Article

Three Identities of the Catalan-Qi Numbers

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Abstract: In the paper, the authors find three new identities of the Catalan-Qi numbers and provide alternative proofs of two identities of the Catalan numbers. The three identities of the Catalan-Qi numbers generalize three identities of the Catalan numbers.

Keywords: identity; Catalan number; Catalan-Qi number; Catalan-Qi function; alternative proof; hypergeometric series; generalization

MSC: Primary 05A19; Secondary 11B75, 11B83, 33B15, 33C05, 33C20

1. Introduction

It is stated in [1] that the Catalan numbers $C_n$ for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as “In how many ways can a regular $n$-gon be divided into $n-2$ triangles if different orientations are counted separately?” (for other examples, see [2,3]) the solution of which is the Catalan number $C_{n-2}$. The Catalan numbers $C_n$ can be generated by

$$\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$$

(1)

Three of explicit equations of $C_n$ for $n \geq 0$ read that

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+2)} = \binom{2n}{n} = 2F_1(1-n,-n;2;1)$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0$$

is the classical Euler gamma function and

$$pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

is the generalized hypergeometric series defined for complex numbers $a_i \in \mathbb{C}$ and $b_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, for positive integers $p, q \in \mathbb{N}$, and in terms of the rising factorials $(x)_n$ defined by

$$(x)_n = \begin{cases} x(x+1)(x+2)\ldots(x+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

and
\[ (-x)_n = (-1)^n (x - n + 1)_n \]

A generalization of the Catalan numbers \( C_n \) was defined in [4–6] by
\[
p^d_n = \frac{1}{n} \binom{pn}{n-1} = \frac{1}{(p-1)n+1} \binom{pn}{n}
\]
for \( n \geq 1 \). The usual Catalan numbers \( C_n = 2d_n \) are a special case with \( p = 2 \).

In combinatorial mathematics and statistics, the Fuss-Catalan numbers \( A_n(p, r) \) are defined in [7,8] as numbers of the form
\[
A_n(p, r) = r^n \Gamma(np + r) \Gamma(n + r + 1) / \Gamma(n + 1) \Gamma((p-1)n + r + 1)
\]
It is obvious that \( A_n(2, 1) = C_n, \quad n \geq 0 \) and \( A_{n-1}(p, p) = p^d_n, \quad n \geq 1 \)

There have existed some literature such as [8–20] on the investigation of the Fuss-Catalan numbers \( A_n(p, r) \).

In (Remark 1 [21]), an alternative and analytical generalization of the Catalan numbers \( C_n \) and the Catalan function \( C_x \) was introduced by
\[
C(a, b; z) = \frac{\Gamma(b) b^z \Gamma(z + a) \Gamma(z + b)}{\Gamma(a) \Gamma(a + b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \geq 0
\]
In particular, we have
\[
C(a, b; n) = \left( \frac{b}{a} \right)^n \frac{(a)_n}{(b)_n}
\]
For the uniqueness and convenience of referring to the quantity \( C(a, b; x) \), we call the quantity \( C(a, b; x) \) the Catalan-Qi function and, when taking \( x = n \geq 0 \), call \( C(a, b; n) \) the Catalan-Qi numbers.

It is clear that
\[
C\left(\frac{1}{2}, 2; n\right) = C_n, \quad n \geq 0
\]

In (Theorem 1.1 [22]), among other things, it was deduced that
\[
A_n(p, r) = p^n \prod_{k=1}^{n} C\left(\frac{k+r-1}{p}, 1; n\right) / \prod_{k=1}^{n-1} C\left(\frac{k+r}{p}, 1; n\right)
\]
for integers \( n \geq 0, p > 1, \) and \( r > 0 \). In the recent papers [21–31], some properties, including the general expression and a generalization of an asymptotic expansion, the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, and connections with the Bessel polynomials and the Bell polynomials of the second kind, of the Catalan numbers \( C_n \), the Catalan function \( C_x \), and the Catalan-Qi function \( C(a, b; x) \) were established.

In 1928, J. Touchard ([32] p. 472) and ([33] p. 319) derived an identity
\[
C_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-2k} C_k
\]
where \( \lfloor x \rfloor \) denotes the floor function the value of which is the largest integer less than or equal to \( x \). For the proof of Equation (2) by virtue of the generating function (1), see ([33] pp. 319–320).
In 1987, when attending a summer program at Hope College, Holland, Michigan in USA, D. Jonah ([34] p. 214) and ([33] pp. 324–326) presented that
\[
\binom{n + 1}{m} = \sum_{k=0}^{m} \binom{n - 2k}{m - k} C_k, \quad n \geq 2m, \quad n \in \mathbb{N}
\]
(3)

In 1990, Hilton and Pedersen ([34] p. 214) and ([33] p. 327) generalized Identity (3) for an arbitrary real number \( n \) and any integer \( m \geq 0 \).

