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Microtubules Nonlinear Models Dynamics Investigations through the $\exp(-\Phi(\xi))$-Expansion Method Implementation

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Academic Editor: Reza Abedi

Received: 24 November 2015; Accepted: 20 January 2016; Published: 4 February 2016

Abstract: In this research article, we present exact solutions with parameters for two nonlinear model partial differential equations(PDEs) describing microtubules, by implementing the $\exp(-\Phi(\xi))$-Expansion Method. The considered models, describing highly nonlinear dynamics of microtubules, can be reduced to nonlinear ordinary differential equations. While the first PDE describes the longitudinal model of nonlinear dynamics of microtubules, the second one describes the nonlinear model of dynamics of radial dislocations in microtubules. The acquired solutions are then graphically presented, and their distinct properties are enumerated in respect to the corresponding dynamic behavior of the microtubules they model. Various patterns, including but not limited to regular, singular kink-like, as well as periodicity exhibiting ones, are detected. Being the method of choice herein, the $\exp(-\Phi(\xi))$-Expansion Method not disappointing in the least, is found and declared highly efficient.

Keywords: The $\exp(-\Phi(\xi))$-Expansion Method; models of microtubules; exact solutions; periodic solutions; rational solutions; solitary solutions; trigonometric solutions

1. Introduction

Microtubules (MTs) are major cytoskeletal proteins. MTs are cytoskeletal biopolymers shaped as nanotubes. They are hollow cylinders formed by Proto-Filaments (PFs) representing a series of proteins known as tubulin dimers. Each dimer is an electric dipole. These dimers are in a straight position within the PFs or placed in radial positions pointing out of the cylindrical surface. MTs compriseaninteresting type of protein structure that may be a good candidate for designing and manufacturing electronic nano-devices. MTs dynamical behavior is modeled by nonlinear partial differential equations (NPDEs). These equations are mathematical models of physical circumstances that emerge in various fields of engineering, plasma physics, solid state physics, optical fibers, chemistry, hydrodynamics, biology, fluid mechanics and geochemistry. To date solving NPDEs exactly or approximately, a plethora of methods have been in use. These include, but are not limited to, (G'/G)-expansion [1–6], Frobenius decomposition [7], local fractional variation iteration [8], local fractional series expansion [9], multiple exp-function algorithm [10,11], transformed rational function [12], exp-function method [13,14], trigonometric series function [15], inverse scattering [16], homogeneous balance [17,18], first integral [19–22], F-expansion [23–25], Jacobi function [26–29], Sumudu transform [30–32], solitary wave ansatz [33–36], novel (G'/G) -expansion [37–42], modified direct algebraic method [43,44], and last but not least, the $\exp(-\Phi(\xi))$-Expansion Method [45–50].

The objective of this paper is to apply the latter method, namely the $\exp(-\Phi(\xi))$-Expansion Method, to construct the exact solutions for the following two NPDEs modeling MT dynamics, [51–59]. In particular, in presenting the questions to be solved, for comparison purposes, we follow the initial set up established by Zayed and Alurrfi [56], solving the extended Riccatti equations (see Equations (1) and (2)). We then depart generically from their development by using an entirely distinct method, albeit we compare our final results with theirs in [56], keeping in focus the developments in [57–59], as well.

(i) The model of nonlinear dynamics of microtubules assuming a single longitudinal degree of freedom per tubulin dimer is described by the nonlinear PDE (see [59]),

$$m\frac{\partial^2 z(x,\,t)}{\partial t^2} - kl^2\frac{\partial^2 z(x,\,t)}{\partial x^2} - qE - Az(x\,,t) + Bz^3(x,t) + \gamma\frac{\partial z(x,\,t)}{\partial t} = 0 \tag{1}$$

where A, and B are positive parameters, m, is the mass of the dimer, $z(x,t)$, is the traveling wave, E is the magnitude of intrinsic electric field, l, is the MT length, $q > 0$, is the excess charge within the dipole, γ, is the viscosity coefficient and, k, is a harmonic constant describing the nearest-neighbor interaction between the dimers belonging to the same PFs. In [48], authors have used the Jacobi elliptic function method to find the exact solutions of Equation (1), the physical details and derivations of which were discussed there, although omitted here for obvious reasons.

