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Article

Free W^*-Dynamical Systems From p-Adic Number Fields and the Euler Totient Function

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Abstract: In this paper, we study relations between free probability on crossed product W^*-algebras with a von Neumann algebra over p-adic number fields $\mathbb{Q}_p$ (for primes p), and free probability on the subalgebra Φ, generated by the Euler totient function ϕ, of the arithmetic algebra $\mathcal{A}$, consisting of all arithmetic functions. In particular, we apply such free probability to consider operator-theoretic and operator-algebraic properties of W^*-dynamical systems induced by $\mathbb{Q}_p$ under free-probabilistic (and hence, spectral-theoretic) techniques.

Keywords: p-Adic number fields $\mathbb{Q}_p$; p-Adic von neumann algebras $\mathfrak{M}_p$; dynamical systems induced by $\mathbb{Q}_p$; arithmetic functions; the arithmetic algebra $\mathcal{A}$; the euler totient function ϕ

1. Introduction

While in standard probability spaces, the random variables are functions (measurable with respect to a prescribed σ-algebra), and hence their analysis entails only abelian algebras of functions. By contrast, in *free probability*, one studies (both) noncommutative (and commutative) random variables (on algebras) in terms of fixed linear functionals. In the classical case, independence is fundamental, and we get the notion of products of probability spaces. The analogous concept in the noncommutative setting is freeness and free products. Freeness (or free independence) is then studied in connected with free products. The free probability theory was pioneered by D. Voiculescu (e.g., [1,2]) and motivated by a question in von Neumann algebra (alias W^*-algebra) theory, the free-group factors isomorphism problem (e.g., [2,3]). There has been a recent renewed interest in analysis on free probability spaces, especially in connection with free random processes (e.g., [4,5])

In this paper, we consider connections between the two independent free-probabilistic models induced from number-theoretic objects, (i) free probability spaces $(\mathfrak{M}_p, \varphi_p)$ of the von Neumann algebras $\mathfrak{M}_p$ generated by p-adic number fields $\mathbb{Q}_p$ and the corresponding integrations φ_p on $\mathfrak{M}_p$, (e.g., [6–8]) and (ii) free probability spaces $(\mathcal{A}, g_p)$ of the algebra $\mathcal{A}$ consisting of all arithmetic functions, equipped with the usual functional addition $(+)$ and the convolution $(*)$, and the point-evaluation linear functionals g_p on $\mathcal{A}$, for all primes p (e.g., [9–12]). And we apply such relations to study W^*-dynamical systems induced by $\mathbb{Q}_p$ (e.g., [9]).

In particular, for the later models (ii), we construct free-probabilistic sub-structures (Φ_p, g_p) of $(\mathcal{A}, g_p)$ (under suitable quotient) for primes p. Here, Φ_p is an subalgebra of $\mathcal{A}$ (under quotient) generated by the *Euler totient function* $\phi \in \mathcal{A}$, defined by

$$\phi(n) \overset{def}{=} \left| \left\{ k \in \mathbb{N} \;\middle|\; \begin{matrix} 1 \leq k \leq n \\ \gcd(k, n) = 1 \end{matrix} \right\} \right|$$

for all $n \in \mathbb{N}$.

The main purpose of this paper is to show the free probability on W^*-dynamical systems induced by $\mathbb{Q}_p$ is related to the free probability on the corresponding W^*-dynamical systems acted by Φ_p. Our results not only relate the calculus on $\mathbb{Q}_p$ with the free probability on Φ_p (Also, see [9]), but also provide better tools for studying non-Archimedean p-adic (or Adelic) dynamical systems.

We considered how *primes* (or *prime numbers*) act on operator algebras, in particular, on von Neumann algebras. The relations between primes and operator algebra theory have been studied in various different approaches. For instance, in [11], we studied how primes act "on" certain von Neumann algebras generated by p-adic and Adelic measure spaces. Also, the primes as operators in certain von Neumann algebras, have been studied in [8].

The main results deal with explicit computations for our free-dynamical systems in Sections 5 and 6, and structure theorems in Sections 8–10. The first four sections deal with some preliminaries (free probability systems generated by arithmetic functions, and their prime components), which we need in the proofs of main results (Theorems 5.1, 6.3, 8.6, 9.3, 9.4, and 10.2).

We address-and-summarize the main theorems, (i) in a given free probability space, either global, or one of the prime factors, how do we identify mutually free sub-systems? See, for example, Theorem 8.6; and (ii) how do our global systems factor in terms of the prime free probability spaces? See especially Theorem 9.3; and (iii) how do we apply the above results from (i) and (ii), see Theorem 10.2.

Independently, in [9,10], we have studied primes as *linear functionals* acting on arithmetic functions. *i.e.*, each prime p induces a free-probabilistic structure $(\mathcal{A}, g_p)$ on arithmetic functions $\mathcal{A}$. In such a case, one can understand arithmetic functions as *Krein-space operators*, via certain *representations* (See [11,12]).

These studies are all motivated by well-known number-theoretic results (e.g., [13–17]) with help of free probability techniques (e.g., [8,11,12]).

In modern number theory and its application, *p-adic analysis* provides an important tool for studying geometry at small distance (e.g., [18]). it is not only interested in various mathematical fields but also in related scientific fields (e.g., [11,12,15,19]). The *p-adic number fields* $\mathbb{Q}_p$ and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*, *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [9,19,20]).

In earlier papers [11,12], the authors studied harmonic analysis of arithmetic functions, leading to free probability spaces $(\mathcal{A},\ g_p)$ indexed by the prime numbers p. In [21], we considered von Neumann algebras $L^\infty(\mathbb{Q}_p)$ induced by p-adic number fields $\mathbb{Q}_p$, and realized the connection between non-Archimedean calculus on $L^\infty(\mathbb{Q}_p)$ and free probability on $(\mathcal{A},\ g_p)$, liked via Euler totient function ϕ. The purpose of the present paper is to enlarge such connections between them, and apply such connections to non-Archimedean p-adic or Adelic dynamical systems.

In [8], the first-named author constructed W^*-dynamical systems induced by $\mathbb{Q}_p$, by understanding the σ-algebra $\sigma(\mathbb{Q}_p)$ as a semigroup $(\sigma(\mathbb{Q}_p),\ \cap)$ under set-intersection $\cap$. By acting this semigroup $\sigma(\mathbb{Q}_p)$ on an arbitrary von Neumann algebra M via a semigroup-action α, one can establish a W^*-dynamical system $(\sigma(\mathbb{Q}_p),\ M,\ \alpha)$. Then the corresponding crossed product algebra $M \times_\alpha \sigma(\mathbb{Q}_p)$ is constructed and it is $*$-isomorphic to the conditional tensor product algebra $M \otimes_\alpha L^\infty(\mathbb{Q}_p)$. The free probability on such von Neumann algebras was studied in [8].

In [21], the author and Jorgensen considered the connection between calculus (in particular, integration) on $L^\infty(\mathbb{Q}_p)$ and free probability on Φ_p (inherited from the free probability on $\mathcal{A}$ under the linear functional g_p). We realized that, for any $f \in L^\infty(\mathbb{Q}_p)$, there exists $h \in \Phi_p$ (under quotient), such that

$$\int_{\mathbb{Q}_p} f d\rho_p = g_p(h)$$

and vice versa.

We here apply the results of [21] to the study of W^*-dynamical systems.

In Section 2, we introduce basic concepts for the paper. In Sections 3–6, we briefly consider main results of [8]. The main results of [21] are reviewed in Sections 7 and 8. In Sections 9 and 10, we re-construct free probability on the W^*-dynamical systems induced by $\mathbb{Q}_p$ in terms of W^*-dynamical systems induced by Φ_p.

2. Definitions and Background

For related themes from W^*-Dynamical Systems, see [22]. For useful themes from harmonic analysis of number fields, both commutative and noncommutative, see [23–26]. Some related themes from mathematical physics are found in [17,27,28].

In this section, we introduce basic definitions and backgrounds of the paper.

2.1. p-Adic Number Fields $\mathbb{Q}_p$

Throughout this section, let p be a fixed *prime*, and let $\mathbb{Q}_p$ be the *p-adic number field* for p. This set $\mathbb{Q}_p$ is by definition the completion of the *rational numbers* $\mathbb{Q}$ with respect to the *p-adic norm*

$$|q|_p = \left|p^k \frac{a}{b}\right| = \left(\frac{1}{p}\right)^k$$

for $q = p^k \frac{a}{b} \in \mathbb{Q}$, for some $k \in \mathbb{Z}$. Remark here that the norm $|.|_p$ satisfies that

$$|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$$

and hence, it is *non-Archimedean*. The *topology for* $\mathbb{Q}_p$ is induced by the non-Archimedean metric d_p induced by the p-adic norm $|.|_p$

$$d_p(q_1, q_2) = |q_1 - q_2|_p$$

for all $q_1, q_2 \in \mathbb{Q}$

Under topology, $\mathbb{Q}_p$ is locally compact and totally disconnected as a *topological space*, and contains a maximal compact subring,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

We call $\mathbb{Z}_p$, the *unit disk of* $\mathbb{Q}_p$, and all elements of $\mathbb{Z}_p$ are said to be the *p-adic integers* in $\mathbb{Q}_p$. The unit disk $\mathbb{Z}_p$, as an algebraic object, is a *discrete valuation ring*, in the sense that: it is a principal ideal domain with a unique non-zero prime ideal (generated by p). The ideal (p) is also a maximal ideal, and hence, the quotient

$$\mathbb{Z}_p/(p) \overset{\text{Field}}{=} \mathbb{Z}/p\mathbb{Z}$$

forms a field, called the *residue field* of $\mathbb{Z}_p$. Similarly, one can verify that

$$\mathbb{Z}_p/(p^k) \overset{\text{Ring}}{=} \mathbb{Z}/p^k\mathbb{Z}, \text{ for } k \in \mathbb{N}$$

Using powers of the ideal (p), we obtain a particularly nice description for the topology of $\mathbb{Q}_p$. It has *neighborhood bases of zero* consisting of the *compact open* (*additive*) *subgroups*

$$p^k\mathbb{Z}_p = \{p^k x : x \in \mathbb{Z}_p\}, \text{ for } k \in \mathbb{Z}$$

In fact, set-theoretically, one has

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\cup} p^k\mathbb{Z}_p$$

In other words, if we consider $\mathbb{Q}_p$ as an additive group, then it is *locally profinite*.

Recall that an arbitrary group is called *profinite*, if it is both locally profinite and compact. So, the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is profinite, since $\mathbb{Q}_p$ is locally profinite and $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$.

Recall also that any profinite group can be realized as the *inverse limit* of finite groups. Since $\mathbb{Z}_p$ is compact and has a neighborhood base of zero consisting of compact open subgroups obtained by taking k to be a natural number above, there exists an isomorphism φ,

$$\varphi : \mathbb{Z}_p \to \varprojlim \left(\mathbb{Z}_p \,/\, (p^k)\right) = \varprojlim \left(\mathbb{Z} \,/\, p^k\mathbb{Z}\right)$$

such that

$$\varphi(x) = (x \text{mod}(p^k))_{k \in \mathbb{N}}$$

for all $x \in \mathbb{Z}_p$. This inverse limit runs over finite groups since

$$\mathbb{Z}_p/(p^k) \cong \mathbb{Z}/p^k\mathbb{Z}$$

are finite groups, for any $k \in \mathbb{N}$.

For a fixed prime p, note that the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is then a compact group with the induced $(+)$-operation on $\mathbb{Q}_p$, passed to the projective limit. Hence, $\mathbb{Z}_p$ has a unique normalized Haar measure ρ_p, satisfying

$$\rho_p(\mathbb{Z}_p) = 1$$

and

$$\rho_p(x + S) = \rho_p(S + x) = \rho_p(A)$$

for all Borel subsets $S \subseteq \mathbb{Z}_p$, and $x \in \mathbb{Z}_p$. Here,

$$x + S = \{x + a : a \in S\}$$

where $(+)$ is the p-adic addition on $\mathbb{Z}_p$ (inherited from that on $\mathbb{Q}_p$).

One can check that the *dual character group* $\mathbb{Z}_p^*$ of $\mathbb{Z}_p$,

$$\mathbb{Z}_p^* = \underset{k\in\mathbb{N}}{\cup} p^{-k}\mathbb{Z}$$

and it is an injective limit of the group inducing

$$p^{-k}\mathbb{Z} \hookrightarrow p^{-(k+1)}\mathbb{Z}$$

So, there is an associated Fourier transform

$$f \in L^2(\mathbb{Z}_p, \rho_p) \longmapsto \widehat{f} \in l^2(\mathbb{Z}_p^*)$$

such that

$$\widehat{f}(\xi) = \int_{\mathbb{Z}_p} \overline{< \xi, x >} f(x) d\rho_p(x)$$

for all $\xi \in \mathbb{Z}_p^*$. Moreover, we have

$$\sum_{\xi \in \mathbb{Z}_p^*} \left| \widehat{f}(\xi) \right|^2 = \int_{\mathbb{Z}_p} |f|^2 \, d\rho_p$$

The *boundary* U_p *of the unit disk* $\mathbb{Z}_p$ is defined by

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p$$

We call U_p, the *unit circle of* $\mathbb{Q}_p$.

Under the Haar measure ρ_p on $\mathbb{Q}_p$, we have

$$\rho_p \left(a + p^k\mathbb{Z}_p \right) = \rho_p \left(p^k\mathbb{Z}_p \right) = \frac{1}{p^k}$$

and

$$\rho_p \left(a + p^k U_p \right) = \rho_p \left(p^k U_p \right) = \frac{1}{p^k} - \frac{1}{p^{k+1}} = p^k \left(1 - \frac{1}{p} \right)$$

for all $a \in \mathbb{Q}_p$ and for all $k \in \mathbb{Z}$, where

$$p^k X = \{p^k x : x \in X\}, \text{ for all subsets } X \text{ of } \mathbb{Q}_p$$

2.2. Free Probability

In this section, we briefly introduce *free probability*. Free probability is one of a main branch of *operator algebra theory*, establishing noncommutative probability theory on noncommutative (and hence, on commutative) algebras (e.g., pure algebraic algebras, topological algebras, topological $*$-algebras, *etc.*).

Let $\mathfrak{A}$ be an arbitrary algebra over the complex numbers $\mathbb{C}$, and let $\psi : \mathfrak{A} \to \mathbb{C}$ be a linear functional on $\mathfrak{A}$. Then the pair $(\mathfrak{A},\ \psi)$ is called a free probability space (*over* $\mathbb{C}$). All operators $a \in (\mathfrak{A},\ \psi)$ are called *free random variables*. Remark that free probability spaces are dependent upon the choice of linear functionals.

