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Article

Coefficient Inequalities of Second Hankel Determinants for Some Classes of Bi-Univalent Functions

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Abstract: In this paper, we investigate two sub-classes $S^*(\theta, \beta)$ and $K^*(\theta, \beta)$ of bi-univalent functions in the open unit disc Δ that are subordinate to certain analytic functions. For functions belonging to these classes, we obtain an upper bound for the second Hankel determinant $H_2(2)$.

Keywords: analytic functions; univalent functions; bi-univalent functions; second Hankel determinants

1. Introduction

Let A be the class of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Further, by S we shall denote the class of all functions in A that are univalent in Δ.

Let P denote the family of functions $p(z)$, which are analytic in Δ such that $p(0) = 1$, and $\Re p(z) > 0$ $(z \in \Delta)$ of the form

$$P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2}$$

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, and we write it as $f(z) \prec g(z)$ if there exists a Schwarz function ω, which is analytic in Δ with $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \Delta)$ such that

$$f(z) = g(\omega(z)). \tag{3}$$

Indeed, it is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta). \tag{4}$$

Every function $f \in S$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z$, $(z \in \Delta)$

$$\text{and } f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); r_0(f) \geqslant \frac{1}{4}\right). \tag{5}$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots.$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ.

Let $\sum$ denote the class of bi-univalent functions defined in the unit disc Δ.

We notice that $\sum$ is non empty .One of the best examples of bi-univalent functions is $f(z) = \log\left(\frac{1+z}{1-z}\right)$, which maps the unit disc univalently onto a strip $|\mathrm{Im}w| < \frac{\pi}{2}$, which in turn contains the unit disc. Other examples are z, $\frac{z}{1-z}$, $-\log(1-z)$.

However, the Koebe function is not a member of $\sum$ because it maps unit disc univalent onto the entire complex plane minus a slit along $-\frac{1}{4}$ to $-\infty$. Hence, the image domain does not contain the unit disc.

Other examples of univalent function that are not in the class $\sum$ are $z - \frac{z^2}{2}$, $\frac{z}{1-z^2}$.

In 1967, Lewin [1] first introduced class $\sum$ of bi-univalent function and showed that $|a_2| \leqslant 1.51$ for every $f \in \sum$. Subsequently, in 1967, Branan and Clunie [2] conjectured that $|a_2| \leqslant \sqrt{2}$ for bi-star like functions and $|a_2| \leqslant 1$ for bi-convex functions. Only the last estimate is sharp; equality occurs only for $f(z) = \frac{z}{1-z}$ or its rotation.

Later, Netanyahu [3] proved that $\max_{f\in\sum} |a_2| = \frac{4}{3}$. In 1985, Kedzierawski [4] proved Brannan and Clunie's conjecture for bi-starlike functions. In 1985, Tan [5] obtained that $|a_2| < 1.485$, which is the best known estimate for bi-univalent functions. Since then, various subclasses of the bi-univalent function classes $\sum$ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin's series expansion were found in several investigations. The coefficient estimate problem for each of $|a_n|$ $(n \in N\,\{2, 3\})$ is still an open problem.

In 1976, Noonan and Thomas [6] defined q^{th}Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$, which is stated by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

These determinants are useful, for example, in showing that a function of bounded characteristic in Δ, *i.e.*, a function that is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficient is rational.

The Hankel determinant plays an important role in the study of singularities (for instance, see [7] Denies, p.329 and Edrei [8]).A Hankel determinant plays an important role in the study of power series with integral coefficients. In 1966, Pommerenke [9] investigated the Hankel determinants of areally mean p-valent functions, univalent functions as well as of starlike functions, and, in 1967 [10], he proved that the Hankel determinants of univalent functions satisfy $H_q(n) < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$ $(n = 1, 2, \dots, q = 2, 3, \dots)$ where $\beta > \frac{1}{4000}$ and K depend only on q. Later, Hayman [11] proved that $H_2(n) < An^{\frac{1}{2}}$ $(n = 1, 2, \dots; A$ an absolute constant) for areally mean univalent functions. The estimates for the Hankel determinant of areally mean p-valent functions have been investigated [12–14]. Elhosh [15,16] obtained bounds for Hankel determinants of univalent functions with a positive Hayman index α and k-fold symmetric and close to convex functions. Noor [9] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary.

Ehrenborg [17] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [18].