(ii) The nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs:

$$I\frac{\partial^2 z(x,\,t)}{\partial t^2} - kl^2\frac{\partial^2 z(x,\,t)}{\partial x^2} + pEz(x\,,t) - \frac{pE}{6}z^3(x,t) + \Gamma\frac{\partial z(x,\,t)}{\partial t} = 0 \tag{2}$$

Here, $z(x,t)$, is the corresponding angular displacement when the whole dimer rotates and, l, is the MT length, p is the magnitude of intrinsic electric field, k, stands for inter-dimer bonding interaction within the same PFs, I, is the moment of inertia of the single dimer and Γ is the viscosity coefficient. In [57], authors have used the simple equation method to find the exact solutions of Equation (2), after relating physical aspects and equation derivation being omitted here.

This paper is organized as follows: In Section 2, we give the description of the $\exp(-\Phi(\xi))$-Expansion Method, while in Section 3, we apply the said method to solve the given NPDEs, Equations (1) and (2). In Section 4, physical explanations are given, followed by the conclusion in Section 5. The paper ends with relevant acknowledgments, and a rich list of references for interested readers.

2. Description of the $\exp(-\Phi(\xi))$-Expansion Method

Following th initial setup in [56], we consider the nonlinear evolution equation in the form,

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots\cdots) = 0 \tag{3}$$

where, F, is a polynomial in, $u(x,t)$, and its partial derivatives, involving nonlinear terms and highest order derivatives. The focal steps of the method are as follows:

Step 1. It is well known that, for a given wave equation, a travelling wave, $u(\xi)$, is a solution which depends upon, x, and, t, only through a unified variable, ξ, such that,

$$u(x,t) = u(\xi), \xi = k_1 x + \omega t \tag{4}$$

where, k_1 and ω, are constants. Based on this we have,

$$\frac{\delta}{\delta t} = \omega\frac{\delta}{\delta\xi}, \frac{\delta^2}{\delta t^2} = \omega^2\frac{\delta^2}{\delta\xi^2}, \frac{\delta}{\delta x} = k_1\frac{\delta}{\delta\xi}, \text{ and, } \frac{\delta^2}{\delta x^2} = {k_1}^2\frac{\delta^2}{\delta\xi^2} \tag{5}$$

and so on, for other derivatives.

We reduce Equation (3) to the following ODE:

$$Q(u, u', u'', \cdots\cdots) = 0 \tag{6}$$

Here, Q is a polynomial in, $u(\xi)$, and its total derivatives, such that $' = \frac{d}{d\xi}$.

Step 2. We assume that Equation (6) has the formal solution:

$$u(\xi) = \sum_{i=0}^{N} A_i(\exp(-\Phi(\xi)))^i \tag{7}$$

where, the A_i's are constants to be determined, such that $A_N \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following ODE:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda \tag{8}$$

Consequently, we get the following possibilities for Equation (8):

Cluster 1: When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$, we get,

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + E)\right) - \lambda}{2\mu}\right) \tag{9}$$

Cluster 2: When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$, we get,

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)}\tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2}(\xi + E)\right) - \lambda}{2\mu}\right) \tag{10}$$

Cluster 3: When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$, we obtain,

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1}\right) \tag{11}$$

Cluster 4: When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$, we obtain

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi + E) + 2)}{\lambda^2(\xi + E)}\right) \tag{12}$$

Cluster 5: When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$, we then have,

$$\Phi(\xi) = \ln(\xi + E) \tag{13}$$

where $A_N, \cdots\cdots, V, \lambda, \mu$, are constants to be determined, such that $A_N \neq 0$. The positive integer, m, can be determined by considering the homogeneous balance between nonlinear terms and the highest order derivatives occurring in the ODE in Equation (6), after using Equation (7).

Step 3. We interchange Equation (7) into Equation (6) and then we expand the function $\exp(-\Phi(\xi))$. As a result of this interchange, we get a polynomial of $\exp(-\Phi(\xi))$. We equate all the coefficients of same power of $\exp(-\Phi(\xi))$ to zero. This procedure yields a system of algebraic equations which could be solved to obtain the values of $A_N, \cdots\cdots, V, \lambda, \mu$ which after substitution into Equation (7) along with general solutions of Equation (8) completes the setup for getting the traveling wave solutions of the NPDE in Equation (3).

3. Applications

In this section, we will apply the $\exp(-\Phi(\xi))$-Expansion Method described in Section 2 to find the exact solutions of the NPDE Equations (1) and (2).