Let $a_1,\ ...,\ a_s$ be a free random variable in a $(\mathfrak{A},\ \psi)$, for $s \in \mathbb{N}$. The *free moments of* $a_1,\ ...,\ a_s$ are determined by the quantities

$$\psi(a_{i_1} ... a_{i_n})$$

for all $(i_1,\ ...,\ i_n) \in \{1,\ ...,\ s\}^n$, for all $n \in \mathbb{N}$

and the *free cumulants* $k_n(a_{i_1},\ ...,\ a_{i_n})$ *of* $a_1,\ ...,\ a_s$ is determined by the *Möbius inversion*,

$$\begin{aligned} k_n(a_{i_1},\ ...,\ a_{i_n}) &= \sum_{\pi \in NC(n)} \psi_\pi(a_{i_1},\ ...,\ a_{i_n}) \mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi_V(a_{i_1},\ ...,\ a_{i_n}) \mu\left(0_{|V|},\ 1_{|V|}\right) \right) \end{aligned}$$

for all $(i_1,\ ...,\ i_n) \in \{1,\ ...,\ s\}^n$, for all $n \in \mathbb{N}$, where $\psi_\pi(...)$ means the *partition-depending moments*, and $\psi_V(...)$ means the *block-depending moment*, for example, if

$$\pi_0 = \{(1,5,7), (2,3,4), (6)\} \text{ in } NC(7)$$

with three blocks $(1,\ 5,\ 7)$, $(2,\ 3,\ 4)$, and (6), then

$$\begin{aligned} \psi_{\pi_0}\left(a_{i_1}^{r_1},\ ...,\ a_{i_7}^{r_7}\right) &= \psi_{(1,5,7)}(a_{i_1}^{r_1}, ..., a_{i_7}^{r_7}) \psi_{(2,3,4)}(a_{i_1}^{r_1}, ..., a_{i_7}^{r_7}) \psi_{(6)}(a_{i_1}^{r_1}, ..., a_{i_7}^{r_7}) \\ &= \psi(a_{i_1}^{r_1} a_{i_5}^{r_5} a_{i_7}^{r_7}) \psi(a_{i_2}^{r_2} a_{i_3}^{r_3} a_{i_4}^{r_4}) \psi(a_{i_6}^{r_6}) \end{aligned}$$

Here, the set $NC(n)$ means the *noncrossing partition set* over $\{1,\ ...,\ n\}$, which is a *lattice* with the inclusion $\leq$, such that

$$\theta \leq \pi \overset{def}{\Longleftrightarrow} \forall V \in \theta, \exists B \in \pi,\ \text{s.t.}\ , V \subseteq B$$

where $V \in \theta$ or $B \in \pi$ means that V is a *block of* θ, respectively, B is a block of π, and $\subseteq$ means the usual set inclusion, having its minimal element $0_n = \{(1),\ (2),\ ...,\ (n)\}$, and its maximal element $1_n = \{(1,\ ...,\ n)\}$.

Especially, a partition-depending free moment $\psi_\pi(a,\ ...,\ a)$ is determined by

$$\psi_\pi(a, ..., a) = \prod_{V \in \pi} \psi\left(a^{|V|}\right)$$

where $|V|$ means the cardinality of V.

Also, μ is the *Möbius functional* from $NC \times NC$ into $\mathbb{C}$, where $NC = \overset{\infty}{\underset{n=1}{\cup}} NC(n)$. *i.e.*, it satisfies that

$$\mu(\pi, \theta) = 0,\ \text{for all}\ \pi > \theta in NC(n)$$

and

$$\mu(0_n, 1_n) = (-1)^{n-1} c_{n-1}, \text{ and } \sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0$$

for all $n \in \mathbb{N}$, where

$$c_k = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{1}{k+1} \frac{(2k)!}{k!k!}$$

means the *k-th Catalan numbers*, for all $k \in \mathbb{N}$. Notice that since each $NC(n)$ is a well-defined lattice, if $\pi < \theta$ are given in $NC(n)$, one can decide the "interval"

$$[\pi, \theta] = \{\delta \in NC(n) : \pi \leq \delta \leq \theta\}$$

and it is always lattice-isomorphic to

$$[\pi, \theta] = NC(1)^{k_1} \times NC(2)^{k_2} \times \ldots \times NC(n)^{k_n}$$

for some k_1, ..., $k_n \in \mathbb{N}$, where $NC(l)^{k_t}$ means "l blocks of π generates k_t blocks of θ," for $k_j \in \{0, 1$, ..., $n\}$, for all $n \in \mathbb{N}$. By the multiplicativity of μ on $NC(n)$, for all $n \in \mathbb{N}$, if an interval $[\pi, \theta]$ in $NC(n)$ satisfies the above set-product relation, then we have

$$\mu(\pi, \theta) = \prod_{j=1}^{n} \mu(0_j, 1_j)^{k_j}$$

(For details, see [11,12]).

By the very definition of free cumulants, one can get the following equivalent Möbius inversion,

$$\psi\left(a_{i_1} a_{i_2} \ldots a_{i_n}\right) = \sum_{\pi \in NC(n)} k_\pi\left(a_{i_1}, \ldots, a_{i_n}\right)$$

where $k_\pi(a_{i_1}, ..., a_{i_n})$ means the *partition-depending free cumulant*, for all $(a_{i_1}, ..., a_{i_n}) \in \{a_1, ..., a_s\}^n$, for $n \in \mathbb{N}$, where a_1, ..., $a_s \in (A, \psi)$, for $s \in \mathbb{N}$. Under the same example,

$$\pi_0 = \{(1, 5, 7), (2, 3, 4), (6)\} \text{ in } NC(7)$$

we have

$$\begin{aligned} k_{\pi_0}(a_{i_1}, \ldots, a_{i_7}) &= k_{(1,5,7)}\left(a_{i_1}, \ldots, a_{i_7}\right) k_{(2,3,4)}\left(a_{i_1}, \ldots, a_{i_7}\right) k_{(6)}\left(a_{i_1}, \ldots, a_{i_7}\right) \\ &= k_3\left(a_{i_1}, a_{i_5}, a_{i_7}\right) k_3\left(a_{i_2}, a_{i_3}, a_{i_4}\right) k_1(a_{i_6}) \end{aligned}$$

In fact, the free moments of free random variables and the free cumulants of them provide equivalent free distributional data. For example, if a free random variable a in $(\mathfrak{A}, \psi)$ is a self-adjoint operator in the von Neumann algebra $\mathfrak{A}$ in the sense that $a^* = a$, then both free moments $\{\psi(a^n)\}_{n=1}^{\infty}$ and free cumulants $\{k_n(a, ..., a)\}_{n=1}^{\infty}$ give its spectral distributional data.

However, their uses are different case-by-case. For instance, to study the free distribution of fixed free random variables, the computation and investigation of free moments is better, and to study the freeness of distinct free random variables in the structures, the computation and observation of free cumulants is better (See [12]).

Definition 2.1. *We say two subalgebras A_1 and A_2 of $\mathfrak{A}$ are* free in $(\mathfrak{A},\, \psi)$, *if all "mixed" free cumulants of A_1 and A_2 vanish.. Similarly, two subsets X_1 and X_2 of $\mathfrak{A}$ are* free in $(\mathfrak{A},\, \psi)$*, if two subalgebras A_1 and A_2, generated by X_1 and X_2 respectively, are free in $(\mathfrak{A},\, \psi)$. Two free random variables x_1 and x_2 are* free in $(\mathfrak{A},\, \psi)$, *if $\{x_1\}$ and $\{x_2\}$ are free in $(\mathfrak{A},\, \psi)$.*

Suppose A_1 and A_2 are free subalgebras in $(\mathfrak{A},\, \psi)$. Then the subalgebra A generated both by these free subalgebras A_1 and A_2 is denoted by

$$A \overset{denote}{=} A_1 \star_{\mathbb{C}} A_2$$

Inductively, assume that $\mathfrak{A}$ is generated by its family $\{A_i\}_{i\in\Lambda}$ of subalgebras, and suppose the subalgebras A_i are free from each other in $(\mathfrak{A},\, \psi)$, for $i \in \Lambda$. Then we call $\mathfrak{A}$, the *free product algebra of* $\{A_i\}_{i\in\Lambda}$ *(with respect to* ψ*), i.e.*,

$$\mathfrak{A} = \underset{i\in\Lambda}{\star_{\mathbb{C}}} A_i$$

is the free product algebra of $\{A_i\}_{i\in\Lambda}$ (with respect to ψ).

In the above text, we concentrated on the cases where $(A,\, \psi)$ is a "pure-algebraic" free probability space. Of course, one can take A as a topological algebra, for instance, A can be a *Banach algebra*. In such a case, ψ is usually taken as a "bounded (or continuous)" linear functional (under topology). Similarly, A can be taken as a $*$-*algebra*, where $(*)$ means here the *adjoint on* A, satisfying that

$$\begin{aligned}
&a^{**} = a, forall a \in A \\
&(a_1 + a_2)^* = a_1^* + a_2^* \\
&(a_1 a_2)^* = a_2^* a_1^*
\end{aligned}$$

for all $a_1,\, a_2 \in A$. Then we put an additional condition on ψ, called the $(*)$-relation on ψ,

$$\psi(a^*) = \overline{\psi(a)}, \text{ for all } a \in A$$

where $\overline{z}$ means the conjugate of z, for all $z \in \mathbb{C}$.

Finally, the algebra A can be taken as a topological $*$-algebra, for example, a C^*-*algebra* or a *von Neumann algebra*. Then usually we take a linear functional ψ satisfying both the boundedness and the $(*)$-relation on it.

In the following, to distinguish the differences, we will use the following terms.

(i) If A is a Banach algebra and if ψ is bounded, then $(A,\, \psi)$ is said to be a *Banach probability space*.
(ii) If A is a $*$-algebra and if ψ satisfies the $(*)$-relation, then $(A,\, \psi)$ is called a $*$-*probability space*.
(iii) If A is a C^*-algebra and if ψ is bounded with $(*)$-relation, then $(A,\, \psi)$ is a C^*-*probability space*.
(iv) If A is a von Neumann algebra and if ψ is bounded with $(*)$-relation, then $(A,\, \psi)$ is a W^*-*probability space*.

2.3. The Arithmetic Algebra $\mathcal{A}$

In this section, we introduce an algebra $\mathcal{A}$, consisting of all arithmetic functions. Recall that an *arithmetic function* f is nothing but a $\mathbb{C}$-valued function whose domain is $\mathbb{N}$. *i.e.*,

$$\mathcal{A} = \{f : \mathbb{N} \to \mathbb{C} : f \text{ is a function}\}$$

set-theoretically. It is easy to check that $\mathcal{A}$ forms a vector space over $\mathbb{C}$. Indeed, the functional addition $(+)$ is well-defined on $\mathcal{A}$, since $f + h$ is a well-defined arithmetic function whenever f and h are arithmetic functions, and the scalar product is well-defined on $\mathcal{A}$, because $r\, f$ is a well-defined arithmetic function whenever f is an arithmetic function and $r \in \mathbb{C}$.

Moreover, one can define the convolution $(*)$ on $\mathcal{A}$ by

$$f * h(n) \overset{def}{=} \sum_{d|n} f(d)h\left(\frac{n}{d}\right) = \sum_{d_1,d_2 \in \mathbb{N}\ s.t.,\ n=d_1d_2} f(d_1)h(d_2)$$

for all $n \in \mathbb{N}$, for all $f,\, h \in \mathcal{A}$, where "$d \mid n$" means "d is a divisor of n," or "d divides n," or "n is divisible by d," for $d,\, n \in \mathbb{N}$.

Then $f * h \in \mathcal{A}$, too. Also, we have that

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

and

$$(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$$

for all $f_1,\, f_2,\, f_3 \in \mathcal{A}$.

Thus, equipped with this vector multiplication $(*)$ on $\mathcal{A}$, the vector space $\mathcal{A}$ forms an algebra over $\mathbb{C}$.

Definition 2.2. *The algebra $\mathcal{A} = (\mathcal{A},\, +,\, *)$ over $\mathbb{C}$ is called the arithmetic algebra.*

This algebra $\mathcal{A}$ has its $(+)$-identity $0_{\mathcal{A}}$, the arithmetic function,

$$0_{\mathcal{A}}(n) = 0, \text{ for all } n \in \mathbb{N}$$

and the $(*)$-identity $1_{\mathcal{A}}$, the arithmetic function,

$$1_{\mathcal{A}}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$.

Note the difference between the constant arithmetic function 1 and the $(*)$-identity $1_{\mathcal{A}}$,

$$1(n) = 1, \text{ for all } n \in \mathbb{N}$$

It is not difficult to check that, in fact, the algebra $\mathcal{A}$ is commutative under $(*)$, *i.e.*,

$$f * h = h * f, \text{ for all } f, h \in \mathcal{A}$$

2.4. The Euler Totient Function ϕ

In this section, we consider a special element, the Euler totient function ϕ of the arithmetic algebra $\mathcal{A}$. Let ϕ be an arithmetic function,

$$\phi(n) \overset{def}{=} \left| \left\{ k \in \mathbb{N} \,\middle|\, \begin{array}{c} 1 \leq k \leq n, \\ \gcd(k,\ n) = 1 \end{array} \right\} \right|$$

where $\gcd(n_1, n_2)$ means the *greatest common divisor of* n_1 *and* n_2, for all $n_1, n_2 \in \mathbb{N}$. This function ϕ is a well-defined arithmetic function, as an element of $\mathcal{A}$.

Definition 2.3. *The above arithmetic function ϕ is called the Euler totient function in $\mathcal{A}$.*

The Euler totient function ϕ is so famous, important, and applicable in both classical and modern number theory that we cannot help emphasizing the importance of this function not only in mathematics but also in other scientific areas (e.g., [4,5,22,23,29]).

For any fixed prime p, and $k \in \mathbb{N}$, one can have $\phi(1) = 1$, and

$$\phi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right)$$

in particular, with $\phi(p) = p - 1$.

Recall that an arithmetic function f is *multiplicative*, if

$$f(nm) = f(n)f(m), \text{ whenever } \gcd(n, m) = 1$$

for all $n, m \in \mathbb{N}$.

The Euler totient function ϕ is multiplicative by definition. Thus, we have that

$$\phi(n) = \phi\left(\prod_{p:prime,\, p|n} p^{k_p}\right) = n \prod_{p:prime,\, p|n} \left(1 - \frac{1}{p}\right)$$

for all $n \in \mathbb{N}$, whenever n is prime-factorized by $\prod_{p|n} p^{k_p}$, with $\phi(1) = 1$.

Furthermore, the arithmetic function ϕ satisfies the following functional equation in general,

$$\phi(nm) = \phi(n)\phi(m) \frac{\gcd(n,\ m)}{\phi\left(\gcd(n,\ m)\right)}$$

for all $n, m \in \mathbb{N}$.

The above Formula generalizes the multiplicativity of ϕ. So, one can have that

$$\phi(2m) = \begin{cases} 2\phi(m) & \text{if } m \text{ is even} \\ \phi(m) & \text{if } m \text{ is odd} \end{cases}$$

for all $m \in \mathbb{N}$.

We also obtain that

$$\phi(n^k) = n^{k-1}\phi(n)$$

for all $n,\, k \in \mathbb{N}$.

Recall the *Möbius inversion on* $\mathcal{A}$,

$$h = f * 1 \Longleftrightarrow f = h * \mu$$

where μ is the *arithmetic Möbius function* (different from the Möbius functional in the incidence algebra in Section 2.2), *i.e.*,

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } \omega(n) = \Omega(n) \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, where

$$\omega(n) = \text{ the number of “distinct” prime, as factors of } n$$

and

$$\Omega(n) = \text{ the number prime factors of } n$$

for all $n \in \mathbb{N}$.

It is well-known that

$$\phi = 1 * \mu \Longleftrightarrow 1 = \phi * \mu.$$

3. Free Probability on Von Neumann Algebras $L^{\infty}(\mathbb{Q}_p)$

Let's establish von Neumann algebras $\mathfrak{M}_p$ induced by the p-adic number fields $\mathbb{Q}_p$, for primes p. Since $\mathbb{Q}_p$ is an unbounded Haar-measured non-Archimedean Banach field, for each fixed prime p, we naturally obtain the corresponding von Neumann algebra $L^{\infty}(\mathbb{Q}_p)$, induced by a Haar-measure space

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p) \tag{1}$$

where $\sigma(\mathbb{Q}_p)$ means the σ-algebra of $\mathbb{Q}_p$, consisting of all ρ_p-measurable subsets of $\mathbb{Q}_p$.

Then there exists a natural linear functional, denoted by φ_p, on the von Neumann algebra $\mathfrak{M}_p$, satisfying that

$$\varphi_p\left(\chi_S\right) = \int_{\mathbb{Q}_p} \chi_S d\rho_p = \rho_p(S) \tag{2}$$

for all $S \in \sigma(\mathbb{Q}_p)$, where χ_S means the characteristic function of S.

