_2\int_\Omega u_x^2\mathrm{d}x-g(0)\delta_2\int_\Omega(g'\Box u_x)\mathrm{d}x
\end{aligned}
\tag{23}
$$

Inserting (20)–(23) into (19), we get (17).

By the same method, taking the derivative of $\mathscr{F}_1(t)$ with respect to t, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t)=&-\int_{L_1}^{L_2}q(x)v_{xt}v_t\mathrm{d}x-\int_{L_1}^{L_2}q(x)v_xv_{tt}\mathrm{d}x\\
=&\left[-\frac{1}{2}q(x)v_t^2\right]_{L_1}^{L_2}+\frac{1}{2}\int_{L_1}^{L_2}q'(x)v_t^2\mathrm{d}x+\frac{1}{2}\int_{L_1}^{L_2}bq'(x)v_x^2\mathrm{d}x+\left[-\frac{b}{2}q(x)v_x^2\right]_{L_1}^{L_2}\\
\le&-\frac{L_1+L_3-L_2}{4(L_2-L_1)}\left(\int_{L_1}^{L_2}v_t^2\mathrm{d}x+\int_{L_1}^{L_2}bv_x^2\mathrm{d}x\right)+\frac{L_1}{4}v_t^2(L_1)+\frac{L_3-L_2}{4}v_t^2(L_2)\\
&+\frac{b}{4}\Big((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\Big)
\end{aligned}
$$

Thus, the proof of Lemma 6 is finished. □

In [1], the authors pointed out that if $\mu_2<\mu_1$, then the energy is non-increasing. Thus, the negative term $-\int_\Omega u_t^2\mathrm{d}x$ appeared in the derivative energy can be used to stabilize the system. However, in this paper, the energy is not non-increasing. In this case, we need some additional negative term $-\int_\Omega u_t^2\mathrm{d}x$. For this purpose, let us introduce the functional

$$
\mathscr{F}_3(t)=-\int_\Omega u_t(g\diamond u)\mathrm{d}x
$$

Then, we have the following estimate.

Lemma 7. *The functionals $\mathscr{F}_3(t)$ satisfies*

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_3(t)\le&-\left(\int_0^t g(s)\mathrm{d}s-\frac{\alpha_4}{2}\right)\int_\Omega u_t^2(x,t)\mathrm{d}x+\left[\delta_4+\delta_4\left(\int_0^t g(s)\mathrm{d}s\right)^2\right]\int_\Omega u_x^2(x,t)\mathrm{d}x\\
&+\delta_4|\mu_2|\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x+\left[\left(\delta_4+\frac{1}{2\delta_4}+\frac{a^2}{4\delta_4}+\frac{|\mu_2|L^2}{4\delta_4}\right)\int_0^t g(s)\mathrm{d}s\right]\int_\Omega(g\Box u_x)\mathrm{d}x\\
&-\frac{g(0)L^2}{2\alpha_4}\int_\Omega(g'\Box u_x)\mathrm{d}x
\end{aligned}
\tag{24}
$$

Proof. Taking the derivative of $\mathscr{F}_3(t)$ with respect to t and using (2), we get

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_3(t)=&-\int_\Omega u_{tt}(g\diamond u)\mathrm{d}x-\int_0^t g(s)\mathrm{d}s\int_\Omega u_t^2(x,t)\mathrm{d}x-\int_\Omega u_t(g'\diamond u)\mathrm{d}x\\
=&-\int_\Omega\left(au_{xx}(x,t)-\int_0^t g(t-s)u_{xx}(x,s)\mathrm{d}s-|\mu_2|u_t(x,t-\tau)\right)(g\diamond u)\mathrm{d}x\\
&-\int_0^t g(s)\mathrm{d}s\int_\Omega u_t^2(x,t)\mathrm{d}x-\int_\Omega u_t(g'\diamond u)\mathrm{d}x\\
=&-\int_\Omega\left(\int_0^t g(t-s)u_x(x,s)\mathrm{d}s\right)(g\diamond u_x)\mathrm{d}x+a\int_\Omega u_x(g\diamond u_x)\mathrm{d}x\\
&+|\mu_2|\int_\Omega u_t(x,t-\tau)(g\diamond u)\mathrm{d}x-\int_0^t g(s)\mathrm{d}s\int_\Omega u_t^2(x,t)\mathrm{d}x-\int_\Omega u_t(g'\diamond u)\mathrm{d}x
\end{aligned}
\tag{25}
$$