3.1. Exact Solutions of the NPDE Equation(1)

In this subsection, we find the exact wave solutions of Equation (1). To this end, we use the transformation (4) to reduce Equation (1) into the nonlinear ordinary differential equation (NODE),

$$P\psi''(\xi) - Q\psi'(\xi) - \psi(\xi) + \psi^3(\xi) - R = 0 \tag{14}$$

where,

$$P = \frac{m\omega^2 - kl^2{k_1}^2}{A}, Q = \frac{\gamma\omega}{A}, R = \frac{qE}{A\sqrt{A/B}} \tag{15}$$

and,

$$z(\xi) = \sqrt{\frac{A}{B}}\psi(\xi) \tag{16}$$

Balancing, $\psi''(\xi)$, with, $\psi^3(\xi)$, in Equation (14), we get $N = 1$. Consequently, we have,

$$\psi(\eta) = A_0 + A_1(\exp(-\Phi(\xi))) \tag{17}$$

where A_0, A_1 are constants to be determined such that $A_N \neq 0$, while λ, μ, are arbitrary.

Substituting Equation (17) into Equation (14) and equating the coefficients of $\exp(-\Phi(\xi))^3$, $\exp(-\Phi(\xi))^2$, $\exp(-\Phi(\xi))^1$, $\exp(-\Phi(\xi))^0$, to zero, we respectively obtain,

$$\exp(-\Phi(\xi))^3 : 2PA_1 + {A_1}^3 = 0 \tag{18}$$

$$\exp(-\Phi(\xi))^2 : 3A_0{A_1}^2 + QA_1 + 3PA_1\lambda = 0 \tag{19}$$

$$\exp(-\Phi(\xi))^1 : 2PA_1\mu + P\lambda^2A_1 - A_1 + QA_1\lambda + 3{A_0}^2A_1 = 0 \tag{20}$$

and,

$$\exp(-\Phi(\xi))^0 : A_0 - R + PA_1\mu\lambda + QA_1\mu + {A_0}^3 = 0 \tag{21}$$

Now, solving Equations (18)–(21) yields,

$$\begin{gathered} A_0 = A_0,\ A_1 = \alpha,\ \lambda = -\frac{1}{3P}(3A_0\alpha + Q),\ \text{and,} \\ \mu = \frac{1}{18P^2}(3A_0\alpha Q + 2Q^2 + 9P - 9{A_0}^2P),\ R = \frac{1}{27P^2}\{Q\alpha(2Q^2 + 9P)\} \end{gathered} \tag{22}$$

where, $\alpha = \pm\sqrt{-2P}$, and A_0, P, and, Q, are arbitrary constants.

Substituting Equation (22) into Equation (17), we obtain

$$\psi(\xi) = A_0 + \alpha(\exp(-\Phi(\xi))) \tag{23}$$

Now, substituting Equations (9)–(13) into Equation (23) respectively, we get the following five traveling wave solutions of the NPDE Equation (1).

When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$z_1(\xi) = \sqrt{\frac{A}{B}}\{A_0 - \alpha(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)) + \lambda}\} \tag{24}$$

where E is an arbitrary constant.

When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$z_2(\xi) = \sqrt{\frac{A}{B}}\{A_0 + \alpha(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)) - \lambda}\} \tag{25}$$

where, E, is an arbitrary constant.

When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$z_3(\xi) = \sqrt{\frac{A}{B}}\{A_0 + \alpha(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1})\} \tag{26}$$

where, E, is an arbitrary constant.

When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$z_4(\xi) = \sqrt{\frac{A}{B}}\{A_0 - \alpha(\frac{\lambda^2(\xi + E)}{2(\lambda(\xi + E)) + 2)})\} \tag{27}$$

where, E, is an arbitrary constant.

When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$z_5(\xi) = \sqrt{\frac{A}{B}}\{A_0 + \alpha(\frac{1}{\xi + E})\} \tag{28}$$

where, E, is an arbitrary constant.