I.e., one has a well-defined W^*-probability space $(\mathfrak{M}_p,\, \varphi_p)$, in terms of the integration φ_p.

3.1. p-Adic Von Neumann Algebras $\mathfrak{M}_p$

Throughout this section, let's fix a prime p. As a measure space, the field $\mathbb{Q}_p$ has its corresponding L^2-*Hilbert space* H_p, defined by

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p,\ \rho_p\right) \tag{3}$$

We call H_p, the p-*adic Hilbert space*. *i.e.*, all elements of H_p are the square ρ_p-integrable functions on $\mathbb{Q}_p$. Remark that all elements of H_p are the functions approximated by *simple functions*

$$\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$$

with $t_S \in \mathbb{C}$ (under limit), generated by *characteristic functions* χ_X

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Q}_p$. So, one can understand each element f of H_p as an expression,

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ (a finite or infinite sum)}$$

The *inner product*, denoted by $<, >_p$, on H_p is naturally defined by

$$< f_1, f_2 >_p \overset{def}{=} \int_{\mathbb{Q}_p} f_1 \overline{f_2} d\rho_p$$

for all $f_1,\ f_2 \in H_p$, having the corresponding norm $\|.\|_p$ on H_p,

$$\|f\|_p \overset{def}{=} \sqrt{< f,\ f >_p} = \sqrt{\int_{\mathbb{Q}_p} |f|^2\ d\rho_p}$$

for all $f \in H_p$. Thus, if $f = \sum\limits_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$ in H_p, then

$$\int_{\mathbb{Q}_p} f d\rho_p = \sum_{X \in \sigma(\mathbb{Q}_p)} t_X \rho_p(X)$$

Now, let $L^\infty(\mathbb{Q}_p, \rho_p)$ be the L^∞-Banach space, consisting of all essentially bounded functions on $\mathbb{Q}_p$. Let's now fix a function

$$h \in L^\infty\left(\mathbb{Q}_p,\ \rho_p\right)$$

Similar to H_p-case, one can / may understand h as the approximation of simple functions, since

$$hf \in H_p, \text{ for all } f \in H_p \tag{4}$$

Moreover, one can define the vector multiplication on $L^\infty(\mathbb{Q}_p,\ \rho_p)$ by the usual functional multiplication. Then it is well-defined because $h_1,\ h_2 \in L^\infty(\mathbb{Q}_p,\ \rho_p)$, then $h_1 h_2 \in L^\infty(\mathbb{Q}_p,\ \rho_p)$, too. *I.e.*, it becomes a well-defined von Neumann algebra over $\mathbb{C}$. We denote this von Neumann algebra by $\mathfrak{M}_p$. *i.e.*,

$$\mathfrak{M}_p \overset{def}{=} L^\infty(\mathbb{Q}_p, \rho_p)$$

More precisely, all elements of $\mathfrak{M}_p$ are understood as multiplication operators on H_p, by Equation (4).

Definition 3.1. *The von Neumann subalgebras* $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p,\ \rho_p)$ *acting on* H_p *are called the* p*-adic von Neumann algebras, for all primes* p.

By locally compactness, and Hausdorff property of $\mathbb{Q}_p$, for any $x \in \mathbb{Q}_p$, there exist $a \in \mathbb{Q}$, and $n \in \mathbb{Z}$, such that $x \in a + p^n U_p$ (e.g., [21]). Therefore, we obtain the following result.

Proposition 3.1. *Let* χ_S *be a characteristic function for* $S \in \sigma(\mathbb{Q}_p)$. *Then there exist* $N \in \mathbb{N} \cup \{\infty\}$, *and* $k_1, ..., k_N \in \mathbb{Z}$, $r_1, ..., r_N \in (0, 1]$ *in* $\mathbb{R}$, *such that*

$$\int_{\mathbb{Q}_p} \chi_S d\rho_p = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}} \right) \tag{5}$$

Proof. The detailed proof of Equation (5) can be found in [8]. □

The above Formula (5) characterizes the identically distributedness under the integral in $\mathfrak{M}_p$.

3.2. p-Prime W^*-Probability Spaces $(\mathfrak{M}_p, \varphi_p)$

In this section, on the p-adic von Neumann algebras $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ we define canonical linear functionals φ_p, and establish corresponding W^*-probability spaces $(\mathfrak{M}_p, \varphi_p)$. Throughout this section, we fix a prime p, and corresponding p-adic von Neumann algebra $\mathfrak{M}_p$, acting on the p-adic Hilbert space $H_p = L^2(\mathbb{Q}_p, \rho_p)$.

Define a linear functional

$$\varphi_p : \mathfrak{M}_p \to \mathbb{C}$$

on the p-adic von Neumann algebra $\mathfrak{M}_p$ by the integration,

$$\varphi_p(h) \overset{def}{=} \int_{\mathbb{Q}_p} h d\rho_p, \text{ for all } h \in \mathbb{Q}_p \tag{6}$$

Then the pair $(\mathfrak{M}_p, \varphi_p)$ forms a well-defined W^*-probability space in the sense of Section 2.2.

Definition 3.2. *The* W^**-probability space* $(\mathfrak{M}_p, \varphi_p)$ *of a* p*-adic von Neumann algebra* $\mathfrak{M}_p$ *and a linear functionals* φ_p *of Equation (6) is called the* p*-prime* W^**-probability spaces, for all primes* p.

We concentrate on studying free-distributional data of characteristic functions χ_S, for $S \in \sigma(\mathbb{Q}_p)$, or simple functions

$$\sum_{k=1}^{m} t_k \chi_{S_k}, \text{ with } t_k \in \mathbb{C}, S_k \in \sigma(\mathbb{Q}_p)$$

for $m \in \mathbb{N}$.

Proposition 3.2. *Let* $S \in \sigma(\mathbb{Q}_p)$, *and let* $\chi_S \in (\mathfrak{M}_p, \varphi_p)$. *Then*

$$\varphi_p\left(\chi_S^n\right) = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}} \right) \tag{7}$$

for some $N \in \mathbb{N} \cup \{\infty\}$, *where* $r_j \in [0, 1]$ *in* $\mathbb{R}$, $k_j \in \mathbb{Z}$, *for* $j = 1, ..., N$, *for all* $n \in \mathbb{N}$.

Proof. The Formula (7) is proven by Equation (5). The detailed proof can be found in [8,21]. □

The above Formula (7) shows not only free-moment computation for χ_S, but also the identically free-distributedness of $\{\chi_S^k\}_{k=1}^{\infty}$ in $(\mathfrak{M}_p, \varphi_p)$, too.

More generally, one can obtain the following joint free moment computation formula.

Theorem 3.3. *Let $S_j \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_j} \in (\mathfrak{M}_p, \varphi_p)$, for $j = 1, ..., n$, for $n \in \mathbb{N}$. Let $k_1, ..., k_n \in \mathbb{N}$, and $s_1, ..., s_n \in \{1, *\}$. Then*

$$\varphi_p\left(\chi_{S_1}^{k_1 s_1}\, \chi_{S_2}^{k_2 s_2} \cdots \chi_{S_n}^{k_n s_n}\right) = \varphi_p\left(\chi_{\overset{n}{\underset{i=1}{\cap}} S_i}\right) \tag{8}$$

So, if the ρ_p-measurable subset $S = \overset{n}{\underset{i=1}{\cap}} S_i$ and its corresponding free random variable χ_S satisfies Equation (7), then

$$\varphi_p\left(\chi_{S_1}^{k_1 s_1}\, \chi_{S_2}^{k_2 s_2}\ \cdots\ \chi_{S_n}^{k_n s_n}\right) = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j}+1}\right) \tag{9}$$

Proof. The proofs of Equations (8) and (9) are by Equation (7) under linearity. See [8,21] for more details. □

4. Free Probability on $\mathcal{A}$ Determined by Primes

Let $\mathcal{A}$ be the arithmetic algebra consisting of all arithmetic functions under the usual functional addition and convolution. In [9–12], we define the point-evaluation linear functionals g_p on $\mathcal{A}$, determined by fixed primes p. As before, throughout this section, we fix a prime p.

Define a linear functional $g_p : \mathcal{A} \to \mathbb{C}$ by

$$g_p(f) = f(p), \text{ for all } f \in \mathcal{A} \tag{10}$$

as the *point evaluation at p*. It is a well-defined linear functional on $\mathcal{A}$, inducing a (pure-algebraic) free probability space $(\mathcal{A}, g_p)$.

Definition 4.1. *The pure-algebraic free probability space $(\mathcal{A}, g_p)$ is said to be the arithmetic p-prime probability space.*

For convenience, we denote the n-th convolution

$$\underbrace{f * \cdots\cdots * f}_{n\text{-times}}$$

by $f^{(n)}$, for all $n \in \mathbb{N}$.

For $f_1, f_2 \in \mathcal{A}$, one can get that

$$\begin{aligned} g_p(f_1 * f_2) &= f_1(1) f_2(p) + f_1(p) f_2(1) \\ &= f_1(1) g_p(f_2) + g_p(f_1) f_2(1) \end{aligned} \tag{11}$$

Therefore, we can verify that the free-distributional data on $\mathcal{A}$ (for a fixed prime p) is determined by quantities

$$\{f(1), f(p) : f \in \mathcal{A}\}$$

(See [9])

Proposition 4.1. *(See [9,10]) Let* $(\mathcal{A},\, g_p)$ *be the arithmetic* p*-prime probability space* $(\mathcal{A},\, g_p)$*.*

$$g_p\left(f^{(n)}\right) = nf(1)^{n-1}g_p(f), \text{ for all } n \in \mathbb{N}, andf \in \mathcal{A} \tag{12}$$

For f_1, ..., $f_n \in (\mathcal{A},\, g_p)$, *for* $n \in \mathbb{N}$, *we have*

$$g_p\left(\prod_{j=1}^{n} f_j\right) = \sum_{j=1}^{n} g_p(f_j)\left(\prod_{l \neq j \in \{1,\ldots,n\}} f_l(1)\right) \tag{13}$$

For f_1, ..., $f_n \in (\mathcal{A},\, g_p)$, *for* $n \in \mathbb{N}$, *we have*

$$k_n^{(p)}\left(f_1,\, ...,\, f_n\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\sum_{j \in V} g_p(f_j)\left(\prod_{l \in V \setminus \{j\}} f_l(1)\right)\mu\left(0_{|V|},\, 1_{|V|}\right)\right)\right) \tag{14}$$

where $k_n^{(p)}(...)$ *means the free cumulant in terms of* g_p *in the sense of Section 2.2.*

The above Formulas (12)–(14) provide ways to consider free-distributional data on $\mathcal{A}$, for a fixed prime p. Again, they demonstrate that the quantities $\{f(1),\, f(p)\}_{f \in \mathcal{A}}$ determine free distributions of arithmetic functions in $(\mathcal{A},\, g_p)$. Also, the Formulas (13) and (14) provide equivalent free-distributional data for f_1, ..., f_n (See Section 2.2, and [1]), under Möbius inversion (in the sense of Section 2.2).

By [9], we can define an equivalence relation $\mathcal{R}_p$ on $\mathcal{A}$ by

$$f_1 \mathcal{R}_p f_2 \overset{def}{\Longleftrightarrow} (f_1(1), f_1(p)) = (f_2(1), f_2(p)) \tag{15}$$

as pairs in the 2-dimensional $\mathbb{C}$-vector space $\mathbb{C}^2$.

Construct now a quotient algebra $\mathcal{A}/\mathcal{R}_p$ naturally. *i.e.*, it is a set

$$\{[f]_{\mathcal{R}_p} : f \in \mathcal{A}\} \tag{16}$$

where

$$[f]_{\mathcal{R}_p} = \{h \in \mathcal{A} : f\mathcal{R}_p h\}, \text{ for all } f \in \mathcal{A}$$

Without loss of generality, we keep writing $[f]_{\mathcal{R}_p}$ simply by f in $\mathcal{A}/\mathcal{R}_p$.

We obtain the following classification theorem.

Theorem 4.2. *(See [9]) Let* $(\mathcal{A},\, g_p)$ *be the arithmetic* p*-prime probability space. Then*

$$\mathcal{A} = \underset{(t_1,\, t_2) \in \mathbb{C}^2}{\sqcup} [t_1, t_2] \tag{17}$$

set-theoretically, where $\sqcup$ *means the disjoint union and*

$$[t_1, t_2] = \{f \in \mathcal{A} : f(1) = t_1, f(p) = t_2\} \tag{18}$$

for all $(t_1, t_2) \in \mathbb{C}^2$.

Clearly, one can inherit the linear functional g_p on $\mathcal{A}$ to a linear functional, also denoted by g_p, on $\mathcal{A}/\mathcal{R}_p$, defined by

$$g_p(f) = g_p\left([f]_{\mathcal{R}_p}\right) = f(p) \tag{19}$$

for all $f = [f]_{\mathcal{R}_p} \in \mathcal{A}/\mathcal{R}_p$. Then, under the linear functional g_p of Equation (19), the pair $(\mathcal{A}/\mathcal{R}_p, g_p)$ forms a pure-algebraic free probability space, too.

As in [11,12], we put a suitable topology on $\mathcal{A}/\mathcal{R}_p$. By Equations (17) and (19), whenever we choose an element $f \in \mathcal{A}/\mathcal{R}_p$, it is represented as a pair

$$(f(1), f(p)) \text{ of } \mathbb{C}^2$$

Now, let's define an indefinite inner product $[,]$ on $\mathbb{C}^2$ by

$$[(t_1, t_2), (s_1, s_2)] = t_1\overline{s_2} + t_2\overline{s_1} \tag{20}$$

for all $(t_1, t_2), (s_1, s_2) \in \mathbb{C}^2$, and define the corresponding norm $\|.\|$ by

$$\|(t_1,\ t_2)\| = \sqrt{|[(t_1, t_2),\ (t_1, t_2)]|} = \sqrt{\left|2\,\mathrm{Re} t_1\overline{t_2}\right|} \tag{21}$$

for all $(t_1, t_2) \in \mathbb{C}^2$, where $|.|$ in the second equality means the modulus on $\mathbb{C}$, and $|.|$ in the third equality means the absolute value on $\mathbb{R}$.

Then the pair $(\mathbb{C}^2, \|.\|)$ is a well-defined Banach space, denoted by $\mathbb{C}^2_{A_0}$.

Notice that we may / can understand this Banach space $\mathbb{C}^2_{A_0}$ as the 2-dimensional $\mathbb{C}$-algebra $\mathbb{C}^{\oplus 2}$, equipped with $[,]$ of Equation (22) and $\|.\|$ of Equation (23), with its multiplication,

$$(t_1, t_2)(s_1, s_2) = (t_1 s_1,\ t_1 s_2 + t_2 s_1) \tag{22}$$

for all $(t_1, t_2), (s_1, s_2) \in \mathbb{C}^{\oplus 2}_{A_0}$. The multiplication Equation (22) is a well-defined vector-multiplication on $\mathbb{C}^{\oplus 2}$, by [11,12].

Notation. We denote such an algebra $\mathbb{C}^{\oplus 2}$ equipped with vector-multiplication Equation (22), with $[,]$ of Equation (20) and $\|.\|$ of Equation (21), by $\mathfrak{C}^2$.

Define now a norm $\|.\|_p$ on the quotient algebra $\mathcal{A}/\mathcal{R}_p$ by

$$\|f\|_p = \|(f(1),\ f(p))\| = \sqrt{\left|2\,\mathrm{Re}\left(f(1)\overline{f(p)}\right)\right|} \tag{23}$$

for all $f \in \mathcal{A}/\mathcal{R}_p$, where $\|.\|$ is the norm Equation (21) on $\mathbb{C}^2$.