One can easily observe that the Fekete-Szego functional $\left|a_3 - a_2^2\right| = H_2(1)$. This function was further generalized with μ real as well as complex. Fekete-Szego gave a sharp estimate of $\left|a_3 - \mu a_2^2\right|$ for μ real. The well-known results due to them is

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} 4\mu - 3 & \mu \leqslant 1 \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & 0 \leqslant \mu \leqslant 1 \\ 3 - 4\mu & \mu \geqslant 0 \end{cases}.$$

On the other hand, Zaprawa [19,20] extended the study on Fekete-Szego problem to some classes of bi-univalent functions. Ali [21] found sharp bounds on the first four coefficients and a sharp estimate for the Fekete-Szego functional $\left|\gamma_3 - t\gamma_2^2\right|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{k=2}^{\infty} \gamma_k w^k$ to the class of strongly starlike functions of order α $(0 < \alpha \leqslant 1)$.

Recently S.K. Lee *et al.* [22] obtained the second Hankel determinant $H_2(2) = a_2 a_4 - a_3^2$ for functions belonging to subclasses of Ma-Minda starlike and convex functions. T. Ram Reddy [23] obtained the Hankel determinants for starlike and convex functions with respect to symmetric points. T. Ram Reddy *et al.* [24,25] also obtained the second Hankel determinant for subclasses of p-valent functions and p-valent starlike and convex function of order α.

Janteng [26] has obtained sharp estimates for the second Hankel determinant for functions whose derivative has a positive real part.Afaf Abubaker [27] studied sharp upper bound of the second Hankel determinant of subclasses of analytic functions involving a generalized linear differential operator. In 2015, the second Hankel determinant for bi-starlike and bi-convex function of order β was obtained by Erhan Deniz [28].

2. Preliminaries

Motivated by above work, in this paper, we introduce certain subclasses of bi-univalent functions and obtained an upper bound to the coefficient functional $a_2 a_4 - a_3^2$ for the function f in these classes defined as follows:

Definition 2.1.: A function $f \in A$ is said to be in the class $S^*(\theta, \beta)$ if it satisfies the following conditions:

$$\Re\left\{e^{i\theta}\left\{\frac{zf'(z)}{f(z)}\right\}\right\} > \beta\cos\theta \quad (\forall z \in \Delta) \tag{6}$$

$$\Re\left\{e^{i\theta}\left\{\frac{wg'(w)}{g(w)}\right\}\right\} > \beta\cos\theta \quad (\forall w \in \Delta) \tag{7}$$

where g is an extension of f^{-1} to Δ.

Note: 1. For $\theta = 0$, the class $S^*(\theta, \beta)$ reduces to the class $S_\sigma^*(\beta)$, and, for this class, coefficient inequalities of the second Hankel determinant were studied by Deniz *et al* [28].

2. For $\theta = 0$ and $\beta = 0$, the class $S^*(\theta, \beta)$ reduces to the class S_σ^*, and, for this class, coefficient inequalities of the second Hankel determinant were studied by Deniz *et al* [28].

Definition 2.2.: A function $f \in A$ is said to be in the class $K^*(\theta, \beta)$ if it satisfies the following conditions:

$$\Re\left\{e^{i\theta}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right\} > \beta\cos\theta \quad (\forall z \in \Delta) \tag{8}$$

$$\Re\left\{e^{i\theta}\left\{1 + \frac{wg''(w)}{g'(w)}\right\}\right\} > \beta\cos\theta \quad (\forall w \in \Delta) \tag{9}$$

where g is an extension of f^{-1} in Δ.

Note: 1. For $\theta = 0$, the class $K^* (\theta, \beta)$ reduces to the class $K^*_\sigma (\beta)$, and, for this class, coefficient inequalities ofthe second Hankel determinantwere studied by Deniz *et al* [28].

2. For $\theta = 0$ and $\beta = 0$ the class $K^* (\theta, \beta)$ reduces to the class K^*_σ, and, for this class, coefficient inequalities ofthe second Hankel determinantwere studied by Deniz *et al* [28].

To prove our results, we require the following Lemmas:

Lemma 2.1. [14] Let the function $p \in P$ be given by the following series:

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \qquad (z \in \Delta). \tag{10}$$

Then the sharp estimate is given by .