3.2. Exact Solutions of the NPDE Equation (2)

In this subsection, we find the exact solutions of Equation (2). To this end, we use the transformation Equation (4) to reduce Equation (2) into the following NODE,

$$S\psi''(\xi) - T\psi'(\xi) + \psi(\xi) - \psi^3(\xi) = 0 \tag{29}$$

where,

$$S = \frac{I\omega^2 - kl^2{k_1}^2}{pE}, T = \frac{\Gamma\omega}{pE} \tag{30}$$

and,

$$z(\xi) = \sqrt{6}\psi(\xi) \tag{31}$$

Balancing $\psi''(\xi)$ with $\psi^3(\xi)$ in Equation (29), we get $N = 1$. Consequently, we have the formal solution of Equation (29), as follows:

$$\psi(\xi) = A_0 + A_1(\exp(-\Phi(\xi))) \tag{32}$$

where A_0, A_1 are constants to be determined such that $A_N \neq 0$, while λ, μ, are arbitrary. Substituting Equation (32) into Equation (29) and equating the coefficients of $\exp(-\Phi(\xi))^3$, $\exp(-\Phi(\xi))^2$, $\exp(-\Phi(\xi))^1$, $\exp(-\Phi(\xi))^0$ to zero, we respectively obtain

$$\exp(-\Phi(\xi))^3 : \ 2SA_1 - {A_1}^3 = 0 \tag{33}$$

$$\exp(-\Phi(\xi))^2 : 3SA_1\lambda - 3A_0{A_1}^2 + TA_1 = 0 \tag{34}$$

$$\exp(-\Phi(\xi))^1 : \ A_1 + 2SA_1\mu + S\lambda^2 A_1 + TA_1\lambda - 3{A_0}^2 A_1 = 0 \tag{35}$$

and,

$$\exp(-\Phi(\xi))^0 : A_0 + SA_1\mu\lambda + TA_1\mu - {A_0}^3 = 0 \tag{36}$$

Solving the Equation (33)–(36) yields:

Cluster 1: We have,

$$A_0 = A_0, A_1 = \frac{2}{3}T, \lambda = \frac{3}{2T}(2A_0 - 1), \mu = \frac{9}{4T^2}({A_0}^2 - A_0), S = \frac{2}{9}T^2 \tag{37}$$

Of course, A_0, T, are arbitrary constants.

Cluster 2 : We have,

$$A_0 = A_0,\ A_1 = -\frac{2}{3}T, \lambda = -\frac{3}{2T}(2A_0 + 1), \mu = \frac{9}{4T^2}({A_0}^2 + A_0), S = \frac{2}{9}T^2 \tag{38}$$

where A_0, T are arbitrary constants.

For cluster 1, substituting Equation (37) into Equation (32), we obtain

$$u(\xi) = A_0 + \frac{2T}{3}(\exp(-\Phi(\xi))) \tag{39}$$

while, for cluster 2, substituting Equation (38) into Equation (32), we obtain

$$u(\xi) = A_0 - \frac{2T}{3}(\exp(-\Phi(\xi))) \tag{40}$$

Now, substituting Equations (9)–(13) into Equation (39), respectively, we get the following five traveling wave solutions of the NPDE Equation (2).

When, $\mu \neq 0, \lambda^2 - 4\mu > 0$,

$$z_1(\xi) = \sqrt{6}\{A_0 - \frac{2T}{3}(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)) + \lambda}\} \tag{41}$$

where E is an arbitrary constant.

When $\mu \neq 0, \lambda^2 - 4\mu < 0$,

$$z_2(\xi) = \sqrt{6}\{A_0 + \frac{2T}{3}(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)) - \lambda})\} \tag{42}$$

where E is an arbitrary constant.

When, $\mu = 0, \lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$z_3(\xi) = \sqrt{6}\{A_0 + \frac{2T}{3}(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1})\} \tag{43}$$

where E is an arbitrary constant.

When $\mu \neq 0, \lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$z_4(\xi) = \sqrt{6}\{A_0 - \frac{2T}{3}(\frac{\lambda^2(\xi + E)}{2(\lambda(\xi + E)) + 2)})\} \tag{44}$$

where E is an arbitrary constant.

When $\mu = 0, \lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$z_5(\xi) = \sqrt{6}\{A_0 + \frac{2T}{3}(\frac{1}{\xi + E})\} \tag{45}$$

where E is an arbitrary constant.

At this point, inserting Equations (9)–(13) into Equation (40), respectively, we get the following other five traveling wave solutions of the NPDE Equation (2).

When, $\mu \neq 0, \lambda^2 - 4\mu > 0$,

$$z_6(\xi) = \sqrt{6}\{A_0 + \frac{2T}{3}(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E)) + \lambda})\} \tag{46}$$

where, E, is an arbitrary constant.