Then, under this norm $\|.\|_p$ of Equation (23), the quotient algebra $\mathcal{A}/\mathcal{R}_p$ is understood as a topological space, moreover, embedded in the 2-dimensional $\mathbb{C}$-algebra

$$\mathfrak{C}^2 \overset{\text{homeo}}{=} \mathbb{C}^2$$

where $\overset{\text{homeo}}{=}$ means "being homeomorphic." *i.e.*, $\mathfrak{C}^2$ is a Banach algebra.

Theorem 4.3. *The normed quotient algebra $\mathcal{A}/\mathcal{R}_p = (\mathcal{A}/\mathcal{R}_p, \|.\|_p)$ is Banach-isomorphic to $\mathfrak{C}^2$.*

One can define a morphism

$$F : \mathcal{A}/\mathcal{R}_p \to \mathfrak{C}^2$$

by

$$F(f) \overset{def}{=} (f(1), f(p)), \text{ for all } f \in \mathcal{A}/\mathcal{R}_p. \tag{24}$$

Then it is surjective, by Equation (17). And, again by Equation (17), it is injective. *i.e.*, if

$$(f_1(1), f_1(p)) \neq (f_2(1), f_2(p)) \text{ in } \mathbb{C}^{\oplus 2}$$

then $f_1 \neq f_2$ in $\mathcal{A}/\mathcal{R}_p$, as equivalent classes in the sense of Equation (16). So, it is injective, too. *i.e.*, F of Equation (24) is a bijective morphism.

Now, let $f_1,\, f_2 \in \mathcal{A}/\mathcal{R}_p$, and $t_1,\, t_2 \in \mathbb{C}$. Then

$$\begin{aligned} &F(t_1 f_1 + t_2 f_2) \\ &\quad = ((t_1 f_1 + t_2 f_2)(1),\ (t_1 f_1 + t_2 f_2)(p)) \\ &\quad = (t_1 f_1(1),\ t_1 f_1(p)) + (t_2 f_2(1),\ t_2 f_2(p)) \\ &\quad = t_1 (f_1(1),\ f_1(p)) + t_2 (f_2(1),\ f_2(p)) \\ &\quad = t_1 F(f_1) + t_2 F(f_2) \end{aligned} \tag{25}$$

The identity Equation (25) guarantees the linearity of F.

Also, F satisfies that, for all $f_1,\, f_2 \in \mathcal{A}/\mathcal{R}_p$,

$$F(f_1 * f_2) = (f_1(1) f_2(1),\ f_1(1) f_2(p) + f_1(p) f_2(1))$$

by Equation (11)

$$= (f_1(1), f_1(p))(f_2(1), f_2(p))$$

by the multiplication Equation (22) on

$$\mathbb{C}^{\oplus 2}_{A_0} = F(f_1) F(f_2) \tag{26}$$

Thus, the morphism F is multiplicative, by Equation (26). So, by Equations (25) and (26), the bijective morphism F is an algebra-isomorphism from $\mathcal{A}/\mathcal{R}_p$ onto $\mathfrak{C}^2$.

Furthermore, one has that

$$\|F(f)\| = \|(f(1),\ f(p)\| = \sqrt{\left|2\,\mathrm{Re} f(1)\overline{f(p)}\right|} = \|f\|_p \tag{27}$$

for all $f \in \mathcal{A}/\mathcal{R}_p$. The relation Equation (27) shows that the algebra-isomorphism F is isometric. *i.e.*, it is a Banach-algebra isomorphism from $\mathcal{A}/\mathcal{R}_p$ onto $\mathfrak{C}^2$. It shows that the normed-algebra $\mathcal{A}/\mathcal{R}_p$ is isometrically isomorphic to the Banach algebra $\mathfrak{C}^2$.

The above topological-algebraic characterization is motivated both by the set-theoretic classification in [9] and by the Krein-space representations in [11,12].

Definition 4.2. *We denote the Banach algebra $\mathcal{A}/\mathcal{R}_p$ by $\mathfrak{A}_p$, and we call $\mathfrak{A}_p$, the p-prime Banach algebra. Moreover, $\mathfrak{A}_p$ is characterized by*

$$\mathfrak{A}_p \overset{\text{Banach}}{=} \mathfrak{C}^2 \tag{28}$$

by the above theorem.

Define now a linear functional π_2 on $\mathfrak{C}^2$ by

$$\pi_2\left((t_1,\ t_2)\right) = t_2, \text{ for all } (t_1, t_2) \in \mathfrak{C}^2 \tag{29}$$

as a natural projection on $\mathbb{C}^2$. Then the pair $(\mathfrak{C}^2,\ \pi_2)$ forms a *Banach probability space* (e.g., [12]).

Recall that two arbitrary free probability spaces $(A_1,\ \varphi_1)$ and $(A_2,\ \varphi_2)$ are said to be *equivalent* (*in the sense of Voiculescu*), if (i) there exists an isomorphism h from A_1 onto A_2; and (ii) h satisfies that

$$\varphi_2\left(h(a)\right) = \varphi_1(a), \text{ for all } a \in A_1$$

If A_1 and A_2 are topological algebras (or, topological $*$-algebras), then h of the condition (i) and (ii) should be continuous (respectively, both continuous and preserving $*$-relation, $h(a^*) = h(a)^*$ in A_2, for all $a \in A_1$, where $(*)$ here means *adjoint*).

Theorem 4.4. *The Banach probability spaces $(\mathfrak{A}_p,\ g_p)$ and $(\mathfrak{C}^2,\ \pi_2)$ are equivalent, i.e.,*

$$(\mathfrak{A}_p, g_p) \overset{\text{equivalent}}{=} (\mathfrak{C}^2, \pi_2) \tag{30}$$

Proof. By Equation (28) and by the above theorem, there exists a Banach-algebra isomorphism F of Equation (24) from $\mathfrak{A}_p$ onto $\mathfrak{C}^2$. For any $f \in \mathfrak{A}_p$, we obtain that

$$\pi_2\left(F(f)\right) = \pi_2\left((f(1),\ f(p))\right) = f(p) = g_p(f)$$

for all $f \in \mathfrak{A}_p$. $\square$

The above equivalence Equation (30) shows that the study of free probability on $\mathfrak{A}_p$ (or on $\mathcal{A}$), for a fixed prime p, is to investigate that on $\mathfrak{C}^2$ under π_2.

In [11,12], indeed, we showed that each element $f \in \mathfrak{A}_p$ is understood as a Krein space operator Θ_f on the Krein space $\mathbb{C}^2_{A_o}$,

$$\Theta_f = \begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix} \tag{31}$$

and

$$\mathfrak{C}^2 \overset{\text{Banach-}*}{=} \mathbb{C}\left[\{\Theta_f : f \in \mathfrak{A}_p\}\right]$$

Note here that

$$\Theta_{f_1}\Theta_{f_2} = \Theta_{f_1 * f_2} on \mathbb{C}^2_{A_o}, \text{ for all } f_1, f_2 \in \mathfrak{A}_p \tag{32}$$

and

$$\Theta^*_f = \Theta_{f^*} on \mathbb{C}^2_{A_o}, \text{ for all } f \in \mathfrak{A}_p \tag{33}$$

(See [11,12]).

5. Euler Subalgebras Φ_p of p-Prime Banach Algebras $\mathfrak{A}_p$

In this section, we consider a certain subalgebra of our p-prime Banach algebra $\mathfrak{A}_p$, for a fixed prime p. In Section 4, we showed that the Banach probability space $(\mathfrak{A}_p,\, g_p)$ is well-determined under quotient, and it is equivalent to the 2-dimensional Banach probability space $(\mathfrak{C}^2,\, \pi_2)$.

Let's fix the Euler totient function ϕ in $\mathfrak{A}_p$ (*i.e.*, understand $\phi = [\phi]_{\mathcal{R}_p}$). Define now the subalgebra Φ_p of $\mathfrak{A}_p$ by the Banach subalgebra generated by ϕ. *i.e.*,

$$\Phi_p \overset{def}{=} \mathbb{C}_*[\{\phi\}] = \overline{\mathbb{C}_*\left[\{\phi\}\right]}^{\|.\|_p} in \mathfrak{A}_p \tag{34}$$

where $\mathbb{C}_*[X]$ means that the subalgebra generated by X under $(+)$ and $(*)$ in $\mathfrak{A}_p$, and $\overline{Y}^{\|.\|_p}$ means the $\|.\|_p$-norm-closure of Y, where $\|.\|_p$ is in the sense of Equation (25). Thus, by Equation (34), we have

$$\Phi_p = \left\{ \sum_{k=0}^{n} t_k \phi^{(k)} \,\middle|\, \begin{array}{c} n \in \mathbb{N},\ t_k \in \mathbb{C}, \\ \text{with identity, } \phi^{(0)} = 1_{\mathfrak{A}_p} \end{array} \right\}$$

where $1_{\mathfrak{A}_p} = 1_{\mathcal{A}}/\mathcal{R}_p$, where $1_{\mathcal{A}}$ is the identity element of $\mathcal{A}$.

Definition 5.1. *We call the subalgebra Φ_p of the p-prime Banach algebra $\mathfrak{A}_p$, the (p-prime) Euler subalgebra of $\mathfrak{A}_p$.*

Since $(\mathfrak{A}_p,\, g_p)$ and $(\mathfrak{C}^2,\, \pi_2)$ are equivalent by Equation (30), under the subspace topology, the Euler subalgebra Φ_p is a Banach subalgebra of $\mathfrak{C}^2$.

Also, one can consider the adjoint $(*)$ on Φ_p as a unary operation on Φ_p such that

$$\left(\sum_{k=0}^{n} t_k \phi^{(k)}\right)^* \overset{def}{=} \sum_{k=0}^{n} \overline{t_k} \phi^{(k)\,*}$$

where $\overline{z}$ means the conjugate of z, for all $z \in \mathbb{C}$. Note that, in fact,

$$\phi^{(k)*}(m) = \overline{\phi^{(k)}(m)}, \text{ for all } m \in \mathbb{N}$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, since $\phi(\mathbb{N}) \subseteq \mathbb{N}$ in $\mathbb{R}$, *i.e.*,

$$\phi^{(k)*} = \phi^{(k)}, \text{ for all } k \in \mathbb{N}_0$$

It shows that the adjoint $(*)$ is well-defined on Φ_p, and hence, the Euler subalgebra Φ_p is understood as a *Banach $*$-algebra*. Remark that Φ_p is a $*$-subalgebra of the finite-dimensional algebra $\mathfrak{C}^2$. So, this Banach $*$-algebra Φ_p can be understood as a C^*-algebra or a von Neumann algebra, too, because all topologies on an arbitrary finite-dimensional space are equivalent from each other.

Assumption. From now on, we understand our Euler subalgebra Φ_p as a von Neumann algebra acting on $\mathbb{C}^2_{A_0}$.

Definition 5.2. *The W^*-probability space (Φ_p, g_p) is called the (p-prime) Euler W^*-probability space.*

Observe that, for any $n \in \mathbb{N}$, we have in general that

$$g_p\left(\phi^{(n)}\right) = n\phi(1)^{n-1}\phi(p)$$

by Equation (12). By the very definition of the Euler totient function ϕ,

$$\phi(1) = 1, \text{ and } \phi(p) = p\left(1 - \frac{1}{p}\right)$$

and hence, one can get that

$$g_p(\phi^{(n)}) = n\phi(p) = np\left(1 - \frac{1}{p}\right).$$

Therefore, one has that

$$g_p\left(\phi^{(n)}\right) = np\left(1 - \frac{1}{p}\right), \text{ for all } n \in \mathbb{N} \tag{35}$$

The above Formula (35) not only provides a recursive formula to compute n-th free moments of ϕ, but also shows that our linear functional g_p is additive on Φ_p, in the sense that

$$g_p\left(\phi^{(n)}\right) = \underbrace{g_p(\phi) + \cdots + g_p(\phi)}_{n\text{-times}} = ng_p(\phi)$$

for all $n \in \mathbb{N}$.

By applying Equation (35), we obtain the following general free-moment formula.

Theorem 5.1. *Let $T \in (\Phi_p, g_p)$ be a free random variable,*

$$T = \sum_{j=1}^{N} t_j \phi^{(n_j)}, \text{ with } t_j \in \mathbb{C}, n_j \in \mathbb{N} \cup \{0\}$$

Then the n-th free moments of T are determined by

$$g_p\left(T^{(n)}\right) = (g_p(\phi))\left(\sum_{(j_1,\dots,j_n)\in\{1,\dots,N\}^n}\left(\prod_{i=1}^{n} t_{j_i}\right)\left(\Sigma_{i=1}^{n} n_{j_i}\right)\right)$$
$$= p\left(1-\frac{1}{p}\right)\left(\sum_{(j_1,\dots,j_n)\in\{1,\dots,N\}^n}\left(\prod_{i=1}^{n} t_{j_i}\right)\left(\Sigma_{i=1}^{n} n_{j_i}\right)\right) \tag{36}$$

where $T^{(n)} = T * \cdots * T$ *in* Φ_p, *for all* $n \in \mathbb{N}$.

Proof. The detailed proof is found in [9]. □

The above Formula (36) characterizes the free-distributional data on Φ_p. Also, the Formula (36) with Formula (35) shows the free-momental data for $T \in (\Phi_p,\, g_p)$ are determined by certain scalar-multiples of

$$g_p(\phi) = p\left(1-\frac{1}{p}\right)$$

The following corollary is the direct consequence of Formulas (35) and (36).

By Formula (36), we obtain the following proposition.

Proposition 5.2. *For any* $n \in \mathbb{N}$, *we have that*

$$g_p\left(\phi^{(n)}\right) = np^{k+1}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) \tag{37}$$

for all $k \in \mathbb{Z}$.

6. Free-Distributional Data on $(\mathfrak{M}_p,\, \varphi_p)$ and $(\Phi_p,\, g_p)$

In this section, we consider identically free-distributedness on our two distinct free probability spaces $(\mathfrak{M}_p,\, \varphi_p)$ and $(\Phi_p,\, g_p)$. By Sections 3–5, one can realize that

$$\varphi_p\left(\chi_{p^k U_p}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}} = \frac{1}{np^{k+1}} g_p\left(\phi^{(n)}\right)$$

for all $k \in \mathbb{Z}$, and $n \in \mathbb{N}$.

Proposition 6.1. *(See [21]) Let* $S \in \sigma(\mathbb{Q}_p)$ *and* $\chi_S \in (\mathfrak{M}_p,\, \varphi_p)$. *Then there exist* $N \in \mathbb{N} \cup \{\infty\}$, $r_j \in [0,\, 1]$ *in* $\mathbb{R}$, *and* $k_j \in \mathbb{Z}$, *for* $j = 1, ..., N$, *such that*

$$g_p\left(\chi_S^n\right) = \frac{1}{m}\sum_{j=1}^{N}\frac{r_j}{p^{k_j+1}} g_p\left(\phi^{(m)}\right) \tag{38}$$

for all $n,\, m \in \mathbb{N}$.

As the converse of Equation (38), one can have the following proposition, too.

Proposition 6.2. *(See [21]) For all $m \in \mathbb{N}$, we have that*

$$g_p\left(\phi^{(m)}\right) = mp^{k+1}\varphi_p\left(\chi^{\;n}_{p^k U_p}\right) \tag{39}$$

for all $n \in \mathbb{N}$.

Therefore, by Equations (38) and (39), we obtain the following theorem.

