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Singular Bilinear Integrals in Quantum Physics

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Abstract: Bilinear integrals of operator-valued functions with respect to spectral measures and integrals of scalar functions with respect to the product of two spectral measures arise in many problems in scattering theory and spectral analysis. Unfortunately, the theory of bilinear integration with respect to a vector measure originating from the work of Bartle cannot be applied due to the singular variational properties of spectral measures. In this work, it is shown how "decoupled" bilinear integration may be used to find solutions X of operator equations $AX - XB = Y$ with respect to the spectral measure of A and to apply such representations to the spectral decomposition of block operator matrices. A new proof is given of Peller's characterisation of the space $L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ of double operator integrable functions for spectral measures P, Q acting in a Hilbert space $\mathcal{H}$ and applied to the representation of the trace of $\int_{\Lambda \times \Lambda} \varphi \, d(PTP)$ for a trace class operator T. The method of double operator integrals due to Birman and Solomyak is used to obtain an elementary proof of the existence of Krein's spectral shift function.

Keywords: bilinear integration; tensor products; operator equations; double operator integrals; spectral measure

1. Introduction

Since its inception, the mathematical treatment of quantum theory has generated many problems in measure and integration theory, some of which are still being worked out. At the forefront is the spectral theory of self-adjoint operators in Hilbert space, where the decomposition $A = \sum_{\lambda \in \sigma(A)} \lambda . P_\lambda$ of a Hermitian matrix $A = \{a_{jk}\}_{j,k=1}^n$ with respect to the orthogonal projections P_λ onto the eigenspace of

the eigenvalues $\lambda \in \sigma(A)$ of A is replaced by the spectral decomposition $T = \int_{\sigma(T)} \lambda \, dP(\lambda)$ with respect to the self-adjoint spectral measure P associated with the self-adjoint linear operator T. The spectrum $\sigma(T)$ of T is the complement of the set of all numbers $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ is a bounded linear operator on $\mathcal{H}$, so any eigenvalue of T belongs to $\sigma(T)$.

For a quantum system in a state $\psi \in \mathcal{H}$, the conventional interpretation of quantum measurement suggests that the number $\|P(E)\psi\|^2$ is the probability that an observation of the quantity represented by the self-adjoint operator T has its value in the Borel set $E \subseteq \sigma(T)$. If ψ is also an eigenvector of T for the eigenvalue $\lambda \in \mathbb{R}$, then:

$$\|P(E)\psi\| = \begin{cases} 1, & \lambda \in E \\ 0, & \lambda \notin E \end{cases}$$

that is, in the state ψ, an observation of T yields the value λ with certainty, so explaining such facts as the quantisation of energy levels in an atom. Clearly, the operator-valued spectral measure uniquely associated with a quantum observable is a fundamental concept in quantum theory.

Another problem of integration theory arising from quantum physics is the Feynman-Kac formula:

$$e^{-it(H_0 + Q(V))} = \int_\Omega e^{-i \int_0^t V \circ X_s \, ds} \, dM_t, \quad t > 0 \tag{1}$$

The left-hand side represents the dynamics of a quantum system described by a free Hamiltonian H_0 perturbed by a potential V where $Q(V)$ is the operator of multiplication by V. The finitely-additive operator valued set functions M_t, $t > 0$, are manufactured from the free evolution e^{-itH_0}, $t \geq 0$, and the spectral measure Q associated with the configuration or position operators; see [1–4] and the extensive references in these monographs. Although we shall need to consider integration with respect to certain finitely additive set functions manufactured from a pair of spectral measures, the operator valued integrals considered in the present work are orders of magnitude more tractable than the singular path integral on the right hand side of equation (1). Even at the basic level of integration theory considered here, Grothendieck's inequality [26] provides essential insights.

A number of problems in scattering theory, spectral theory and their applications in the context of a Hilbert space $\mathcal{H}$ are treated by considering integrals of the form:

$$\int_\Sigma \Phi(\sigma) E(d\sigma), \quad \int_\Sigma E(d\sigma)\Phi(\sigma) \text{ and } \int_{\Sigma \times \Lambda} \varphi(\sigma, \lambda) \, E(d\sigma) T F(d\lambda) \tag{2}$$

for spectral measures E and F, $T \in \mathcal{L}(\mathcal{H})$ and operator valued functions $\Phi : \Sigma \to \mathcal{L}(\mathcal{H})$ and scalar functions φ. Such integrals are *bilinear* in the functions Φ and operator valued measures E and F. Unfortunately, the well-developed theories of bilinear integration of Bartle [5] and Dobrakov [6–9] do not apply in this situation due to the variational properties of spectral measures acting on a Hilbert space.