Young's inequality implies that for any $\delta_4 > 0$:

$$\begin{aligned}
&-\int_\Omega \left(\int_0^t g(t-s)u_x(x,s)\mathrm{d}s\right)(g\diamond u_x)\mathrm{d}x \\
\leq &\frac{\delta_4}{2}\int_\Omega \left(\int_0^t g(t-s)(u_x(t)-u_x(s)-u_x(t))\mathrm{d}s\right)^2\mathrm{d}x + \frac{1}{2\delta_4}\int_\Omega (g\diamond u_x)^2\mathrm{d}x \\
\leq &\delta_4\left(\int_0^t g(s)\mathrm{d}s\right)^2\int_\Omega u_x^2(x,t)\mathrm{d}x + \left(\delta_4+\frac{1}{2\delta_4}\right)\int_\Omega (g\diamond u_x)^2\mathrm{d}x \\
\leq &\delta_4\left(\int_0^t g(s)\mathrm{d}s\right)^2\int_\Omega u_x^2(x,t)\mathrm{d}x + \left(\delta_4+\frac{1}{2\delta_4}\right)\int_0^t g(s)\mathrm{d}s\int_\Omega (g\Box u_x)\mathrm{d}x
\end{aligned} \tag{26}$$

and

$$a\int_\Omega u_x(g\diamond u_x)\mathrm{d}x \leq \delta_4\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{a^2}{4\delta_4}\int_0^t g(s)\mathrm{d}s\int_\Omega (g\Box u_x)\mathrm{d}x \tag{27}$$

By Young's inequality and (12), we get for any $\delta_4 > 0$, $\alpha_1 > 0$

$$|\mu_2|\int_\Omega u_t(x,t-\tau)(g\diamond u)\mathrm{d}x \leq \delta_4|\mu_2|\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x + \frac{|\mu_2|L^2}{4\delta_4}\int_0^t g(s)\mathrm{d}s\int_\Omega (g\Box u_x)\mathrm{d}x \tag{28}$$

and

$$\begin{aligned}
-\int_\Omega u_t(g'\diamond u)\mathrm{d}x &\leq \frac{\alpha_4}{2}\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{1}{2\alpha_4}\int_0^t(-g'(s))\mathrm{d}s\int_\Omega(-g'\Box u)\mathrm{d}x \\
&\leq \frac{\alpha_4}{2}\int_\Omega u_x^2(x,t)\mathrm{d}x - \frac{g(0)L^2}{2\alpha_4}\int_\Omega (g'\Box u_x)\mathrm{d}x
\end{aligned} \tag{29}$$

Inserting (26)–(29) into (25), we get (24). □

Now, we are ready to prove Theorem 3.

Proof. We define the Lyapunov functional:

$$L(t) = N_1E(t) + N_2\mathscr{D}(t) + \mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + N_5\mathscr{F}_3(t) \tag{30}$$

where N_1, N_2, N_4 and N_5 are positive constants that will be fixed later.

Since g is continuous and $g(0) > 0$, then for any $t \geq t_0 > 0$, we obtain

$$\int_0^t g(s)\mathrm{d}s \geq \int_0^{t_0} g(s)\mathrm{d}s = g_0 \tag{31}$$

Taking the derivative of (30) with respect to t and making the use of the above lemmas, we have