When $\mu \neq 0, \lambda^2 - 4\mu < 0$,

$$z_7(\xi) = \sqrt{6}\{A_0 - \frac{2T}{3}(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + E)) - \lambda})\} \tag{47}$$

where, E, is an arbitrary constant.

When, $\mu = 0, \lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$z_8(\xi) = \sqrt{6}\{A_0 - \frac{2T}{3}(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1})\} \tag{48}$$

where, E, is an arbitrary constant.

When, $\mu \neq 0, \lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$z_9(\xi) = \sqrt{6}\{A_0 + \frac{2T}{3}(\frac{\lambda^2(\xi + E)}{2(\lambda(\xi + E)) + 2)})\} \tag{49}$$

where, E, is an arbitrary constant.

When, $\mu = 0, \lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$z_{10}(\xi) = \sqrt{6}\{A_0 - \frac{2T}{3}(\frac{1}{\xi + E})\} \tag{50}$$

where, E, is an arbitrary constant.

4. Comparison

The papers [58,59] by Zdravkovic *et al.* are key to our present work. They collectively considered solutions of the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs using the simplest equation method. The solutions of the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs obtained by the $\exp(-\Phi(\xi))$-Expansion Method are different from those of the simplest equation method. It is n oteworthy to point out that some of our solutions coincide with already published results, if parameters taken particular values which authenticate our solutions. Moreover, Zdravkovic *et al.* [58] investigated the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs using the simplest equation method to obtain exact solutions via the simplest equation method and achieved only two solutions (see Appendix). Furthermore, ten solutions of the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs are constructed by applying the $\exp(-\Phi(\xi))$-Expansion Method. Zdravkovic *et al.* [58] (see also [59]) apply the simplest equation method to the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs, and they only solve kink type solutions, but we apply the $\exp(-\Phi(\xi))$-Expansion Method to the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs and solve kink type solutions, singular kink type solutions and plane periodic type solutions. On the other hand, the auxiliary equation used in this paper is different, so obtained solutions are also different. Similarly, for any nonlinear evolution equation, it can be shown that the $\exp(-\Phi(\xi))$-Expansion Method is much more direct and user-friendly than other methods.

5. Physical Interpretations of Some Obtained Solutions

In this section, attempting to shed lights on the corresponding physical behavior, we to discuss nonlinear dynamics of MTs whether as nano-bioelectronics transmission lines like or radially dislocated MTs, based on the obtained traveling wave solutions, from Equations (24)–(28), and (41)–(50), respectively. We examine the nature of some obtained solutions of Equations (1) and (2) by selecting particular values of the parameters and graphing the resulting exact solutions using mathematical software Maple 13, represented in Figures 1–6.

From our obtained solutions, we observe that Equations (24)–(28), and (41)–(50), exude kink type solitons, singular kink shape solitons, and periodic solutions. Equation (24) shows kink shaped soliton profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 1$, $\lambda = 3$, $E = 1$, within the interval $-10 \leqslant x, t \leqslant 10$ which is represented in Figures 1 and 2. Equation (25) provides a periodic solution profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 3$, $\lambda = 1$, $E = 5$ within the interval $-1 \leqslant x, t \leqslant 1$, which is represented in Figures 3 and 4. Equation (26) provides a singular kink soliton profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 0$, $\lambda = 2$, $E = 1$, within the interval $-10 \leqslant x, t \leqslant 10$, which is represented in Figures 5 and 6. Equations (27) and (28) also represent singular kink type wave solutions which are similar to Figures 5 and 6. Equations (41) and (46) provide kink soliton profile, for $A_0 = 2$, $T = \frac{3}{2}$, $\omega = -1$, $k_1 = 1$, $\mu = 1$, $\lambda = 3$, and $E = 1$, within the interval, $-10 \leqslant x, t \leqslant 10$, as in Figures 1 and 2. Equations (42) and (47) provide periodic solutions for, $A_0 = 2$, $T = \frac{3}{2}$, $\omega = -1$, $k_1 = 1$, $\mu = 3$, $\lambda = 1$, $E = 5$, within the interval, $-1 \leqslant x, t \leqslant 1$, as in Figures 3 and 4. Equations (43) and (48), provide singular kink soliton profiles for, $A_0 = 2$, $T = \frac{3}{2}$, $\omega = -1$, $k_1 = 1$, $\mu = 0$, $\lambda = 2$, and, $E = 1$, within the interval $-10 \leqslant x, t \leqslant 10$, as in Figures 5 and 6. Equations (44) and (45), as well as Equations (49) and (50), also represent singular Kink type wave solutions which are similar to Figures 5 and 6.