Lemma 2.2. [29] The power series for the function $p \in P$ is given (10) converges in the unit disc Δ to a function in P if and only if Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \rightleftharpoons & c_n \\ c_{-1} & 2 & c_1 & \rightleftharpoons & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \rightleftharpoons & 2 \end{vmatrix}, \quad n \in N$$

and $c_{-k} = \bar{c}_k$ are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k P_0 \left(e^{it_k} z\right)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for k $\neq$ j, where $P_0(z) = \left(\dfrac{1+z}{1-z}\right)$; in this case, $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geqslant m$.

This necessary and sufficient condition found in the literature [29] is due to Caratheodary and Toeplitz. We may assume without any restriction that $c_1 > 0$. On using Lemma (2.2) for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = \left[8 + 2\mathrm{Re}\left\{c_1^2 c_2\right\} - 2|c_2|^2 - 4c_1^2\right] \geqslant 0.$$

It is equivalent to

$$2c_2 = \left\{c_1^2 + x\left(4 - c_1^2\right)\right\}, \text{ for some } x, \quad |x| \leqslant 1 \tag{11}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geqslant 0.$$

Then $D_3 \geqslant 0$ is equivalent to

$$\left|\left(4c_3 - 4c_1c_2 + c_1^3\right)\left(4 - c_1^2\right) + c_1\left(2c_2 - c_1^2\right)^2\right| \leqslant 2\left(4 - c_1^2\right)^2 - 2\left|\left(2c_2 - c_1^2\right)\right|^2. \tag{12}$$

From the relations (2.6) and (2.7), after simplifying, we get

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z,$$

for some x, z with

$$|x| \leqslant 1 \text{and } |z| \leqslant 1. \tag{13}$$

3. Main Results

We now prove our main result for the function f in the class $S^* (\theta, \beta)$.

Theorem 3.1. Let the function f given by (1.1) be in the class $S^*(\theta,\ \beta)$. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \begin{cases} \dfrac{16(1-\beta)^4\cos^4\theta}{3} + \dfrac{4}{3}(1-\beta)^2\cos^2\theta, & \beta \in \left[0, 1 - \dfrac{1}{2\sqrt{2}\cos\theta}\right] \\ \dfrac{3(1-\beta)^2\cos^2\theta}{2\left[1-2(1-\beta)^2\cos^2\theta\right]}, & \beta \in \left(1 - \dfrac{1}{2\sqrt{2}\cos\theta}, 1\right) \end{cases}.$$

Proof: Let $f \in S(\theta,\ \beta;\ h)$ and $g = f^{-1}$. From (6) and (7) it follows that

$$e^{i\theta}\left\{\frac{zf'(z)}{f(z)}\right\} = [(1-\beta)p(z) + \beta]]\cos\theta + i\sin\theta \tag{14}$$

$$e^{i\theta}\left\{\frac{wg'(w)}{g(w)}\right\} = [(1-\beta)q(w) + \beta]]\cos\theta + i\sin\theta \tag{15}$$

where $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ...\quad \in P,\ (z \in \Delta)$ and $q(w) = 1 + d_1w + d_2w^2 + d_3w^3 + ...\quad \in P,\ \ (w \in \Delta)$. Now, equating the coefficients in (14) and (15), we have

$$e^{i\theta}a_2 = c_1(1-\beta)\cos\theta \tag{16}$$

$$e^{i\theta}\left(2a_3 - a_2^2\right) = c_2(1-\beta)\cos\theta \tag{17}$$

$$e^{i\theta}\left(3a_4 - 3a_2a_3 + a_2^3\right) = c_3(1-\beta)\cos\theta \tag{18}$$

and

$$-e^{i\theta}a_2 = d_1(1-\beta)\cos\theta \tag{19}$$

$$e^{i\theta}\left(3a_2^2 - 2a_3\right) = d_2(1-\beta)\cos\theta \tag{20}$$

$$e^{i\theta}\left(-3a_4 + 12a_2a_3 - 10a_2^3\right) = d_3(1-\beta)\cos\theta \tag{21}$$

Now from (16) and (19) we get

$$c_1 = -d_1 \tag{22}$$

and

$$c_2 = e^{-i\theta}p_1(1-\beta)\cos\theta. \tag{23}$$

Now, from (17) and (20), we get

$$a_3 = e^{-2i\theta}c_1^2(1-\beta)^2\cos^2\theta + \frac{e^{-i\theta}(1-\beta)\cos\theta\,(c_2-d_2)}{4}. \tag{24}$$