Theorem 6.3. *Let $T = \sum_{j=1}^{N} t_j\ \phi^{(k_j)} \in (\Phi_p,\, g_p)$. Then there exist $s_1, ..., s_N \in \mathbb{C}$, and*

$$h = \sum_{j=1}^{N} s_j \chi_{p^{k_j} U_p} \in (\mathfrak{M}_p, \varphi_p)$$

such that T and h are identically free-distributed, in the sense that

$$g_p(T^{(n)}) = \varphi_p\left(h^n\right), \text{ for all } n \in \mathbb{N} \tag{40}$$

Proof. Let T be given as above in $(\Phi_p,\, g_p)$. Then, by Equations (38) and (39), T and

$$h = \sum_{j=1}^{N} s_j \chi_{p^{k_j} U_p}$$

with

$$s_j = t_j m_j p^{k_j+1} \in \mathbb{C}, \text{ for all } j = 1, ..., N$$

satisfy

$$g_p(T) = \varphi_p(h)$$

Also, for any $n \in \mathbb{N}$,

$$g_p(T^{(n)}) = \varphi_p(h^n)$$

by Equations (36) and (37). Therefore, two free random variables $T \in (\Phi_p,\, g_p)$ and $h \in (\mathfrak{M}_p,\, \varphi_p)$ are identically free-distributed. □

By the identically free-distributedness Equation (40), we obtain the following theorem, by Equation (38).

Theorem 6.4. *Let $h = \sum_{j=1}^{N} t_j\ \chi_{S_j} \in (\mathfrak{M}_p,\, \varphi_p)$, with $t_j \in \mathbb{C}$, for $N \in \mathbb{N}$. Then there exists $T \in (\Phi_p,\, g_p)$ such that h and T are identically free-distributed in the sense that*

$$\varphi_p(h^n) = g_p(T^{(n)}), \text{ for all } n \in \mathbb{N} \tag{41}$$

Proof. Let h be given as above in $(\mathfrak{M}_p,\, \varphi_p)$. Then, for each summand χ_{S_k}, there exist $N_k \in \mathbb{N} \cup \{\infty\}$, $r_{k:j} \in [0,\, 1]$ in $\mathbb{R}$, and $k_{k:j} \in \mathbb{Z}$, for $j = 1, ..., N_k$, such that χ_{S_k} and

$$h_k = \sum_{j=1}^{N_k} r_{k:j} \chi_{p^{k_{k:j}} U_p} \in (\mathfrak{M}_p, \varphi_p) \tag{42}$$

are identically distributed in the sense of Equation (7), for $k = 1, ..., N_k$, for $k = 1, ..., N$.

And each $\chi_{p^{k_{k:j}}U_p}$ in the right-hand side of Equation (42) is identically free-distributed with $\frac{1}{p^{k_{k:j}}}\,\phi \in (\Phi_p,\, g_p)$, by Equation (39). So, for h_k of Equation (42) and

$$T_k = \sum_{j=1}^{N_k} \frac{r_{k:j}}{p^{k_{k:j}}}\phi \in (\Phi_p, g_p) \tag{43}$$

are identically free-distributed, by Equation (39). Equivalently, T_k and χ_{S_k} are identically free-distributed, again by Equation (39), for all $k = 1$, ..., N. Thus, one can determine a free random variable,

$$T = \sum_{k=1}^{N} t_k T_k \text{ in } (\Phi_p, g_p) \tag{44}$$

where T_k are in the sense of Equation (43), such that

$$\varphi_p(h) = g_p\,(T)$$

By Equations (36) and (37), we have

$$\varphi_p(h^n) = g_p(T^{(n)}), \text{ for all } n \in \mathbb{N}$$

Therefore, there exists $T \in (\Phi_p,\, g_p)$, such that h and T are identically free-distributed. □

7. p-Adic W^*-Dynamical Systems

Let's now establish W^*-dynamical systems on a fixed von Neumann algebra M, by acting the σ-algebra $\sigma(\mathbb{Q}_p)$ of the p-adic number field $\mathbb{Q}_p$. Throughout this section, we fix a von Neumann subalgebra M acting on a Hilbert space H, and a prime p.

7.1. p-Adic Semigroup W^*-Dynamical Systems

Now, let M be a fixed von Neumann algebra in the operator algebra $B(H)$ on a Hilbert space H, and $\mathbb{Q}_p$, a fixed p-adic number field, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p,\, \rho_p)$ be the p-adic von Neumann algebra.

Let $\mathcal{H}_p$ be the tensor product Hilbert space $H_p \otimes H$ of the p-adic Hilbert space $H_p = L^2(\mathbb{Q}_p,\, \rho_p)$, and the Hilbert space H where M acts, where $\otimes$ means the Hilbertian tensor product. *i.e.*,

$$\mathcal{H}_p = H_p \otimes H$$

Define an action α of the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ acting on M "in $B(\mathcal{H}_p)$" by

$$\alpha(S)(m) \overset{def}{=} \chi_S m \chi_S^* = \chi_S m \chi_S \tag{45}$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$, in $B(\mathcal{H}_p)$, by understanding

$$\chi_S = \chi_S \otimes 1_M, \text{ and } m = 1_{\mathfrak{M}_p} \otimes m \text{ in } B(\mathcal{H}_p)$$

where $1_{\mathbb{Q}_p}$ is the identity map $\chi_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and 1_M is the identity element of M. *i.e.*, one can understand $\alpha(S)(m)$ as compressions of m (on $\mathcal{H}_p$), with respect to projections χ_S on $\mathcal{H}_p$. Then α is an action on M satisfying

$$\begin{aligned} alpha(S_1 \cap S_2)(m) &= \chi_{S_1 \cap S_2} m \chi_{S_1 \cap S_2} \\ &= \chi_{S_1}\chi_{S_2} m \chi_{S_1}\chi_{S_2} = \chi_{S_1}\chi_{S_2} m \chi_{S_2}\chi_{S_1} \\ &= \chi_{S_1} \left(\alpha(S_2)(m)\right) \chi_{S_1} = \alpha(S_1)\left(\alpha(S_2)(m)\right) \\ &= (\alpha(S_1) \circ \alpha(S_2))(m) \end{aligned}$$

for all $m \in M$, and S_1, $S_2 \in \sigma(\mathbb{Q}_p)$, *i.e.*,

$$\alpha(S_1 \cap S_2) = \alpha(S_1) \circ \alpha(S_2), \text{ for all } S_1, S_2 \in \sigma(\mathbb{Q}_p) \tag{46}$$

Observe now that the algebraic structure $(\sigma(\mathbb{Q}_p), \cap)$ forms a *semigroup*. Indeed, the intersection $\cap$ is well-defined on $\sigma(\mathbb{Q}_p)$, and it is associative,

$$S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$$

for $S_j \in \sigma(\mathbb{Q}_p)$, for all $j = 1,\ 2,\ 3$. Moreover, this semigroup $\sigma(\mathbb{Q}_p)$ contains $\mathbb{Q}_p$, acting as the semigroup-identity satisfying that

$$S \cap \mathbb{Q}_p = S = \mathbb{Q}_p \cap S$$

for all $S \in \sigma(\mathbb{Q}_p)$, and hence, this semigroup $\sigma(\mathbb{Q}_p)$ forms a *monoid* with its identity $\mathbb{Q}_p$.

Lemma 7.1. *The action α of $\sigma(\mathbb{Q}_p)$ in the sense of Equation (45) acting on a von Neumann algebra M is a monoid action, and hence, the triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ forms a monoid dynamical system.*

Proof. The action α of Equation (45) is indeed a well-defined action acting on M, by Equation (46). And, by the above discussion, $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$ forms a semigroup with the identity $\mathbb{Q}_p$. Moreover,

$$\alpha(\mathbb{Q}_p)m = m, \text{ for all } m \in M$$

So, the triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ forms a well-defined monoid dynamical system. □

Recall that all elements f of the p-adic von Neumann algebra $\mathfrak{M}_p$ is generated by the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$, in the sense that all elements $f \in \mathfrak{M}_p$ has its expression, $\sum\limits_{S \in Supp(f)} t_S \chi_S$. So, the action α of Equation (45) can be extended to a linear morphism, also denoted by α, from $\mathfrak{M}_p$ into $B(\mathcal{H}_p)$, acting on M, with

$$\begin{aligned} alpha(f)(m) &= \alpha\left(\sum_{S \in Supp(f)} t_S \chi_S\right)(m) \\ oversetdef&= \sum_{S \in Supp(f)} t_S \alpha(S)(m) = \sum_{S \in Supp(f)} t_S \chi_S m \chi_S \end{aligned} \tag{47}$$

for all $f \in \mathfrak{M}_p$.

Proposition 7.2. *Let $\mathfrak{M}_p$ be the p-prime von Neumann algebra, and let M be a von Neumann subalgebra of $B(H)$. Then there exists an action α of $\mathfrak{M}_p$ acting on M in $B(\mathcal{H}_p)$.*

Proof. It is proven by Equations (45) and (47). $\square$

Definition 7.1. *Let $\sigma(\mathbb{Q}_p)$ be the σ-algebra of the p-adic number field $\mathbb{Q}_p$, understood as a monoid $(\sigma(\mathbb{Q}_p), \cap)$, and let α be the monoid action of $\sigma(\mathbb{Q}_p)$ on a von Neumann algebra M in the sense of Equation (45). Then monoid dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$ is called the p-adic(-monoidal) W^*-dynamical system. For a p-adic W^*-dynamical system, define the crossed product algebra*

$$\mathcal{M}_p \overset{def}{=} M \times_\alpha \sigma(\mathbb{Q}_p) \tag{48}$$

by the von Neumann subalgebra of $B(\mathcal{H}_p)$ generated by M and $\chi\left(\sigma(\mathbb{Q}_p)\right)$, satisfying Equation (47). The von Neumann subalgebra $\mathcal{M}_p$ of $B(\mathcal{H}_p)$ is called the p-adic dynamical W^*-algebra induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

Note that, all elements of the p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ have their expressions,

$$\sum_{S \in \sigma \mathbb{Q}_p)} m_S \chi_S, \text{ with } m_S \in M$$

Define the *support* $Supp(T)$ of a fixed element $T = \sum_{S \in \sigma(\mathbb{Q}_p)} m_S\, \chi_S$ of $\mathcal{M}_p$ by

$$Supp(T) \overset{def}{=} \{S \in \alpha(\mathbb{Q}_p) : m_S \neq 0_M\} \tag{49}$$

Now, let $m_1\chi_{S_1}, m_2\chi_{S_2} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. Then

$$\begin{aligned}(m_1\chi_{S_1})(m_2\chi_{S_2}) &= m_1\chi_{S_1}m_2\chi_{S_1}\chi_{S_2} \\ &= m_1\chi_{S_1}m_2\chi_{S_1}^2\chi_{S_2} = m_1\chi_{S_1}m_2\chi_{S_1}\chi_{S_1}\chi_{S_2}\end{aligned}$$

since $\chi_S = 1_M \otimes \chi_S$ (in $B(\mathcal{H}_p)$) are projections ($\chi_S^2 = \chi_S = \chi_S^*$), for all $S \in \sigma(\mathbb{Q}_p)$

$$= m_1\alpha_{S_1}(m_2)\chi_{S_1}\chi_{S_2} = m_1\alpha_{S_1}(m_2)\chi_{S_1 \cap S_2}$$

Notation. For convenience, if there is no confusion, we denote $\alpha_S(m)$ by m^S, for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$.

I.e., we have

$$(m_1\chi_{S_1})(m_2\chi_{S_2}) = m_1 m_2^{S_1}\chi_{S_1 \cap S_2} \tag{50}$$

for $m_k\, \chi_{S_k} \in \mathcal{M}_p$, for $k = 1, 2$.

Inductive to Equation (50), one has that

$$\begin{aligned}\prod_{j=1}^{N}(m_j\chi_{S_j}) &= m_1 m_2^{S_1} m_3^{S_1 \cap S_2} ... m_N^{S_1 \cap ... \cap S_{N-1}} \chi_{S_1 \cap ... \cap S_N} \\ &= \left(\prod_{j=1}^{N} m_j^{\overset{j-1}{\underset{i=0}{\cap}} S_i}\right)\left(\chi_{\overset{N}{\underset{j=1}{\cap}} S_j}\right)\end{aligned} \tag{51}$$

for all $N \in \mathbb{N}$. Also, we obtain that

$$(m\chi_S)^* = \chi_S m^* \chi_S \chi_S = (m^*)^S \chi_S \tag{52}$$

for all $m\ \chi_S \in \mathcal{M}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

So, let

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, 2$$

where $Supp(T_k)$ is in the sense of Equation (49). Then

$$\begin{aligned} T_1 T_2 &= \sum_{(S_1,S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} \chi_{S_1} m_{S_2} \chi_{S_2} \\ &= \sum_{(S_1,S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} m_{S_2}^{S_1} \chi_{S_1 \cap S_2} \end{aligned} \tag{53}$$

by Equation (51).

Also, if $T = \sum\limits_{S \in Supp(T)} m_S \chi_S$ in $\mathcal{M}_p$, then

$$T^* = \sum_{S \in Supp(T)} (m_S^*)^S \chi_S \tag{54}$$

by Equation (52).

By Equations (53) and (54), one can have that if

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, ..., n$$

for $n \in \mathbb{N}$, then

$$T_1^{r_1} T_2^{r_2} \cdots T_n^{r_n} = \prod_{j=1}^{n} \left(\sum_{S_j \in Supp(T_j)} [m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right)$$

where

$$[m_{S_j}^{r_j}]^{S_j} \overset{def}{=} \begin{cases} m_{S_j} & \text{if } r_j = 1 \\ (m_{S_j}^*)^{S_j} & \text{if } r_j = * \end{cases} \tag{55}$$

for $j = 1, ..., n$

$$= \sum_{(S_1,...,S_n) \in \prod\limits_{j=1}^{n} Supp(T_j)} \left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right) \right)$$

$$= \sum_{(S_1,...,S_n) \in \prod\limits_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap\limits_{i=1}^{j-1} S_i \right)} \right) \left(\chi_{\bigcap\limits_{j=1}^{n} S_j} \right) \right), \text{ for all } (r_1, ..., r_n) \in \{1, *\}^n.$$

Lemma 7.3. *Let* $T_k = \sum\limits_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k}$ *be elements of the* p*-adic semigroup* W^**-algebra* $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ *in* $B(\mathcal{H}_p)$*, for* $k = 1, ..., n$*, for* $n \in \mathbb{N}$*. Then*

$$\prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1,...,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap_{i=1}^{j-1} S_i \right)} \right) \left(\chi_{\bigcap_{j=1}^{n} S_j} \right) \right) \tag{56}$$

for all $r_1, ..., r_n \in \{1, *\}$, *where* $[m_{S_j}^{r_j}]^{S_j}$ *are in the sense of Equation (55).*

The proof of the above lemma is by discussions of the very above paragraphs.

7.2. Structure Theorem of $M \times_\alpha \sigma(\mathbb{Q}_p)$

Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic W^*-algebra induced by the p-adic W^*-dynamical system $\mathcal{Q}(M, p) = (M, \sigma(\mathbb{Q}_p), \alpha)$. In this section, we consider a structure theorem for this crossed product von Neumann algebra $\mathcal{M}_p$.

First, define the usual tensor product W^*-subalgebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M}_p \text{ of } B(\mathcal{H}_p)$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ is the p-prime von Neumann algebra in the sense of Section 7.1, and where $\otimes_{\mathbb{C}}$ is the von Neumann algebraic tensor product over $\mathbb{C}$. By definition, clearly, one can verify that $\mathcal{M}_p$ is a W^*-subalgebra of $\mathcal{M}_0$ in $B(\mathcal{H}_p)$, *i.e.*,

$$\mathcal{M}_p \overset{\text{Subalgebra}}{\subseteq} \mathcal{M}_0$$

Now, define the "*conditional*" *tensor product* W^**-algebra*

$$\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$$

induced by an action α of $\mathfrak{M}_p$ acting on M (in the sense of Equation (48)), by a W^*-subalgebra of $\mathcal{M}_0$ dictated by the α-*relations*,

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2} \tag{57}$$

and

$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^* \tag{58}$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{Q}_p)$. *i.e.*, the W^*-subalgebra $\mathcal{M}_0^p$ of $\mathcal{M}_0$ satisfying the α-relations, expressed by Equations (57) and (58), is the conditional tensor product W^*-algebra $M \otimes_\alpha \mathfrak{M}_p$.