In 2009, J. Koshy ([33] p. 322) provided another recursive equation
\[
C_n = \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}
\]
(4)

We observe that Identity (4) can be rearranged as
\[
\sum_{k=\left\lfloor \frac{n+1}{2} \right\rfloor}^{n} (-1)^{k} \binom{k+1}{n-k} C_k = 0
\]
where \( \lceil x \rceil \) stands for the ceiling function which gives the smallest integer not less than \( x \).

The aims of this paper are to generalize Identities (2)–(4) for the Catalan numbers \( C_n \) to ones for the Catalan-Qi numbers \( C(a, b; n) \).

Our main results can be summarized up as the following theorem.

Theorem 1. For \( a, b > 0, n \in \mathbb{N}, \) and \( n \geq 2m \geq 0 \), the Catalan-Qi numbers \( C(a, b; n) \) satisfy

\[
3F_2\left( a, \frac{1-n}2, -\frac{n+1}2, \frac 12; 1 \right) = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n}{2k} \frac{a^{k}}{b^{k}} C(a, b; k)
\]
(5)

\[
4F_3\left( 1, a, -m, m-n; b, -\frac{n+1}{2}, \frac{n-1}{2a} \right) = \frac{1}{m} \sum_{k=0}^{m} \binom{n-2k}{m-k} C(a, b; k)
\]
(6)

and

\[
3F_2 \left( 1-b-n,-\frac{n+1}{2},-\frac{n}{2}; -n-1,1-a-n,\frac{4a}{b} \right) = \frac{1}{C(a, b; n)} \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor}^{n} (-1)^{n-k} \binom{k+1}{n-k} C(a, b; k)
\]
(7)

As by-products, alternative proofs for Identities (2) and (4) are also supplied in next section.

2. Proofs

We are now in a position to prove Theorem 1 and Identities (2) and (4).

Proof of Identity (5). By the definition (1), we have
\[
3F_2\left( a, \frac{1-n}2, -\frac{n+1}2; b, \frac 12; 1 \right) = \sum_{k=0}^{\infty} \frac{(a)_{k} \left( \frac{1-n}{2} \right)_{k} \left( \frac{-n}{2} \right)_{k}}{(b)_{k} \left( \frac{1}{2} \right)_{k} k!}
\]

Using the relations
\[
\left( \frac{1-n}{2} \right)_{k} = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor, \quad n = 1, 3, 5, \ldots
\]
and
\[
\binom{-\frac{n}{2}}{k} = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor, \quad n = 2, 4, 6, \ldots
\]
we obtain
\[
{\binom{3F_2}{a, \frac{1-n}{2}, \frac{n}{2}, \frac{1}{2}, \frac{1}{2}, 1} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\binom{1-n}{k} \binom{-n}{k} a}{k!} C(a; b; k)}
\]
Further using the relations
\[
\binom{z}{2} \binom{z+1}{2} = 4^{-z}(z)_{2r}, \quad (-z)_r = (-1)^r \binom{z}{r}, \quad \text{and} \quad \binom{1}{r} = \frac{(2r)!}{r!4^r}
\]
we acquire
\[
\frac{\binom{1-n}{k} \binom{-n}{k}}{(\frac{1}{2})_k} = \binom{n}{2k}
\]
The proof of Identity (5) is thus complete. \(\square\)

**Proof of Identity** (6). By the definition (1), we have
\[
{\binom{4F_3}{1, a, -m, m - n; b, \frac{1-n}{2}, \frac{n}{2}, \frac{1}{2}, \frac{1}{2}, 4a} = \sum_{k=0}^{m} \frac{(-m)_k (m-n)_k}{4^k (\frac{1-n}{2})_k (\frac{-n}{2})_k}}
\]
Since
\[
4^k \left(\frac{1-n}{2}\right)_k \left(\frac{-n}{2}\right)_k = \frac{n!}{(n-2k)!}
\]
and
\[
(-m)_k (m-n)_k = \frac{m!(n-m)!}{(m-k)! (n-m-k)!}
\]
it follows that
\[
\frac{(-m)_k (m-n)_k}{4^k (\frac{1-n}{2})_k (\frac{-n}{2})_k} = \binom{n-2k}{m-k}
\]
Hence, we can derive Identity (6). \(\square\)

**Proof of Identity** (7). By the definition (1), we have
\[
{\binom{3F_2}{1-b-n, -\frac{n+1}{2}, -\frac{n}{2}, -n-1, 1-a-n; 4a}{\frac{1}{b}} = 1 - \sum_{k=1}^{\left\lfloor \frac{1-n}{2} \right\rfloor} \frac{(1-b-n)_k (-n-1)_k (\frac{1-n}{2})_k (\frac{-n}{2})_k}{\left(\frac{1-n}{2}-1\right)_k (1-a-n)_k k!} \left(\frac{4a}{b}\right)^k}
\]
where
\[
\binom{-\frac{n}{2}}{k} = 0, \quad k > \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad n = 2, 4, 6, \ldots
\]
and
\[
\binom{-\frac{n+1}{2}}{k} = 0, \quad k > \left\lfloor \frac{n+1}{2} \right\rfloor, \quad n = 1, 3, 5, \ldots
\]
Using the relations
\[
(-z)_r = (-1)^r (z-r+1)_r \quad \text{and} \quad (z)_{r+s} = (z)_r (z+r)_s
\]
we have
\[(1 - a - n)_k = (-1)^k \frac{(a)_n}{(a)_{n-k}}\]