Additionally, from (18) and (21), we get

$$a_4 = \frac{2}{3}e^{-3i\theta}c_1^3(1-\beta)^3\cos^3\theta + \frac{5}{8}e^{-2i\theta}c_1(c_2-d_2)(1-\beta)^2\cos^2\theta + \frac{1}{6}e^{-i\theta}(c_3-d_3)(1-\beta)\cos\theta. \tag{25}$$

Thus, we can easily obtain

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| &\frac{-1}{3}e^{-4i\theta}c_1^4(1-\beta)^4\cos^4\theta + \frac{1}{8}e^{-3i\theta}c_1^2(c_2-d_2)(1-\beta)^3\cos^3\theta + \\ &\frac{1}{6}e^{-2i\theta}c_1(c_3-d_3)(1-\beta)^2\cos^2\theta - \frac{1}{16}e^{-2i\theta}(c_2-d_2)^2(1-\beta)^2\cos^2\theta\Bigg|. \end{aligned} \tag{26}$$

According to Lemma (2.2) and Equation (22), we get

$$\left.\begin{array}{l} 2c_2 = c_1^2 + x\left(4 - c_1^2\right) \\ 2d_2 = d_1^2 + x\left(4 - d_1^2\right) \end{array}\right\} \Rightarrow c_2 - d_2 = 0 \tag{27}$$

and

$$c_3 - d_3 = \frac{c_1^3}{2} - c_1\left(4 - c_1^2\right)x - \frac{c_1\left(4 - c_1^2\right)x^2}{2} \tag{28}$$

$$\begin{array}{c} \left|a_2a_4 - a_3^2\right| = \left| -\frac{1}{3}e^{-4i\theta}c_1^4(1-\beta)^4\cos^4\theta + \frac{1}{12}e^{-2i\theta}c_1^4(1-\beta)^2\cos^2\theta - \right. \\ \left. \frac{1}{6}e^{-2i\theta}c_1^2\left(4 - c_1^2\right)x(1-\beta)^2\cos^2\theta - \frac{1}{12}e^{-2i\theta}c_1^2\left(4 - c_1^2\right)x^2(1-\beta)^2\cos^2\theta \right| \end{array} . \tag{29}$$

Since $p \in P$, so $|c_1| \leqslant 2$. Letting $c_1 = c$, we may assume without any restriction that $c \in [0, 2]$. Thus, applying the triangle inequality on the right-hand side of Equation (29), with $\mu = |x| \leqslant 1$, we obtain

$$\begin{array}{c} \left|a_2a_4 - a_3^2\right| \leqslant \frac{1}{3}c^4(1-\beta)^4\cos^4\theta + \frac{1}{12}c^4(1-\beta)^2\cos^2\theta + \frac{1}{6}c^2\left(4 - c^2\right)\mu(1-\beta)^2\cos^2\theta \\ + \frac{1}{12}c^2\left(4 - c^2\right)\mu^2(1-\beta)^2\cos^2\theta = F(\mu). \end{array} \tag{30}$$

Differentiating $F(\mu)$, we get

$$F'(\mu) = \frac{c^2\left(4 - c^2\right)(1-\beta)^2\cos^2\theta + c^2\left(4 - c^2\right)\mu(1-\beta)^2\cos^2\theta}{6}. \tag{31}$$

Using elementary calculus, one can show that $F'(\mu) > 0$ for $\mu > 0$. This implies that F is an increasing function, and it therefore cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Further, the upper bound for $F(\mu)$ corresponds to $\mu = 1$, in which case $F(\mu) \leqslant F(1)$

$$\frac{1}{3}c^4(1-\beta)^4\cos^4\theta + \frac{1}{12}c^4(1-\beta)^2\cos^2\theta + \frac{1}{4}c^2\left(4 - c^2\right)(1-\beta)^2\cos^2\theta = G(c).$$

Then

$$G'(c) = \frac{2}{3}c(1-\beta)^2\cos^2\theta\left[\left(2(1-\beta)^2\cos^2\theta - 1\right)c^2 + 1\right]. \tag{32}$$

Setting $G'(c) = 0$, the real critical points are $c_{01} = 0$, $c_{02} = \sqrt{\frac{3}{1 - 2(1-\beta)^2\cos^2\theta}}$.