Theorem 7.4. *(See [8]) Let* $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ *be the* p*-adic* W^**-algebra induced by the* p*-adic* W^**-dynamical system* $\mathcal{Q}(M, p)$*, and let* $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ *be the conditional tensor product* W^**-algebra of* M *and the* p*-prime von Neumann algebra* $\mathfrak{M}_p$ *satisfying the* α*-relations Equations (57) and (58). Then these von Neumann algebras* $\mathcal{M}_p$ *and* $\mathcal{M}_0^p$ *are* $*$*-isomorphic in* $B(\mathcal{H}_p)$*, i.e.,*

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_0^p \tag{59}$$

in $B(\mathcal{H}_p)$.

The above characterization Equation (59) shows that our p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$. So, from now on, we identify $\mathcal{M}_p$ with $\mathcal{M}_0^p$.

8. Free Probability on p-Adic Dynamical W^*-Algebras

In this section, we consider free probability on the p-adic dynamical W^*-algebra

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$$

induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

By Equation (59), the von Neumann subalgebra $\mathcal{M}_p$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$. So, throughout this section, we understand $\mathcal{M}_p$ and $\mathcal{M}_0^p$ alternatively.

First, we assume that a fixed von Neumann algebra M is equipped with a well-defined linear functional ψ on it. *i.e.*, the pair (M, ψ) is a W^*-probability space. Moreover, assume that the linear functional ψ is unital on M, in the sense that

$$\psi(1_M) = 1$$

for the identity element 1_M of M.

By understanding $\mathcal{M}_p$ as $\mathcal{M}_0^p$, we obtain a well-defined conditional expectation

$$E_p : \mathcal{M}_0^p \overset{*\text{-iso}}{=} \mathcal{M}_p \to M_p \tag{60}$$

where

$$M_p \overset{def}{=} M \otimes_\alpha \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p),\ S \subseteq U_p\}\right]$$

where U_p is the unit circle of $\mathbb{Q}_p$, satisfying that

$$E_p(m\chi_S) = E_p(m \otimes \chi_S) \overset{def}{=} m\chi_{S \cap U_p}$$

for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Remark that M_p of Equation (60) is indeed a well-determined W^*-subalgebra of $\mathfrak{M}_p$ (and hence, that of $\mathcal{M}_p = \mathcal{M}_0^p$), because

$$M_p = \chi_{U_p}\mathfrak{M}_p\chi_{U_p}$$

is the compressed W^*-subalgebra of $\mathfrak{M}_p$.

Define now a morphism

$$F_p : M_p \to M_p \tag{61}$$

by a linear transformation satisfying that

$$F_p\left(m\chi_S\right) = m\left(r_S\chi_{U_p}\right)$$

for all $m\chi_S \in M_p$, where $r_S \in [0,\,1]$ satisfies that

$$\int_{\mathbb{Q}_p} \chi_S d\rho_p = r_S \int_{\mathbb{Q}_p} \chi_{U_p} d\rho_p = r_S \left(1 - \frac{1}{p}\right) \tag{62}$$

by Equation (5). Of course, the morphism F_p can be directly defined by a linear morphism satisfying

$$F_p\left(m\chi_S\right) = m\chi_{S\cap U_p}$$

Then, by the identically-distributedness, there exists $r_S \in \mathbb{R}$, such that

$$\int_{\mathbb{Q}_p} \chi_{S\cap U_p} d\rho_p = r_S \int_{\mathbb{Q}_p} \chi_{U_p} d\rho_p$$

and then define a linear functional

$$\gamma : M_p \to \mathbb{C}$$

by a linear functional on $\mathcal{M}_p$, satisfying that, for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$,

$$\gamma \overset{def}{=} \left(\psi \otimes \int_{\mathbb{Q}_p} \bullet\, d\rho_p\right) \circ F_p \tag{63}$$

i.e., a linear functional satisfying that

$$\begin{aligned} \gamma\left(m \,\otimes\, \chi_S\right) &\overset{def}{=} \psi(m) \int_{\mathbb{Q}_p} \left(r_S\chi_{U_p}\right) d\rho_p \\ &= r_S\psi(m)\left(1 - \tfrac{1}{p}\right) \end{aligned}$$

where $r_S \in [0,\,1]$ satisfies Equation (62).

And then define a linear functional

$$\gamma_p : \mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0 \to \mathbb{C}$$

by

$$\gamma_p = \gamma \circ E_p \tag{64}$$

where γ and E_p are in the sense of Equations (63) and (60), respectively. *i.e.*, for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$,

$$\begin{aligned} \gamma_p\left(m\,\chi_S\right) &= \gamma\left(E_p(m\,\chi_S)\right) \\ &= \gamma\left(m\,\chi_{S\cap U_p}\right)\right) = \psi(m) \int_{\mathbb{Q}_p} \left(r_S\,\chi_{U_p}\right) d\rho_p \\ &= r_S\psi(m)\left(1 - \tfrac{1}{p}\right) \end{aligned}$$

for some $r_S \in [0,\,1]$, satisfying Equation (62). Then the pair $(\mathcal{M}_p,\,\gamma_p)$ is a W^*-probability space.

Definition 8.1. *The pair* $(\mathcal{M}_p,\,\gamma_p)$ *is called the* p*-adic dynamical* W^**-probability space.*

The following lemma is obtained by the straightforward computations.

Lemma 8.1. *Let $m\chi_S$ be a free random variable in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Then*

$$\gamma_p\left((m\chi_S)^n\right) = r_S \psi\left(m(m^S)^{n-1}\right)\left(1 - \frac{1}{p}\right) \tag{65}$$

for all $n \in \mathbb{N}$, where $r_S \in [0, 1]$ satisfies Equation (62).

Proof. The proof of Equation (65) is from straightforward computations, by Equations (51) and (56). □

More general to Equation (65), we obtain the following lemma.

Lemma 8.2. *(See [8]) Let $m_1\chi_{S_1}$, ..., $m_n\chi_{S_n}$ be free random variables in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, for $k = 1, ..., n$, for $n \in \mathbb{N}$. Then there exists $r_0 \in [0, 1]$, such that*

$$\gamma_p\left(\prod_{j=1}^{n} m_j\chi_{S_j}\right) = r_0\left(\psi\left(\prod_{j=1}^{N} m_j^{\overset{j-1}{\underset{i=0}{\cap}} S_i}\right)\right)\left(1 - \frac{1}{p}\right) \tag{66}$$

By Equations (65) and (66), we obtain the following free-distributional data of free random variables of $(\mathcal{M}_p, \gamma_p)$.

Theorem 8.3. *(See [8]) Let $(\mathcal{M}_p, \gamma_p)$ be the p-adic dynamical W^*-probability space, and let*

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k}\chi_{S_k}, \text{ for } k = 1, ..., n$$

be free random variables in $(\mathcal{M}_p, \gamma_p)$, for $n \in \mathbb{N}$. Then

$$\gamma_p\left(\prod_{j=1}^{n} T_j^{r_j}\right) = \sum_{(S_1,...,S_n) \in \prod_{j=1}^{n} Supp(T_j)} r_{(S_1,...,S_n)}\left(\psi\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\overset{j-1}{\underset{i=1}{\cap}} S_i\right)}\right)\right)\left(1 - \frac{1}{p}\right) \tag{67}$$

where $[m_{S_j}^{r_j}]^{S_j}$ are in the sense of Equation (55), and $r_1, ..., r_n \in \{1, *\}$, and where $r_{(S_1,...,S_n)} \in [0, 1]$ satisfy Equation (62), for all $(S_1, ..., S_n)$.

Let $(\mathcal{M}_p, \gamma_p)$ be the p-adic dynamical W^*-probability space, and let $m_1\chi_{S_1}$, ..., $m_n\chi_{S_n}$ be free random variables in it, for $n \in \mathbb{N}$, where $m_1, ..., m_n \in M$, and $S_1, ..., S_n \in \sigma(\mathbb{Q}_p)$. Then, we have

$$\gamma_p\left(\prod_{j=1}^{n}\left(m_j\chi_{S_j}\right)^{r_j}\right) = \gamma_p\left(\prod_{j=1}^{n}[m_j^{r_j}]^{S_j}\ \chi_{\underset{j=1}{\overset{n}{\cap}} S_j}\right)$$

where $[m_j^{r_j}]^{S_j}$ are in the sense of Equation (55)

$$= r_0\left(\psi\left(\prod_{j=1}^{n}\left([m_j^{r_j}]^{S_j}\right)^{\overset{n-1}{\underset{j=1}{\cap}} S_j}\right)\right)\left(1 - \frac{1}{p}\right) \tag{68}$$

by Equation (67), where $r_0 \in [0, 1]$ satisfies Equation (62).

So, one can obtain that

$$
\begin{aligned}
k_n\left((m_1\chi_{S_1})^{r_1},\ ...,\ (m_n\chi_{S_n})^{r_n}\right) &= \sum_{\pi\in NC(n)} (\gamma_p)_\pi\left([m_1^{r_1}]^{S_1}\chi_{S_1},\ ...,\ [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu(\pi, 1_n) \\
&= \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}(\gamma_p)_V\left([m_1^{r_1}]^{S_1}\chi_{S_1},\ ...,\ [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right)
\end{aligned}
$$

by the Möbius inversion (See Section 2.2)

$$
\begin{aligned}
&= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\dots,i_k)\in\pi}\gamma_p\left([m_{i_1}^{r_{i_1}}]^{S_{i_1}}\chi_{S_{i_1}}\ \cdots\ [m_{i_k}^{r_{i_k}}]^{S_{i_k}}\chi_{S_{i_k}}\right)\mu\left(0_k,\ 1_k\right)\right) \\
&= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\dots,i_k)\in\pi}\left(r_V\left(\psi\left(\prod_{t=1}^{k}\left([m_{i_t}^{r_{i_t}}]^{S_{i_t}}\right)^{\bigcap_{t=1}^{k-1}S_{i_t}}\right)\right)\left(1-\frac{1}{p}\right)\right)\mu\left(0_k,\ 1_k\right)\right)
\end{aligned}
\tag{69}
$$

by Equation (68), where $r_V \in [0,\ 1]$ satisfy Equation (62).

By Equation (69), we obtain the following inner free structure of the p-adic dynamical W^*-algebra $\mathcal{M}_p$, with respect to γ_p.

Proposition 8.4. *(See [8]) Let $m_1\chi_S$, and $m_2\chi_S$ be free random variables in the p-adic dynamical W^*-probability space $(\mathcal{M}_p,\ \gamma_p)$, with $m_1,\ m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p) \setminus \{\varnothing\}$. Also, assume that S is not measure-zero in $\sigma(\mathbb{Q}_p)$. Then $\{m_1,\ m_1^S\}$ and $\{m_2,\ m_2^S\}$ are free in the W^*-probability space $(M,\ \psi)$, if and only if $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathcal{M}_p,\ \gamma_p)$.*

It is not difficult to check that if $S \cap U_p = \varnothing$, then the family

$$\{m\chi_S : m \in M\}$$

and

$$\{m\chi_Y : m \in M, Y \subseteq U_p in \sigma(\mathbb{Q}_p)\}$$

are free in $(\mathcal{M}_p,\ \gamma_p)$.

Proposition 8.5. *Let $S \in \sigma(\mathbb{Q}_p)$ such that $S \cap U_p = \varnothing$. Then the subsets*

$$\{m\chi_S : m \in M\}$$

and

$$\{m\chi_Y : m \in M, Y \subseteq U_p in \sigma(\mathbb{Q}_p)\}$$

are free in $(\mathcal{M}_p,\ \gamma_p)$.

Proof. Let $m_1\ \chi_S$ and $m_2\ \chi_{U_p} \in \mathcal{M}_p$, with $m_1,\ m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Assume that $S \cap U_p$ is empty. Since $S \cap U_p = \varnothing$, all mixed cumulants of $m_1\chi_S$ and $m_2\chi_{U_p}$ have $r_V = 0$, for some $V \in \pi$ in Equation (69), for all $\pi \in NC(n)$. Therefore, one obtains the following inner freeness condition of $(\mathcal{M}_p,\ \gamma_p)$. □

Motivated by the above proposition, we obtain the following general result.

Theorem 8.6. *Let $S_1 \neq S_2 \in \sigma(\mathbb{Q}_p)$ such that $S_1 \cap S_2 = \varnothing$. Then the subsets $\{m\chi_{S_1} : m \in M\}$ and $\{a\chi_{S_2} : a \in M\}$ are free in $(\mathcal{M}_p, \gamma_p)$, i.e.,*

$$S_1 \cap S_2 = \varnothing \tag{70}$$

$\Longrightarrow$

$$\{m\chi_{S_1} : m \in M\} and \{a\chi_{S_2} : a \in M\} \text{ are free in } (\mathcal{M}_p, \gamma_p)$$

Proof. The proof is a little modification of the proof of the above proposition. Indeed, we can check that

$$S_1 \cap S_2 = \varnothing \Longrightarrow (S_1 \cap U_p) \cap (S_1 \cap U_p) = \varnothing$$

So, we can apply the above proposition. $\square$

9. Euler Subalgebras Φ_p on Certain Dynamical W^* -Probability Spaces

Throughout this section, fix a prime p, and let $(\Phi_p,\ g_p)$ be the Euler probability space in the sense of Section 5. Also, as in Sections 7 and 8, we fix an arbitrary W^*-probability space (M, ψ), where M is a von Neumann algebra in $B(H)$. In particular, we will fix a unital linear functional ψ on M by

$$\psi(m) \overset{def}{=} [m(1_H), 1_H]_H, \text{ for all } m \in M \tag{71}$$

where $[,]_H$ is the inner product of the Hilbert space H, where M acts, and 1_H means the *identity element* (or a *vacuum vector*) of H, satisfying

$$\xi 1_H = \xi = 1_H \xi, \text{ for all } \xi \in H$$

(*i.e.*, we restrict our interests to the cases where M is a certain von Neumann algebra acting on H, having its identity element 1_H.)

In this section, we construct certain W^*-dynamical systems induced by the Euler subalgebra Φ_p. Recall that the close relations between (Φ_p, g_p) and $(\mathfrak{M}_p, g_p)$ in Section 6.

As we have discussed in Section 4, each element $f = [f]_{\mathcal{R}_p}$ of $\mathfrak{A}_p$ is understood as a Krein-space operator Θ_f on the Krein space $\mathbb{C}^2_{A_o}$ (See Equation (33)),

$$\Theta_f = \begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix} \tag{72}$$

Recall that the Krein-space operators Θ_f satisfy Equations (32) and (33) on $\mathbb{C}^2_{A_o}$.

Note that, if $\mathfrak{K}$ is an arbitrary Krein space equipped with its indefinite inner product $[,]$, and $\mathcal{H}$ is an arbitrary Hilbert space equipped with its (positive-definite) inner product $<,>$, the *tensor product space* $\mathfrak{K} \otimes \mathcal{H}$ becomes again a Krein space with its indefinite inner product $[[,]]$, defined by

$$[[a_1 \otimes b_1, a_2 \otimes b_2]] = ([a_1, a_1])\,(< b_1, b_2 >) \tag{73}$$

for all $a_1,\ a_2 \in \mathfrak{K}$, and $b_1,\ b_2 \in \mathcal{H}$. Clearly, the inner product $[[,]]$ on $\mathfrak{K} \otimes \mathcal{H}$ is indefinite, by the indefiniteness of $[,]$ on $\mathfrak{K}$.