$$\begin{aligned}
\frac{d}{dt}L(t) \leq & -\left\{N_5\left(g_0-\frac{\alpha_4}{2}\right)-N_1\left(\frac{|\mu_2|}{2}+\frac{\zeta}{2}\right)-N_2-\left(\frac{a}{2}+\frac{M^2}{4\delta_2}\right)\right\}\int_\Omega u_t^2\mathrm{d}x \\
& -\{N_2\beta_0-N_2(\delta_1+L^2\delta_1|\mu_2|)-(2a^2+4\bar{g}^2+\delta_2|\mu_2|\bar{g}^2+\delta_2M^2a^2|\mu_2|+g^2(0)\delta_2) \\
& -N_5(\delta_4+\delta_4\bar{g}^2)\}\int_\Omega u_x^2\mathrm{d}x \\
& +\left\{N_1\left(\frac{|\mu_2|}{2}-\frac{\zeta}{2}e^{-\sigma\tau}\right)+\frac{N_2|\mu_2|}{4\delta_1}+\left(\delta_2|\mu_2|M^2+\frac{|\mu_2|(1+M^2)}{4\delta_2}\right)+N_5\delta_4|\mu_2|\right\}\int_\Omega u_t^2(x,t-\tau)\mathrm{d}x \\
& -\left\{\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4+N_2b\right\}\int_{L_1}^{L_2}v_x^2\mathrm{d}x-\left\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2\right\}\int_{L_1}^{L_2}v_t^2\mathrm{d}x \\
& -(b-N_4)\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right)-(a-N_4)\left[\frac{L_1}{4}v_t^2(L_1,t)+\frac{L_3-L_2}{4}v_t^2(L_2,t)\right] \\
& +\left[\frac{N_2\bar{g}}{4\delta_1}+\left(4\bar{g}+\frac{|\mu_2|\bar{g}}{4\delta_2}\right)+N_5\left(\delta_4+\frac{1}{2\delta_4}+\frac{(a^2+|\mu_2|L^2)}{4\delta_4}\right)\bar{g}\right]\int_\Omega(g\Box u_x)\mathrm{d}x \\
& +\left[\frac{N_1}{2}-g(0)\delta_2-\frac{N_5g(0)L^2}{2\alpha_4}\right]\int_\Omega(g'\Box u_x)\mathrm{d}x
\end{aligned} \tag{32}$$

At this moment, we wish all coefficients except the last two in (32) will be negative. We want to choose N_2 and N_4 to ensure that

$$\begin{cases} a-N_4\geq 0 \\ b-N_4\geq 0 \\ \dfrac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2>0 \end{cases} \tag{33}$$

For this purpose, since $\dfrac{8l(L_2-L_1)}{L_1+L_3-L_2}<\min\{a,b\}$, we first choose N_4 satisfying

$$\frac{8l(L_2-L_1)}{L_1+L_3-L_2}<N_4\leq\min\{a,b\}$$

Then, we pick

$$\alpha_4=g_0,\quad \delta_1<\frac{\beta_0}{8}\quad\text{and}\quad \delta_2<\frac{1}{g^2(0)}$$

such that

$$N_5\left(g_0-\frac{\alpha_4}{2}\right)=\frac{N_5g_0}{2},\quad N_2\beta_0-N_2\delta_1>\frac{7N_2\beta_0}{8},\quad\text{and}\quad g^2(0)\delta_2<1$$

Once δ_2 is fixed, we take N_2 satisfying

$$N_2>\frac{8(2a^2+4\bar{g}+g^2(0)\delta_2)}{\beta_0}$$

such that

$$(2a^2+4\bar{g}+g^2(0)\delta_2)<\frac{N_2\beta_0}{8}$$

Furthermore, we choose N_5 satisfying

$$\frac{N_5g_0}{8}>N_2+\frac{a}{2}+\frac{M^2}{4\delta_2}$$

such that

$$N_2<\frac{N_5\beta_0}{8},\quad \frac{a}{2}<\frac{N_5\beta_0}{8},\quad \frac{M^2}{4\delta_2}<\frac{N_5\beta_0}{8}\quad\text{and}\quad \frac{N_5g_0}{8}-N_2-\left(\frac{a}{2}+\frac{M^2}{4\delta_2}\right)>0$$

Then, we pick δ_4 satisfying

$$\delta_4 < \frac{\beta_0 N_2}{8N_5(1+\bar{g}^2)}$$

such that

$$N_5(\delta_4 + \delta_4 \bar{g}^2) < \frac{N_2\beta_0}{8}$$

Once the above constants are fixed, we choose N_1 satisfying

$$\frac{N_1}{2} > g(0)\delta_2 + \frac{N_5 g(0) L^2}{2\alpha_4}$$

Now, we need to choose suitable $|\mu_2|$ and ζ such that

$$\begin{cases} K_1 - N_1\left(\frac{|\mu_2|}{2} + \frac{\zeta}{2}\right) > 0 \\ \frac{5\beta_0 N_2}{8} - K_2|\mu_2| > 0 \\ K_3\frac{|\mu_2|}{2} - N_1\frac{\zeta}{2}e^{-\sigma\tau} < 0 \end{cases} \tag{34}$$

where

$$K_1 = \frac{N_5 g_0}{8} - N_2 - \left(\frac{a}{2} + \frac{M^2}{4\delta_2}\right), \quad K_2 = N_2\delta_1 L^2 + \delta_2\bar{g}^2 + \delta_2 M^2 a^2$$
$$K_3 = N_1 + \frac{N_2}{2\delta_1} + 2\delta_2 M^2 + \frac{1}{2\delta_2} + \frac{M^2}{2\delta_2} + 2N_5\delta_4$$