After some calculations we obtain the following cases:

Case 1: When $\beta \in \left[0, 1 - \frac{1}{2\sqrt{2}\cos\theta}\right]$, we observe that $c_{02} \geqslant 2$, that is c_{02}, is out of the interval $(0, 2)$. Therefore, the maximum value of $G(c)$ occurs at $c_{01} = 0$ or $c = c_{02}$, which contradicts our assumption of having a maximum value at the interior point of $c \in [0, 2]$. Since G is an increasing function, the maximum point of G must be on the boundary of $c \in [0, 2]$, that is $c = 2$. Thus, we have

$$\max_{0 \leqslant c \leqslant 2} G(c) = G(2) = \frac{16(1-\beta)^4\cos^4\theta}{3} + \frac{4}{3}(1-\beta)^2\cos^2\theta.$$

Case 2: When $\beta \in \left(1 - \dfrac{1}{2\sqrt{2}\cos\theta}, 1\right)$, we observe that $c_{02} < 2$, that is c_{02}, is interior of the interval $[0, 2]$. Since $G''(c_{02}) < 0$, the maximum value of $G(c)$ occurs at $c = c_{02}$. Thus, we have

$$\max_{0\leqslant c\leqslant 2} G(c) = G(c_{02}) = G\left(\sqrt{\frac{1}{\frac{1}{3} - \frac{2}{3}(1-\beta)^2\cos^2\theta}}\right) = \frac{3(1-\beta)^2\cos^2\theta}{2\left[1 - 2(1-\beta)^2\cos^2\theta\right]}.$$

This completes the proof of the theorem.

Corollary 1: Let f given by (1.1) be in the class $S^*_\sigma(\beta)$. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \begin{cases} \dfrac{16(1-\beta)^4}{3} + \dfrac{4}{3}(1-\beta)^2, & \beta \in \left[0, \left(1 - \dfrac{1}{2\sqrt{2}}\right)\right] \\ \dfrac{3(1-\beta)^2}{2\left[1 - 2(1-\beta)^2\right]}, & \beta \in \left(1 - \dfrac{1}{2\sqrt{2}}, 1\right) \end{cases}.$$

Corollary 2: Let f given by (1.1) be in the class S^*_σ. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \frac{20}{3}.$$

These two corollaries coincide with the results of Deniz *et al.* [28].

Remark 3.1: It is observed that for $\theta = 0$, we get the Hankel determinant $\left|a_2a_4 - a_3^2\right|$ for the class $S^*_\sigma(\beta)$ and the Hankel determinant of this class was studied by Deniz *et al.* [28].

4. Hankel Determinants for the Class of Functions $K(\theta, \beta; h)$

We now estimate an upper bound $a_2a_4 - a_3^2$ for the function $f(z)$ in the class $K(\theta, \beta; h)$.

Theorem 4.1. Let the $f(z)$ given by (1.1) be in the class $K(\theta, \beta; h)$. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \begin{cases} \dfrac{1}{6}(1-\beta)^4\cos^4\theta + \dfrac{1}{6}(1-\beta)^2\cos^2\theta, & \beta \in \left[0, 1 - \dfrac{1}{\sqrt{2}\cos\theta}\right] \\ \dfrac{3(1-\beta)^2\cos^2\theta}{8\left[2 - (1-\beta)^2\cos^2\theta\right]}, & \beta \in \left(1 - \dfrac{1}{\sqrt{2}\cos\theta}, 1\right) \end{cases}.$$

Proof: Let $f \in K(\theta, \beta; h)$ and $g = f^{-1}$. From (8) and (9) we have

$$e^{i\theta}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = [(1-\beta)p(z) + \beta]]\cos\theta + i\sin\theta \tag{33}$$

$$e^{i\theta}\left\{1 + \frac{wg''(w)}{g'(w)}\right\} = [(1-\beta)p(w) + \beta]]\cos\theta + i\sin\theta \tag{34}$$

where $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ...$, $(z \in \Delta)$ and $q(w) = 1 + d_1w + d_2w^2 + d_3w^3 + ...$, $(w \in \Delta)$.