Define now a Krein space

$$\mathfrak{K}_H = \mathbb{C}^2_{A_o} \otimes H \tag{74}$$

and construct the tensor product Banach $*$-algebra

$$\Phi_p(M) \overset{def}{=} \Phi_p \otimes_{\mathbb{C}} M \tag{75}$$

acting on $\mathfrak{K}_H$ of Equation (74). Note that, since Φ_p is a Banach $*$-algebra (on $\mathbb{C}^2_{A_o}$), and M is a von Neumann algebra (on H), the topological $*$-algebra $\Phi_p(M)$ is again a Banach $*$-algebra acting on $\mathfrak{K}_H$, under product topology.

Definition 9.1. *The Banach $*$-algebra $\Phi_p(M)$ of Equation (75) acting on the Krein space $\mathfrak{K}_H$ of Equation (74) is called the M-(tensor-)Euler (Banach $*$-)algebra.*

Let $\Phi\ _p(M) = \Phi_p \otimes_{\mathbb{C}} M$ be the M-Euler algebra acting on the Krein space $\mathfrak{K}_H = \mathbb{C}^2_{A_o} \otimes H$, having its indefinite inner product $[,]_{p,H}$ in the sense of Equation (73),

$$[f_1 \otimes x_1, f_2 \otimes x_2]_{p,H} = ([f_1, f_2]_p)\,([x_1,\ x_2]_H) \tag{76}$$

for all $f_j \otimes x_j \in \mathfrak{K}_H$, for $j = 1,\ 2$, where $[,]_p$ is the indefinite inner product on $\mathbb{C}^2_{A_o}$, introduced in [12], such that

$$[f_1, f_2]_p \overset{def}{=} g_p\left(f_1 * f_2^*\right),\ \text{for all } f_1, f_2 \in \mathcal{A}$$

and where $[,]_H$ is the inner product on the Hilbert space H, where M is acting.

Note that the Krein space $\mathfrak{K}_H$ has its identity vector

$$1_{p,M} = 1_{\mathbb{C}^2_{A_o}} \otimes 1_H \tag{77}$$

where, in particular,

$$1_{\mathbb{C}^2_{A_o}} = (0, 1) = 1_{\mathfrak{A}_p}$$

(See Section 2.3). Now, define a linear functional $\psi_{p,M}$ on $\Phi_p(M)$ by the linear morphism satisfying that

$$\psi_{p:M}\left(f \otimes m\right) \overset{def}{=} \left[\left(\Theta_f \otimes m\right)(1_{p,M}),\ 1_{p,M}\right]_{p,H} \tag{78}$$

for all $f \otimes m \in \Phi_p(M)$, where $1_{p,M}$ in Equation (78) is the identity vector of $\mathfrak{K}_H$ in the sense of Equation (77).

Observe the definition Equation (78) more in detail. For $T = f \otimes m \in \Phi_p(M)$,

$$\begin{aligned}
\psi_{p,M}\left(T\right) &= [T(1_{p,M}),\ 1_{p,M}]_{p,H} \\
&= \left[(\Theta_f \otimes m)(1_{\mathfrak{A}_p} \otimes 1_M),\ (1_{\mathfrak{A}_p} \otimes 1_M)\right]_{p,H}
\end{aligned}$$

by Equation (77)

$$\begin{aligned}
&= \left[\Theta_f\left(1_{\mathfrak{A}_p}\right) \otimes m(1_M),\ \left(1_{\mathfrak{A}_p} \otimes 1_M\right)\right]_{p,H} \\
&= \left[f \otimes m(1_M),\ 1_{\mathfrak{A}_p} \otimes 1_M\right]_{p,H} \\
&= \left([f, 1_{\mathfrak{A}_p}]_p\right)\left([m(1_M),\ 1_M]_H\right) \\
&= \left(g_p\left(f * 1^*_{\mathfrak{A}_p}\right)\right)(\psi(m)) = (g_p(f))\,(\psi(m))
\end{aligned}$$

$$= f(p)\psi(m) \tag{79}$$

Proposition 9.1. *Let $f \otimes m$ be an element of the M-Euler algebra $\Phi_p(M)$, for $f \in \Phi_p$ and $m \in (M, \psi)$, and let $\psi_{p,M}$ be the linear functional in the sense of Equation (78) on $\Phi_p(M)$. Then*

$$\psi_{p,M}(f \otimes m) = f(p)\psi(m) \tag{80}$$

Proof. The proof of the Formula (80) is directly from Formula (79). $\square$

By Equations (79) and (80), one has that, for $n \in \mathbb{N}$,

$$\begin{aligned}\psi_{p,M}\left(\phi^{(n)} \otimes m\right) &= \phi^{(n)}(p)\psi(m)\\ &= n\phi(p)\psi(m) = np\left(1 - \tfrac{1}{p}\right)\psi(m)\end{aligned}$$

for all $m \in M$, since

$$\phi^{(n)}(p) = g_p\left(\phi^{(n)}\right) = n\phi(1)^{n-1}\phi(p) = n\phi(p)$$

by Equation (12), for all $n \in \mathbb{N}$.

Corollary 9.2. *Let $\phi^{(n)} \otimes m \in \Phi_p(M)$, for $n \in \mathbb{N}$. Then*

$$\psi_{p,M}\left(\phi^{(n)} \otimes m\right) = np\left(1 - \frac{1}{p}\right)\psi(m) \tag{81}$$

Definition 9.2. *Let $\Phi_p(M)$ be the M-Euler algebra and let $\psi_{p,M}$ be the linear functional on $\Phi_p(M)$ in the sense of Equation (78). Then the Banach $*$-probability space $(\Phi_p(M),\ \psi_{p,M})$ is called the (Krein-space-representational) M-Euler (Banach-$*$-)probability space.*

By Equations (80) and (81) and the results of Section 6, we can find the relations between the free probability on p-adic dynamical W^*-probability spaces and the free probability on the M-Euler probability spaces.

Theorem 9.3. *Let $(M,\ \psi)$ be an arbitrary W^*-probability space in $B(H)$, where ψ is the linear functional on M in the sense of Equation (71), and assume that H has its identity element 1_H. Let $(\Phi_p(M),\ \psi_{p,M})$ be the M-Euler probability space. Let $f \otimes m$ be an arbitrary free random variable of $(\Phi_p(M),\ \psi_{p,M})$. Then there exists a free random variable T of the p-adic dynamical W^*-probability space*

$$\mathcal{M}_p = (M \times_\alpha \sigma(\mathbb{Q}_p),\ \gamma_p)$$

in the sense of Definition 8.1, such that

$$\psi_{p,M}(f \otimes m) = \gamma_p(T) \tag{82}$$

and the converse also holds true.

More precisely, if $\phi^{(n)} \otimes m \in (\Phi_p(M),\ \psi_{p,M})$, then there exists $\alpha\left(U_p\right)(nm) \in (\mathcal{M}_p,\ \gamma_p)$, such that

$$\psi_{p,M}\left(\phi^{(n)} \otimes m\right) = np\left(1 - \frac{1}{p}\right)\psi(m)$$
$$= \gamma_p\left(\alpha\left(p^{-1}U_p\right)(nm)\right) \tag{83}$$

for all $n \in \mathbb{N}$, *and* $m \in (M, \psi)$. *And the converse also holds true.*

Proof. We will prove the relation Equation (83) first. Recall first that, by Equations (40) and (41), if $f \in (\mathfrak{M}_p, \varphi_p)$, then there exists $h \in (\Phi_p, g_p)$, such that

$$\varphi_p(f) = g_p(h)$$

and conversely, if $h \in (\Phi_p, g_p)$, then there exists $f \in (\mathfrak{M}_p, \varphi_p)$, such that

$$g_p(h) = \varphi_p(f)$$

In particular, by Section 6, one has that

$$\varphi_p\left(\chi_{U_p}\right) = \int_{\mathbb{Q}_p} \chi_{U_p} d\rho_p = 1 - \frac{1}{p} = p^{-1}\left(p-1\right) = p^{-1}\phi(p) \tag{84}$$

by Equations (38) and (39).

So, if $\phi^{(n)} \otimes m \in (\Phi_p(M),\ \psi_{p,M})$, for $n \in \mathbb{N}$, and $m \in (M, \psi)$, then

$$\psi_{p,M}\left(\phi^{(n)} \otimes m\right) = p\left(1 - \frac{1}{p}\right)\psi(nm)$$

by Equation (81)

$$= \gamma_p\left(\alpha(p^{-1}U_p)(nm)\right)$$

by Equations (82), (83) and (84). Therefore, the relation Equation (83) holds true.

By Equation (83), and by the facts that (i) Φ_p is generated by $\{\phi^{(n)}\}$, and (ii) $\mathfrak{M}_p$ is generated by $\{\chi_{p^k U_p}\}_{k\in\mathbb{Z}}$, the relation Equation (82) holds true (under tensor-product structures under product topology), by Equations (40) and (41). □

The above characterization Equation (82) (with Equation (83)) characterizes the relation between free probability on our M-Euler probability spaces $(\Phi_p(M),\ \psi_{p,M})$ and free probability on our p-adic dynamical W^*-probability spaces $(\mathcal{M}_p,\ \gamma_p)$, for fixed W^*-probability spaces (M, ψ), where ψ is in the sense of Equation (71).

Theorem 9.4. *Let* $\alpha(S)(m) = \chi_S m \chi_S^* \in (\mathcal{M}_p,\ \gamma_p)$, *for* $S \in \sigma(\mathbb{Q}_p)$ *and* $m \in (M, \psi)$, *where* ψ *is in the sense of Equation (71). Then there exist* $r_0 \in \mathbb{R}$, *such that*

$$\gamma_p\left(\alpha(S)(m)\right) = \psi_{p,M}\left(r_0(\phi \otimes m)\right) \tag{85}$$

for some $\phi \otimes m \in (\Phi_p(M),\ \psi_{p,M})$.

More generally, if $T \in (\mathcal{M}_p,\ \gamma_p)$, *then there exists* $h \in (\Phi_p(M),\ \psi_{p,M})$, *such that*

$$\gamma_p\left(T\right) = \psi_{p,M}\left(h\right) \tag{86}$$

Proof. Recall that, if $S \in \sigma(\mathbb{Q}_p)$, then there exist $N \in \mathbb{N} \cup \{\infty\}$, and $r_1, ..., r_N \in (0, 1]$ in $\mathbb{R}$, and $k_1, ..., k_N \in \mathbb{Z}$, such that

$$\begin{aligned}\int_{\mathbb{Q}_p} \chi_S d\rho_p &= \rho_p(S) = \sum_{j=1}^N r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right) \\ &= \sum_{j=1}^N r_j p^{-k_j}\left(1 - \frac{1}{p}\right) = \sum_{j=1}^N r_j p^{-k_j-1} p \left(1 - \frac{1}{p}\right) \\ &= \sum_{j=1}^N r_j p^{-(k_j+1)} \phi(p)\end{aligned} \tag{87}$$

by Equations (40) and (41).

Observe now that, for $\alpha(S)(m) \in (\mathcal{M}_p,\ \gamma_p)$,

$$\gamma_p\left(\alpha(S)(m)\right) = (\psi(m))\left(\sum_{j=1}^N r_j p^{-(k_j+1)}\phi(p)\right) \tag{88}$$

by Equation (87).

The Formula (88) shows that there exists $r_0 \in \mathbb{R}$, such that

$$\begin{aligned}\gamma_p\left(\alpha(S)(m)\right) &= (r_0\phi(p))\left(\psi(m)\right) \\ &= r_0\left(g_p(\phi)\psi(m)\right) = r_0\psi_{p,M}\left(\phi \otimes m\right)\end{aligned}$$

where, in particular,

$$r_0 = \sum_{j=1}^N r_j p^{-(k_j+1)}$$

where N, r_j and k_j are determined by Equation (87) and where r_0 satisfies Equation (88).

By the Formula (84), the relation Equation (86) holds under linearity and topology. □

The characterization Equation (84) (resp., Equation (86)) is in fact equivalent to Equation (83) (resp., Equation (82)), providing equivalent relation between free probability on p-adic dynamical W^*-probability spaces and free probability on M-Euler probability spaces, whenever a fixed linear functional ψ on M is in the sense of Equation (71).

In Section 10 below, we study special cases where a fixed von Neumann algebra M is a group von Neumann algebra, and ψ is the canonical trace on M.

10. Application Over Group Von Neumann Algebras

In Section 9, we showed the connection between free probability on p-adic dynamical W^*-probability spaces $(\mathcal{M}_p,\ \gamma_p)$, and free probability on M-Euler probability spaces $(\Phi_p(M),\ \psi_{p,M})$ for fixed W^*-probability spaces (M, ψ), where, in particular, ψ is a linear functional in the sense of Equation (71) on M.

Let G be a discrete group and let M_G be the canonical *group von Neumann algebra* acting on the *group Hilbert space* $H_G = l^2(G)$, the l^2-space generated by G, under the *left-regular unitary representation* (H_G, u), where u is the *unitary action of* G *on* H_G, defined by

$$(u(g))\,(h) \overset{denote}{=} u_g(h) \overset{def}{=} gh, \text{ for all } h \in H_G$$

satisfying

$$u_{g_1}u_{g_2} = u_{g_1g_2}, and u_g^* = u_{g^{-1}} \tag{89}$$

where g_1g_2 means the group product in G and g^{-1} means the *group-inverse of* g, for all g_1, g_2, $g \in G$. So, indeed, each u_g induces a unitary on H_G,

$$\begin{aligned} u_g^*u_g &= u_{g^{-1}}u_g = u_{g^{-1}g} = u_{e_G} \\ &= 1_{H_G} = u_{gg^{-1}} = u_gu_{g^{-1}} \\ &= u_gu_g^* \end{aligned}$$

for all $g \in G$, where e_G means the *group-identity of* G, and 1_{H_G} means the identity element of H_G.

Remark that the group Hilbert space H_G has its *orthonormal basis* $\{\xi_g : g \in G\}$ satisfying that

$$\xi_{g_1}\xi_{g_2} = \xi_{g_1g_2}, \text{ for all } g_1, g_2 \in G$$

with the Hilbert-space identity element $\xi_{e_G} = 1_{H_G}$.

The inner product $<,>_G$ on H_G satisfies

$$< \xi_{g_1}, \xi_{g_2} >_G = \delta_{g_1,g_2}, \text{ for all } g_1, g_2 \in G$$

where δ means the *Kronecker delta*.

The group von Neumann algebra M_G has its *canonical trace* tr_G defined by

$$tr_G\left(\sum_{g\in G} t_g u_g\right) \overset{def}{=} t_{e_G}$$

for all $\sum_{g\in G} t_g\, u_g \in M_G$, with $t_g \in \mathbb{C}$.

The trace tr_G is a well-determined linear functional on M_G, moreover, it satisfies

$$tr_G(a_1a_2) = tr_G(a_2a_1), \text{ for all } a_1, a_2 \in M_G$$

and

$$tr_G(u_{e_G}) = 1$$

Definition 10.1. *The* W^**-probability space* $(M_G,\, tr_G)$ *is called the (*canonical*)* group W^*-probability space of G.

Remark that the trace tr_G is understood as

$$tr_G(a) =< a\xi_{e_G}, \xi_{e_G} >_G$$

where ξ_{e_G} is the identity element 1_{H_G} of H_G.