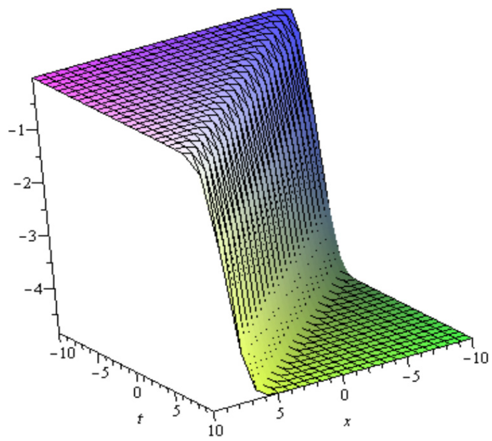

Figure 1. The solitary wave 3D graphics of Equation (24) shows a kink shaped soliton profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 1$, $\lambda = 3$, $E = 1$ within the interval $-10 \leqslant x, t \leqslant 10$.

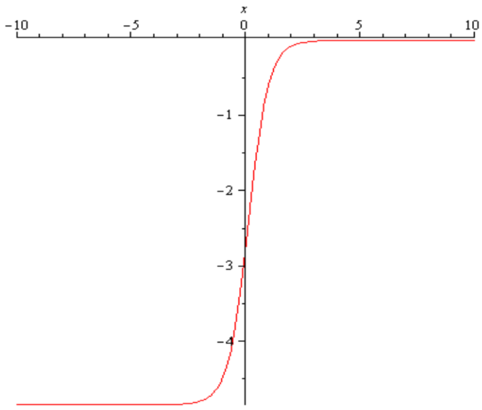

Figure 2. The solitary wave 2D graphics of Equation (24) shows a kink shaped soliton profile for, $A_0 = 1, m = 1, \omega = -1, k_1 = 1, k = 2, l = 2, A = 2, B = 3, \mu = 1, \lambda = 3, E = 1, t = 2$.

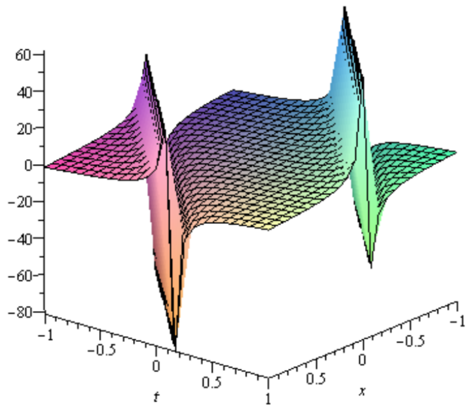

Figure 3. The solitary wave 3D graphics of Equiation (25) provides a periodic solution profile for, $A_0 = 1, m = 1, \omega = -1, k_1 = 1, k = 2, l = 2, A = 2, B = 3, \mu = 3, \lambda = 1, E = 5$ within the interval $-1 \leqslant x, t \leqslant 1$.

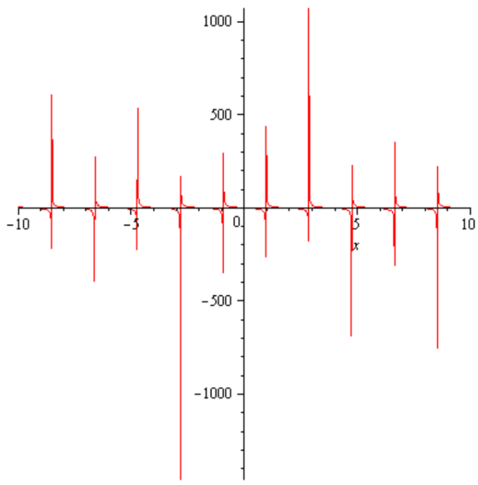

Figure 4. The solitary wave 2D graphics of Equation (25) provides a periodic solution profile for, $A_0 = 1, m = 1, \omega = -1, k_1 = 1, k = 2, l = 2, A = 2, B = 3, \mu = 3, \lambda = 1, E = 5, t = 2$.