Now, equating the coefficients in (33) and (34), we have

$$2e^{i\theta}a_2 = c_1(1-\beta)\cos\theta \tag{35}$$

$$e^{i\theta}\left(6a_3 - 4a_2^2\right) = c_2(1-\beta)\cos\theta \tag{36}$$

$$e^{i\theta}\left(12a_4 - 18a_2a_3 + 8a_2^3\right) = c_3(1-\beta)\cos\theta \tag{37}$$

and

$$-2e^{i\theta}a_2 = d_1(1-\beta)\cos\theta \tag{38}$$

$$e^{i\theta}\left(8a_2^2-6a_3\right)=d_2\left(1-\beta\right)\cos\theta \tag{39}$$

$$e^{i\theta}\left(-32a_2^3+42a_2a_3-12a_4\right)=d_3\left(1-\beta\right)\cos\theta. \tag{40}$$

Now from (35) and (38), we get

$$c_1=-d_1 \tag{41}$$

and

$$a_2=\frac{e^{-i\theta}c_1\left(1-\beta\right)\cos\theta}{2}. \tag{42}$$

Now, from (36) and (39), we get

$$a_3=\frac{e^{-2i\theta}c_1^2\left(1-\rho\right)^2\cos^2\theta}{4}+\frac{e^{-i\theta}\left(c_2-d_2\right)\left(1-\rho\right)\cos\theta}{12}. \tag{43}$$

Additionally, from (37) and (40), we get

$$a_4=\frac{5}{48}e^{-3i\theta}c_1^3\left(1-\beta\right)^3\cos^3\theta+\frac{5}{48}e^{-2i\theta}c_1\left(c_2-d_2\right)\left(1-\beta\right)^2\cos^2\theta+\frac{1}{24}e^{-i\theta}\left(c_3-d_3\right)\left(1-\beta\right)\cos\theta. \tag{44}$$

Thus, we can easily obtain

$$\begin{gathered}\left|a_2a_4-a_3^2\right|=\left|-\frac{c_1^4}{96}e^{-4i\theta}\left(1-\beta\right)^4\cos^4\theta+\frac{c_1^2}{96}e^{-3i\theta}\left(c_2-d_2\right)\left(1-\beta\right)^3\cos^3\theta+\right.\\ \left.\frac{c_1}{48}e^{-2i\theta}\left(c_3-d_3\right)\left(1-\beta\right)^3\cos^3\theta-\frac{e^{-2i\theta}\left(c_2-d_2\right)^2\left(1-\beta\right)^2\cos^2\theta}{144}\right|.\end{gathered} \tag{45}$$

According to Lemma (2.2), and from Equation (41), we get

$$\left.\begin{array}{l}2c_2=c_1^2+x\left(4-c_1^2\right)\\ 2d_2=d_1^2+x\left(4-d_1^2\right)\end{array}\right\}\Rightarrow c_2-d_2=0 \tag{46}$$

and

$$c_3-d_3=\frac{c_1^3}{2}-c_1\left(4-c_1^2\right)x-\frac{c_1\left(4-c_1^2\right)x^2}{2} \tag{47}$$

$$\begin{gathered}\left|a_2a_4-a_3^2\right|=\left|-\frac{c_1^4}{96}e^{-4i\theta}\left(1-\beta\right)^4\cos^4\theta+\frac{c_1^4}{96}e^{-2i\theta}\left(1-\beta\right)^2\cos^2\theta-\right.\\ \left.\frac{e^{-2i\theta}c_1^2\left(4-c_1^2\right)x\left(1-\beta\right)^2\cos^2\theta}{48}-\frac{e^{-2i\theta}c_1^2\left(4-c_1^2\right)x^2\left(1-\beta\right)^2\cos^2\theta}{96}\right|.\end{gathered} \tag{48}$$

Since $p\in P$, $|c_1|\leqslant 2$. Letting $c_1=c$, we may assume without any restriction that $c\in[0,2]$. Thus, applying the triangle inequality on the right-hand side of Equation (4.16), with $\mu=|x|\leqslant 1$, we obtain

$$\begin{gathered}\left|a_2a_4-a_3^2\right|\leqslant\frac{c^4}{96}e^{-4i\theta}\left(1-\beta\right)^4\cos^4\theta+\frac{c^4}{96}e^{-2i\theta}\left(1-\beta\right)^2\cos^2\theta\\ \frac{e^{-2i\theta}c^2\left(4-c^2\right)\mu\left(1-\beta\right)^2\cos^2\theta}{48}+\frac{e^{-2i\theta}c^2\left(4-c^2\right)\mu^2\left(1-\beta\right)^2\cos^2\theta}{96}\\ =F\left(\mu\right).\end{gathered} \tag{49}$$