It shows that the trace tr_G of a given group W^*-probability space $(M_G,\, tr_G)$ satisfies the condition Equation (71) naturally. So, one can construct the M_G-Euler probability space $(\Phi_p(M_G),\, \psi_{p,M_G})$ in the sense of Definition 9.1. For convenience, we will denote

$$\Phi_p(M_G) \overset{denote}{=} \Phi_{p,G} \text{ and } \psi_{p,M_G} \overset{denote}{=} \psi_{p,G}$$

We concentrate on computing free moments of generating elements $\phi^{(n)} \otimes u_g$ of $\Phi_{p,G}$ in terms of $\psi_{p,G}$, for all $n \in \mathbb{N}$, $g \in G$. Again, for convenience, we denote

$$\phi_g^{(n)} \overset{denote}{=} \phi^{(n)} \otimes u_g, \text{ for all } n \in \mathbb{N}, g \in G \tag{90}$$

in $(\Phi_{p,G},\, \psi_{p,G})$.

Observe that

$$\psi_{p,G}\left(\phi_g^{(n)}\right) = \left(\phi^{(n)}(p)\right)\left(tr_G(u_g)\right)$$

by Equation (80)

$$= np\left(1 - \frac{1}{p}\right) tr_G(u_g)$$

by Equation (81)

$$= \delta_{g,e_G} np\left(1 - \frac{1}{p}\right)$$

for all $n \in \mathbb{N}$, $g \in G$. *i.e.*,

$$\psi_{p,G}\left(\phi_g^{(n)}\right) = \delta_{g,e_G} np\left(1 - \frac{1}{p}\right) \tag{91}$$

for all $n \in \mathbb{N}$, $g \in G$.

Motivated by the above observation, we obtain the following proposition.

Proposition 10.1. *Let $\phi_g^{(n)}$ be the generating free random variables of the M_G-Euler probability space $(\Phi_{p,G},\ \psi_{p,G})$ in the sense of Equation (90), where M_G is the group von Neumann algebra of a group G, for all $n \in \mathbb{N}$, $g \in G$. Then*

$$\psi_{p,G}\left(\left(\phi_g^{(n)}\right)^k\right) = \delta_{g^k,e_G} nkp\left(1 - \frac{1}{p}\right) \tag{92}$$

for all $k \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, $g \in G$, and $\phi_g^{(n)}$, a corresponding generating element of $(\Phi_{p,G},\ \psi_{p,G})$ in the sense of Equation (90). Then

$$\begin{aligned}
\psi_{p,G}\left(\left(\phi_g^{(n)}\right)^k\right) &= \psi_{p,G}\left(\left(\phi^{(n)} \otimes u_g\right)^k\right) \\
&= \psi_{p,G}\left(\left(\underbrace{\phi^{(n)} * \cdots\cdot * \phi^{(n)}}_{k\text{-times}}\right) \otimes \left(\underbrace{u_g \cdot \cdots \cdot u_g}_{k\text{-times}}\right)\right) \\
&= \psi_{p,G}\left(\phi^{(nk)} \otimes u_{g^k}\right)
\end{aligned}$$

by Equation (89)

$$= \psi_{p,G}\left(\phi_{g^k}^{(nk)}\right) = \delta_{g^k,e_G} nkp\left(1 - \frac{1}{p}\right)$$

by Equation (91). □

The free-moment computation Equation (92) provides the free-distributional data of generating elements $\phi_g^{(n)}$ of $(\Phi_{p,G},\ \psi_{p,G})$.

By Equations (91) and (92), we obtain the following generalized result.

Theorem 10.2. *Let $\phi_{g_j}^{(n_j)}$ be distinct generating elements of the M_G-Euler probability space $(\Phi_{p,G},\ \psi_{p,G})$ of a group G, in the sense of Equation (90), for $j = 1, ..., N$, for $N \in \mathbb{N}$. Then*

$$\psi_{p,G}\left(\prod_{j=1}^{N} \phi_{g_j}^{(n_j)}\right) = \left(\delta_{\prod\limits_{j=1}^{N} g_j,\, e_G}\right)\left(p\sum_{j=1}^{N} n_j\right)\left(1 - \frac{1}{p}\right) \tag{93}$$

Proof. Observe that

$$\begin{aligned}
\psi_{p,G}\left(\prod_{j=1}^{N} \phi_{g_j}^{(n_j)}\right) &= \psi_{p,G}\left(\left(\phi^{(n_1)} * \cdots * \phi^{(n_N)}\right) \otimes \left(u_{g_1} \cdots u_{g_N}\right)\right) \\
&= \psi_{p,G}\left(\phi^{\left(\Sigma_{j=1}^{N} n_j\right)} \otimes u_{\prod\limits_{j=1}^{N} g_j}\right) = \psi_{p,G}\left(\phi^{\left(\Sigma_{j=1}^{N} n_j\right)}_{\left(\prod\limits_{j=1}^{N} g_j\right)}\right) \\
&= \delta_{\prod\limits_{j=1}^{N} g_j,\, e_G}\left(p\textstyle\sum_{j=1}^{N} n_j\right)\left(1 - \frac{1}{p}\right)
\end{aligned}$$

by Equation (91). □

Let $\phi_{g_j}^{(n_j)}$ be given as in the above theorem in $(\Phi_{p,G}, \psi_{p,G})$, for $j = 1, ..., N$, and let $k_n^{p,G}(...)$ means the free cumulant in terms of $\psi_{p,G}$ obtained by the Möbius inversion in the sense of Section 2.2. Then

$$\begin{aligned}
k_N^{p,G}\left(\phi_{g_1}^{(n_1)},\ \phi_{g_2}^{(n_2)},\ ...,\ \phi_{g_N}^{(n_N)}\right) &= \sum_{\pi \in NC(N)} \left(\prod_{V \in \pi} \psi_{p,G}\left(\prod_{j \in V} \phi_{g_j}^{(n_j)}\right)\right)\mu(\pi, 1_N) \\
&= \sum_{\pi \in NC(N)} \left(\prod_{V \in \pi}\left(\left(\delta_{\prod\limits_{j \in V} g_j,\, e_G}\right)\left(p\sum_{j \in V} n_j\right)\left(1 - \tfrac{1}{p}\right)\right)\right)\mu(\pi, 1_N)
\end{aligned}$$

by Equation (93)

$$= \sum_{\pi \in NC(N)} \delta_\pi \sum_{\pi}\left(p\left(1 - \frac{1}{p}\right)\right)^{|\pi|} \mu(\pi, 1_N)$$

where $|\pi|$ means the number of blocks in the partition π, for all $\pi \in NC(N)$, and where

$$\delta_\pi = \prod_{V \in \pi}\left(\delta_{\prod\limits_{j \in V} g_j,\, e_G}\right), \text{ for all } \pi \in NC(N) \tag{94}$$

and

$$\sum_{\pi} = \prod_{V \in \pi}\left(\sum_{j \in V} n_j\right), \text{ for all } \pi \in NC(N)$$

Proposition 10.3. *Let $\phi_{g_j}^{(n_j)}$ be generating elements of $(\Phi_{p,G},\ \psi_{p,G})$, for $j = 1, ..., N$, for $N \in \mathbb{N}$. Then*

$$k_N^{p,G}\left(\phi_{g_1}^{(n_1)},\ ...,\ \phi_{g_N}^{(n_N)}\right) = \sum_{\pi \in NC(N)} \delta_\pi \sum_\pi \left(p(1 - \frac{1}{p})\right)^{|\pi|} \mu(\pi, 1_N) \tag{95}$$

where δ_π and $\sum_\pi$ are in the sense of Equation (94), for all $\pi \in NC(N)$, for all $N \in \mathbb{N}$.

In the Formula (95), note that

$$\sum_\pi = \underset{V \in \pi}{\Pi} \left(\sum_{j \in V} n_j\right) \geq 1, \text{ and } \left(p\left(1 - \frac{1}{p}\right)\right)^{|\pi|} \geq 1 \tag{96}$$

since $n_j \in \mathbb{N}$, for all $j = 1, ..., N$, and since $V \neq \varnothing$, for all $V \in \pi$, for all $\pi \in NC(N)$, for $N \in \mathbb{N}$.

Now, we define positive quantities θ_π by

$$\theta_\pi \overset{def}{=} \sum_\pi \left(p(1 - \frac{1}{p})\right)^{|\pi|} \geq 1 \tag{97}$$

for all $\pi \in NC(N)$, for all $N \in \mathbb{N}$, where $\sum_\pi$ is determined by Equation (95), satisfying Equation (96).

Then the Formula (95) can be re-written by

$$k_N^{p,G}\left(\phi_{g_1}^{(n_1)},\ ...,\ \phi_{g_N}^{(n_N)}\right) = \sum_{\pi \in NC(n)} \delta_\pi \theta_\pi \mu(\pi, 1_N) \tag{98}$$

by Equation (97).

Now, let $\phi_{g_1}^{(n_1)}$ and $\phi_{g_2}^{(n_2)}$ be fixed two distinct generating free random variables in the M_G-Euler probability space $(\Phi_{p,G},\ \psi_{p,G})$ for a group G, as in Equation (90). Then, for any "mixed" n-tuples $(i_1, ..., i_n) \in \{1, 2\}^n$, for $n \in \mathbb{N} \setminus \{1\}$, we have the mixed free cumulants of $\phi_{g_1}^{(n_1)}$, $\left(\phi_{g_1}^{(n_1)}\right)^*$, $\phi_{g_2}^{(n_1)}$ and $\left(\phi_{g_2}^{(n_2)}\right)^*$ as follows,

$$k_n^{p,G}\left(\phi_{g_{i_1}}^{(n_{i_1}) \atop r_{i_1}},\ ...,\ \phi_{g_{i_n}}^{(n_{i_n}) \atop r_{i_n}}\right) = \sum_{\pi \in NC(\{i_1, ..., i_n\})} \delta_\pi \theta_\pi \mu(\pi, 1_n) \tag{99}$$

by Equations (95) and (98), for $(r_{i_1}, ..., r_{i_n}) \in \{1, -1\}^n$, where δ_π are in the sense of Equation (94), and θ_π are in the sense of Equation (97), for all $\pi \in NC(n)$. Remark that

$$\left(\phi_g^{(n)}\right)^* = \left(\phi^{(n)} \otimes u_g\right)^* = \phi^{(n)*} \otimes u_{g^{-1}} = \phi^{(n)} \otimes u_{g^{-1}} = \phi_{g^{-1}}^{(n)}$$

for all $n \in \mathbb{N}$, $g \in G$. So, the mixed free cumulants of

$$\{\phi_{g_1}^{(n_1)}, \left(\phi_{g_1}^{(n_1)}\right)^*, \phi_{g_2}^{(n_1)} \left(\phi_{g_2}^{(n_2)}\right)^*\}$$

are indeed determined by Equation (99).

Consider further that, if noncrossing partitions π_1 and π_2 have different numbers of blocks in $NC(n)$, then

$$\theta_{\pi_1} \neq \theta_{\pi_2} \text{ in } \mathbb{N}$$

by the inequality of Equation (97).

So, to make the sum Equation (99) be vanishing, in general,

$$\delta_{\pi} = 0, or \delta_{\pi}\theta_{\pi} = 1, \text{ for all } \pi \in NC(n) \tag{100}$$

by Equation (99). Indeed, if $\delta_{\pi} = 0$, for all $\pi \in NC(n)$, then definitely, the right-hand side of Equation (99) vanishes. And if $\delta_{\pi}\theta_{\pi} = 1$, then the right-hand side of Equation (99) becomes

$$\sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0$$

However, note that, by Equations (96) and (97),

$$\theta_{\pi} > 1, \text{ in general, whenever } n > 2, \text{ in Equation } (99) \tag{101a}$$

and hence, in general,

$$\delta_{\pi}\theta_{\pi} \neq 1 \tag{101b}$$

Moreover, by the very definition,

$$\theta_{\pi_1} \neq \theta_{\pi_2} \Longleftrightarrow \pi_1 \neq \pi_2 \tag{102}$$

By Equations (101a) and (101b) the condition Equation (100) can be re-written by that "in general," $\delta_{\pi} = 0$. *i.e.*, to vanish the Formula (99), we in general have to have $\delta_{\pi} = 0$. Recall, by Equation (94), that

$$\delta_{\pi} = \prod_{V \in \pi} \delta_{\prod_{i_j \in V} g_{i_j}^{r_{i_1}},\, e_G}, \text{ in Equation } (99)$$

for all $\pi \in NC(n)$. So, to satisfy $\delta_{\pi} = 0$, for all $\pi \in NC(n)$, for all $n \in \mathbb{N} \setminus \{1\}$

$$\delta_{\prod_{j=1}^{m} g_{i_j}^{r_{i_j}},\, e_G} = 0$$

for all $m \in \mathbb{N} \setminus \{1\}$, for all mixed m-tuples $(i_1, ..., i_m) \in \{1, 2\}^m$ and $(r_{i_1}, ..., r_{i_m}) \in \{1, -1\}^m$, equivalently,

$$\prod_{j=1}^{m} g_{i_j}^{r_{i_j}} \neq e_G \text{ in } G$$

where $(i_1, ..., i_m) \in \{1, 2\}$, and $(r_{i_1}, ..., r_{i_m}) \in \{1, -1\}$ are "mixed," for $m \in \mathbb{N} \setminus \{1\}$.

By the above observation, one can get the following refined result of Equation (95), equivalently Equation (99).

Proposition 10.4. *Let* $T_j = \phi_{g_j}^{(n_j)} \in (\Phi_{p,G}, \psi_{p,G})$ *be in the sense of Equation (90), for* $j = 1, ..., N$, *for* $N \in \mathbb{N}$. *Then*

$$kk_N^{p,G}(T_1, ..., T_N) = \begin{cases} \sum_{\pi \in NC(T_1,...,T_N)} \theta_{\pi}\mu(\pi, 1_N) & \text{if } N \text{ is even} \\ \\ 0 & \text{if } N \text{ is odd} \end{cases} \tag{103}$$

where

$$NC(T_1, ..., T_N) = \{\pi \in NC(N)\,|\delta_\pi \neq 0\}$$

whenever N is even.

Proof. The proof is done by Equations (95), (99) and (100). □

Moreover, by Equation (103), we obtain the following corollary.

Corollary 10.5. *Let $T_j = \phi_{g_j}^{(n_j)} \in (\Phi_{p,G},\ \psi_{p,G})$ be in the sense of Equation (90), for $j = 1, 2$, and let $(S_{i_1}, ..., S_{i_{2n}})$ be a "mixed" $2n$-tuple of $\{T_1, T_1^*, T_2, T_2^*\}$. Then*

$$k_{2n}^{p,G}\,(S_{i_1},\ ...,\ S_{i_{2n}}) = \sum_{\pi \in NC(S_{i_1},\ ...,\ S_{i_{2n}})} \theta_\pi \mu(\pi, 1_{2n}) \tag{104}$$

where θ_π are in the sense of Equation (97), and

$$NC(S_{i_1}, ..., S_{i_{2n}}) = \left\{ \pi \in NC_E(\{i_1, ..., i_{2n}\}) \;\middle|\; \begin{array}{c} \forall V \in \pi,\ V \text{ contains the} \\ \text{same number of } T_1 \text{ and } T_1^*, \\ \text{"or" the same number of} \\ T_2 \text{ and } T_2^* \text{ in it} \end{array} \right\}$$

where

$$NC_E(2n) \overset{def}{=} \{\pi \in NC(2n) : \forall V \in \pi, |V| \in 2\mathbb{N}\}$$

Proof. The proof of Equation (104) is from Equation (103). □

The above Formulas Equations (93), (95), (103) and (104) provide equivalent joint free-distributional data for free random variables $T_j = \phi_{g_j}^{(n_j)}$ and T_j^*, for $j = 1, ..., n$, for $n \in \mathbb{N}$.

Observation *The Formulas* (93), (95), (103) *and* (104) *also provide equivalent free-distributional data of certain free random variables for p-adic dynamical W^*-algebras by Sections 6–9.*

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Conflicts of Interest

The authors declare no conflict of interest.

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