As a result, it follows that
\[3F_2 \left( 1 - b - n, -\frac{n+1}{2}, -\frac{n}{2}; -n - 1, 1 - a - n, \frac{4a}{b} \right) - 1 = \frac{\sum_{k=1}^{n+1} (-1)^k \binom{n-k+1}{k} C(a, b; n-k)}{C(a, b; n)}\]

which can be reformulated as Identity (7). The proof of Identity (7) is complete. \(\square\)

**Proof of Identity (2).** Putting \(a = \frac{1}{2}\) and \(b = 2\) in Equation (5) results in
\[\sum_{k=0}^{n} \frac{n}{2k} 2^{-2k} C_k = 3F_2 \left( \frac{1}{2}, \frac{1-n}{2}, -\frac{n}{2}; 2, 1 \right) = 2F_1 \left( \frac{1-n}{2}, \frac{n}{2}; 2, 1 \right)\]

Now applying Kummer’s transformation equation
\[2F_1(a, \beta; 1 + a - \beta; z) = (1 + z)^{-\alpha} 2F_1 \left( \frac{a}{2}, \frac{a+1}{2}, 1 + a - \beta; \frac{4z}{(z+1)^2} \right)\]
to \(a = -n, \beta = -n - 1,\) and \(z = 1\) leads to
\[2F_1 \left( \frac{1-n}{2}, \frac{n}{2}; 2, 1 \right) = 2^{-n} 2F_1 (-1-n, -n; 2, 1) = 2^{-n} C_{n+1}\]

The proof of Identity (2) is complete. \(\square\)

**Proof of Identity (4).** Putting \(a = \frac{1}{2}\) and \(b = 2\) in Equation (7) gives
\[C_n \left[ 1 - 3F_2 \left( -1-n, -\frac{n+1}{2}, -\frac{n}{2}; -n - 1, \frac{1}{2} - n; 1 \right) \right] = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}\]

that is,
\[3F_2 \left( -1-n, -\frac{n+1}{2}, -\frac{n}{2}; -n - 1, \frac{1}{2} - n; 1 \right) = 2F_1 \left( -\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2} - n; 1 \right)\]

Applying the summation equation
\[2F_1(\ell; h; c; 1) = \frac{\Gamma(c) \Gamma(c - \ell - h)}{\Gamma(c - \ell) \Gamma(c - h)}, \quad \Re(c - \ell - h) > 0\]
to \(c = \frac{1}{2} - n, \ell = -\frac{n+1}{2},\) and \(h = -\frac{n}{2},\) yields
\[2F_1 \left( -\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2} - n; 1 \right) = \frac{\Gamma \left( \frac{1}{2} - n \right)}{\Gamma(1 - \frac{n}{2}) \Gamma(1 - \frac{n}{2})}\]

Further employing the duplication equation
\[\Gamma(z) \Gamma \left( z + \frac{1}{2} \right) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)\]
at $z = \frac{1}{2} - n$ gives us

$$
2F_1\left( -\frac{n+1}{2}, -\frac{n}{2}; -n; 1 \right) = \frac{\Gamma\left(\frac{1}{2} - n\right)}{2^n \sqrt{n} \Gamma(1-n)} = 0, \quad n \in \mathbb{N}
$$

where $\frac{1}{\Gamma(m)}$ has zeros at $m = 0, -1, -2, \ldots$. Identity (4) is thus proved. $\square$

**Remark 1.** From Equations (3) and (6), we can conclude

$$
4F_3\left( 1, \frac{1}{2}, -m, m - n; 2, \frac{1-n}{2}, \frac{n}{2}; 1 \right) = \frac{n+1}{n+1-m}
$$

and

$$
3F_2\left( -\frac{1}{2}, -m - 1, m - n - 1; -1, -\frac{n}{2}, \frac{n+1}{2}; 1 \right) = \frac{n-2m}{n+m},
$$

for $n \geq 2m$ and $n \in \mathbb{N}$.

**Remark 2.** Please note, we recommend a newly-published paper [35] which is closely related to the Catalan numbers $C_n$.

**Remark 3.** This paper is a slightly revised version of the preprint [36] and has been reviewed by the survey article [37].

3. Conclusions

Three new identities for the Catalan-Qi numbers are discovered and alternative proofs of two identities for the Catalan numbers are provided. The three identities for the Catalan-Qi numbers generalize three identities for the Catalan numbers.

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