Differentiating $F\left(\mu\right)$, we get

$$F'\left(\mu\right)=\frac{e^{-2i\theta}c^2\left(4-c^2\right)\left(1-\beta\right)^2\cos^2\theta}{48}+\frac{e^{-2i\theta}c^2\left(4-c^2\right)\mu\left(1-\beta\right)^2\cos^2\theta}{48}. \tag{50}$$

Using elementary calculus, one can show that $F'(\mu) > 0$ for $\mu > 0$. It implies that F is an increasing function and it hence cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Further, the upper bound for $F(\mu)$ corresponds to $\mu = 1$, in which case

$$F(\mu) \leqslant F(1) \begin{array}{l} \leqslant \dfrac{c^4}{96}(1-\beta)^4\cos^4\theta + \dfrac{c^4}{96}(1-\beta)^2\cos^2\theta + \dfrac{c^2\left(4-c^2\right)(1-\beta)^2\cos^2\theta}{48} \\ \quad + \dfrac{c^2\left(4-c^2\right)(1-\beta)^2\cos^2\theta}{96} = G(c) \ \ (\text{say}) \end{array}.$$

Then

$$G'(c) = \frac{c^3}{24}(1-\beta)^4\cos^4\theta + \frac{c^3}{24}(1-\beta)^2\cos^2\theta + \frac{\left[8c-4c^3\right](1-\beta)^2\cos^2\theta}{32}. \tag{51}$$

Setting $G'(c) = 0$, the real critical points are $c_{01} = 0$, $c_{02} = \sqrt{\dfrac{6}{\left[2-(1-\beta)^2\cos^2\theta\right]}}$.

After some calculations we obtain the following cases:

Case 1: When $\beta \in \left[0, 1-\dfrac{1}{\sqrt{2}\cos\theta}\right]$, we observe that $c_{02} \geqslant 2$, that is c_{02}, is out of the interval $(0, 2)$. Therefore, the maximum value of $G(c)$ occurs at $c_{01} = 0$ or $c = c_{02}$, which contradicts our assumption of having the maximum value at the interior point of $c \in [0, 2]$. Since G is an increasing function, the maximum point of G must be on the boundary of $c \in [0, 2]$, that is $c = 2$. Thus, we have

$$\max_{0\leqslant c\leqslant 2} G(c) = G(2) = \frac{1}{6}(1-\beta)^4\cos^4\theta + \frac{1}{6}(1-\beta)^2\cos^2\theta.$$

Case 2: When $\beta \in \left(1-\dfrac{1}{\sqrt{2}\cos\theta}, 1\right)$, we observe that $c_{02} < 2$, that is c_{02}, is interior of the interval $[0, 2]$. Since $G''(c_{02}) < 0$, the maximum value of $G(c)$ occurs at $c = c_{02}$. Thus, we have

$$\begin{aligned} \max_{0\leqslant c\leqslant 2} G(c) = G(c_{02}) &= G\left(\sqrt{\frac{6}{\left[2-(1-\beta)^2\cos^2\theta\right]}}\right) \\ &= \frac{3(1-\beta)^2\cos^2\theta}{8\left[2-(1-\beta)^2\cos^2\theta\right]} \end{aligned}.$$

This completes the proof of the theorem.

Corollary 1: Let f given by (1) be in the class $K^*_\sigma(\beta)$. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \begin{cases} \dfrac{(1-\beta)^4}{6} + \dfrac{(1-\beta)^2}{6}, & \beta \in \left[0, \left(1-\dfrac{1}{\sqrt{2}}\right)\right] \\ \dfrac{3(1-\beta)^2}{8\left[2-(1-\beta)^2\right]}, & \beta \in \left(1-\dfrac{1}{\sqrt{2}}, 1\right) \end{cases}.$$

Corollary 2: Let f given by (1) be in the class K^*_σ. Then

$$\left|a_2a_4 - a_3^2\right| \leqslant \frac{1}{3}.$$

These two corollaries coincide with the results of Deniz *et al.* [28].

5. Conclusion

For specific values of α and β, the results obtained in this paper will generalize and unify the results of the earlier researchers in this direction.

Interested researchers can work upon finding an upper bound for $|a_2a_4 - \mu a_3^2|$ and $|a_n|$ for a real or complex μ.

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