The 2-variation $\sup_{\mathcal{P}} \sum_{B \in \mathcal{P}} \|E(B)\psi\|^2$ for a spectral measure E and a vector $\psi \in \mathcal{H}$ is always finite. The supremum is taken over all finite partitions $\mathcal{P}$ into Borel sets. However, for the spectral measure of multiplication by characteristic functions say, the *total variation* $\sup_{\mathcal{P}} \sum_{B \in \mathcal{P}} \|\chi_B \psi\|$ of the $L^2([0,1])$-valued measure $B \longmapsto \chi_B \psi$, $B \in \mathcal{B}([0,1])$, is infinite on each set of positive measure where $\psi \in L^2([0,1])$ is nonzero, which leads to difficulties interpreting integrals like (2) using the classical theory of bilinear integration utilising semivariation [5].

An integral of the first form in (2) arises in treating the connection between time-dependent scattering theory and stationary state scattering theory [10,11]. Bilinear integrals of this nature have recently been handled by a "decoupling" method in [12]. In this technique, an auxiliary tensor product $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau\mathcal{H}$ is defined, and the tensor product integral:

$$\int_\Sigma \Phi(\sigma) \otimes (Eh)(d\sigma)$$

is an element of $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau\mathcal{H}$ for each $h \in \mathcal{H}$, in the fashion of [13]. The bilinear evaluation map $(T,h) \longmapsto Th$, $T \in \mathcal{L}(\mathcal{H})$, $h \in \mathcal{H}$, uniquely defines a continuous linear map $J : \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau\mathcal{H} \to \mathcal{H}$ for which $J(T \otimes h) = Th$ for every $T \in \mathcal{L}(\mathcal{H})$ and $h \in \mathcal{H}$, so that:

$$\left(\int_\Sigma \Phi(\sigma)E(d\sigma)\right) h := \int_\Sigma \Phi(\sigma)(Eh)(d\sigma) := J\left(\int_\Sigma \Phi(\sigma) \otimes (Eh)(d\sigma)\right)$$

By this means, the variational properties of the spectral measure E play no role in the definition of the first integral in (2).

Similar difficulties arise in the theory of stochastic integration, which is now thoroughly understood. For a Brownian motion process $\langle b_t\rangle_{t\geq 0}$ with respect to a probability measure P, there exists a unique $L^2(P)$-valued orthogonally scattered measure W given by $W([s,t)) = b_t - b_s$, $0 \leq s < t$. The multiplication map $J : X \otimes Y \longmapsto X.Y$ for random variables X, Y is actually continuous into $L^2(P)$ on the closure of $\int_0^t X \otimes dW$ in $L^2(P \otimes P)$ as X runs over all adapted simple processes; see [14], Theorem 5.3. For an adapted process X, the stochastic integral $\int_0^t X_s db_s$ may be viewed as an example of a bilinear integral $J\left(\int_0^t X \otimes dW\right)$. The two-variation of W on a Borel set $B \subset [0, \infty)$ is the Lebesgue measure $|B|$ of B, but $W : \mathcal{B}([0, \infty)) \to L^2(P)$ has infinite variation on any set of positive Lebesgue measure: the variation of a Brownian motion process $\langle b_t\rangle_{t\geq 0}$ on any interval is infinite P-a.e.