We first choose ζ satisfying

$$\frac{2K_1}{N_1} - \zeta > 0$$

Then, we pick $|\mu_2|$ satisfying

$$|\mu_2| < \min\left\{\frac{5\beta_0 N_2}{8K_2}, \frac{N_1\zeta}{K_3 e^{\sigma\tau}}, \frac{2K_1}{N_1} - \zeta\right\} := a_0 \tag{35}$$

From the above, we deduce that there exist two positive constants α_5 and α_6 such that (32) becomes

$$\frac{d}{dt}L(t) \leq -\alpha_5 E(t) + \alpha_6 \int_\Omega (g\Box u_x)\mathrm{d}x \tag{36}$$

Multiplying (36) by ξ, we have

$$\xi\frac{d}{dt}L(t) \leq -\alpha_5\xi E(t) + \alpha_6\xi \int_\Omega (g\Box u_x)\mathrm{d}x$$

On the other hand, by the definition of the functionals $\mathscr{D}(t)$, $\mathscr{F}_1(t)$, $\mathscr{F}_2(t)$, $\mathscr{F}_3(t)$ and $E(t)$, for N_1 large enough, there exists a positive constant α_3 satisfying

$$|N_2\mathscr{D}(t) + N_3\mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + \mathscr{F}_3(t)| \leq \eta_1 E(t)$$

which implies that

$$(N_1 - \eta_1)E(t) \leq L(t) \leq (N_1 + \eta_1)E(t)$$

Exploiting (H2) and (8), we have

$$\xi \int_{\Omega} (g \Box u_x) \mathrm{d}x \leq -\int_{\Omega} (g' \Box u_x) \mathrm{d}x \leq -2\frac{d}{dt}E(t) + \frac{2K_1}{N_1}\int_{\Omega} u_t^2 \mathrm{d}x \tag{37}$$

Thus, (36) becomes

$$\xi \frac{d}{dt}L(t) \leq -\alpha_5 \xi E(t) - 2\alpha_6 \frac{d}{dt}E(t) + \frac{4K_1\alpha_6}{N_1}E(t) \tag{38}$$

We add a restriction condition on ξ, that is, we suppose that

$$\xi > \frac{4K_1\alpha_6}{N_1} := \xi_0 \tag{39}$$

Then, (38) becomes, for some positive constants

$$\xi \frac{d}{dt}L(t) \leq -\alpha_7 \xi E(t) - \alpha_8 \frac{d}{dt}E(t)$$

Now, we define functionals $\mathscr{L}(t)$ as

$$\mathscr{L}(t) = \xi L(t) + \alpha_8 E(t)$$

It is clear that

$$\mathscr{L}(t) \sim E(t) \tag{40}$$

Then, we have

$$\frac{d}{dt}\mathscr{L}(t) \leq -\alpha_7 \xi E(t) \tag{41}$$

A simple integration of (41) over (t_0, t) leads to

$$\mathscr{L}(t) \leq \mathscr{L}(t_0)e^{-c\xi(t-t_0)}, \quad \forall t \geq t_0 \tag{42}$$

Recalling (40), Equation (42) yields the desired result (7). This completes the proof of Theorem 3. □

4. Conclusions

The main purpose of present work is to investigate decay rate for a transmission problem with a viscoelastic term and a delay term but without the frictional damping term. It is based upon our previous work ([1]), in which we studied the well-posedness and general decay rate for a transmission problem in a bounded domain with a viscoelastic term and a delay term. The main difficulty in dealing with the problem here is that in the first equation of system (2), we have no frictional damping term to control the delay term in the estimate of the energy decay. To overcome this difficulty, our basic idea is to control the delay term by making use of the viscoelastic term. In order to achieve this target, a restriction of the size between the parameter μ_2 and the relaxation function g and a suitable energy is needed. This is motivated by Dai and Yang's work [34], in which the viscoelastic wave equation with delay term but without a frictional damping term was considered and an exponential decay result was established. In Section 2, we give some notations and hypotheses needed for our work and state the main results. In Section 3, because the energy is not non-increasing, we introduce the additional functional to produce negative term $-\int_{\Omega} u_t^2 \mathrm{d}x$. Then by introducing suitable Lyaponov functionals, we prove the exponential decay of the solutions for the relaxation function satisfying assumption (H_1) and (H_2).

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