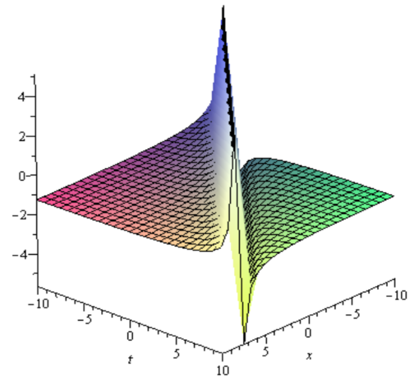

Figure 5. The solitary wave 3D graphics of Equation (26) provides a singular kink soliton profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 0$, $\lambda = 2$, $E = 1$ within the interval $-10 \leqslant x, t \leqslant 10$.

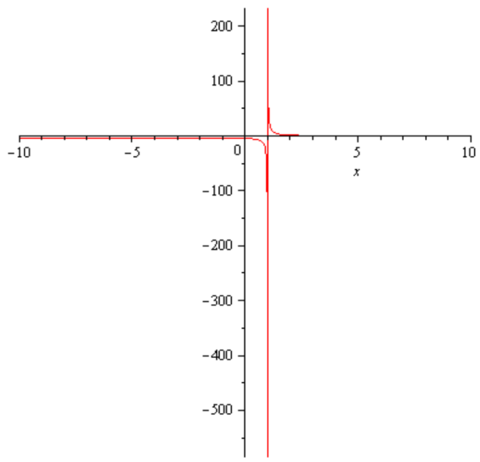

Figure 6. The solitary wave 2D graphics of Equation (26) provides a singular kink soliton profile for, $A_0 = 1$, $m = 1$, $\omega = -1$, $k_1 = 1$, $k = 2$, $l = 2$, $A = 2$, $B = 3$, $\mu = 0$, $\lambda = 2$, $E = 1$, $t = 2$.

6. Conclusions

The $\exp(-\Phi(\xi))$-Expansion Method has been appliedto Equations (1) and (2), which describe the nonlinear dynamics of microtubules assuming a single longitudinal degree of freedom per tubulin dimer [59] and the dynamics of radial dislocations in MTs, respectively. The said method was instrumental in the provision of new analytical solutions such as kink type solutions, singular kink type solutions and plane periodic type solutions which are shown in Figures 1–3. On comparing our results in this paper with the well-known results obtained in [50,58,59], we deduce that our results are new and not published elsewhere. All analytical solutions obtained by The $\exp(-\Phi(\xi))$-Expansion Method in the paper have been controlled, whether they are verified to Equation (1) and Equation (2) with the aid of commercial software Maple, and all new solutions have been verified to the original equations Equations (1) and (2). Zayed and Alurrfi [56] recently solved the two equations but used the alternative generalized Ricatti projective method. There, they also obtained trigonometric, hyperbolic and rational solutions but failed to obtain the exponential ones that we got. Our distinction resides mostly in obtaining extra solution types using our method. Of course, the choice of parameters yields different facets of the solutions and their graphic presentation so as to be type representative, without rendering the paper so voluminous, should more realizations be expected.

Acknowledgments: The authors acknowledge and salute the mathematics editorial board management, and thank the consequent anonymous referees' diligent efforts and critiques that helped improve the flow, style and scientific veracity of this paper. Furthermore, Fethi Bin Muhammad Belgacem wishes to acknowledge the support of the Public Authority for Applied Education and Training, Kuwait, through the Research Department grant No. PAAET RDBE-13-09.

Author Contributions: In light of the recent scholarly interest in nano-bioelctronics Microtubules dynamic behavior, both authors agreed to investigate this phenomenon but with the advent of an original method not alreay used in the literature, which gave the authors the opportunity to compare with existing solutions. The contribution is therefore equally and evenly shared between both authors. Due to the subject importance and the desire that common understanding of these physical phenomena be widely spread, the authors invite interested readers to communicate, request, share relevant pieces of information, and collaborate, if so wished!

Conflicts of Interest: The authors declare no conflict of interest.

Appendix

Zdravkovic *et al.* [56] studied solutions of of the nonlinear PDE describing the nonlinear dynamics of radially dislocated MTs using the simplest equation method and achieved the following exact solutions:

$$\psi_1(x,\, t) = \pm\frac{1}{2}\left[1 + \tanh y + \frac{1}{\cosh^2 y(d + \tanh y)}\right]$$

$$\psi_2(x,\, t) = \pm\frac{1}{2}\left[1 + \tanh(\frac{y}{2}) + \frac{1}{\sinh y)}\right]$$

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