The last type of integral in (2) is a double operator integral studied in a series of papers by Birman and Solomyak [15–20]. With the choice of the function:

$$\varphi_f(\sigma, \lambda) = \begin{cases} \frac{f(\sigma)-f(\lambda)}{\sigma-\lambda}, & \sigma \neq \lambda \\ f'(\lambda), & \sigma = \lambda \end{cases}$$

for a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, the equality:

$$f(A) - f(B) = \left(\int_{\sigma(A)\times\sigma(B)} \varphi_f \, d(E \otimes F)_{\mathcal{C}_1(\mathcal{H})}\right)(A - B)$$

holds if A, B are densely-defined self-adjoint operators in the Hilbert space $\mathcal{H}$ with spectral measures E and F, respectively, and $A - B$ belongs to the operator ideal $\mathcal{C}_1(\mathcal{H})$ of all trace class operators on $\mathcal{H}$. The finitely-additive spectral measure $(E \otimes F)_{\mathcal{C}_1(\mathcal{H})}$ acts on the operator ideal $\mathcal{C}_1(\mathcal{H})$.

Integration with respect to finitely-additive spectral measures is studied in considerable detail in [21], Section 2, where for an algebra $\mathcal{A}$ of subsets of a non-empty set Ω and a complex Banach space X, a finitely-additive set function $M : \mathcal{A} \to \mathcal{L}(X)$ is called a finitely-additive spectral measure if $M(C \cap D) = M(C)M(D)$ for all $C, D \in \mathcal{A}$ and $M(\Omega) = I$, the identity operator on X. In the case that $\mathcal{A}$ is actually a σ-algebra of subsets of Ω and M is countably additive for the strong operator topology of $\mathcal{L}(X)$, the operator-valued measure M is simply called a spectral measure. The spectral theorem for

a self-adjoint operator T acting in a Hilbert space asserts the existence of a unique spectral measure P whose values are self-adjoint projection operators, such that $T = \int_{\sigma(T)} \lambda \, dP(\lambda)$. The finitely-additive spectral measure $(E \otimes F)_{\mathcal{C}_1(\mathcal{H})}$ is defined on the algebra $\mathcal{A}$ generated by measurable rectangles $U \times V$ by the formula:

$$\big((E \otimes F)_{\mathcal{C}_1(\mathcal{H})}(U \times V)\big)(T) = E(U)TF(V), \quad T \in \mathcal{C}_1(\mathcal{H})$$

for $U \in \mathcal{B}(\sigma(A))$, $V \in \mathcal{B}(\sigma(B))$. It is only in trivial cases that $(E \otimes F)_{\mathcal{C}_1(\mathcal{H})}$ is countably additive for the strong operator topology of $\mathcal{L}(\mathcal{C}_1(\mathcal{H}))$; see Proposition 19 below. The finitely-additive spectral measure $(E \otimes F)_{\mathfrak{S}}$ is similarly defined in the case that $\mathfrak{S}$ is a symmetrically-normed operator ideal. The commutative Banach *-algebra $L^1((E \otimes F)_{\mathfrak{S}})$ of all equivalence classes of $(E \otimes F)_{\mathfrak{S}}$-integrable functions is discussed in Proposition 15 below.

An application of the formula above leads to the expression:

$$\mathrm{tr}(f(A) - f(B)) = \int_{\mathbb{R}} f'(\lambda) \, d\Xi(\lambda)$$

for a finite Borel measure Ξ on $\mathbb{R}$. It turns out that Ξ is absolutely continuous with respect to the Lebesgue measure, and its density ξ with respect to the Lebesgue measure is Krein's spectral shift function with respect to the pair (A, B). If $S(\lambda)$ is the scattering operator associated with the self-adjoint operators A and B, then the remarkable formula:

$$\mathrm{Det}(S(\lambda)) = e^{-2\pi i \xi(\lambda)}, \quad \lambda \in \mathbb{R}$$

holds ([22], Chapter 8).

Examples of the other integrals of the form (2) arise in the spectral theory of block operators, resonance and optimal control theory, numerical analysis and the theory of Krein's spectral shift function in scattering theory and non-commutative geometry. The authoritative treatment of the applications of the decoupling approach to bilinear integration is given in the papers referred to in the sections to follow.

Section 2 deals with solutions of operator equations $AX - XB = Y$ where A is a self-adjoint operator, B is a closed linear operator and Y is bounded. Representations for the solution X given in Section 3 in terms of an integral of the second form in (2) leads to estimates for the norm of X in terms of the spectral separation between the linear operators A and B. As in the case of the connection between time-dependent and stationary state scattering theory [10,11], such integrals have been previously referred to as strong operator-valued Stieltjes integrals in [23,24]. A brief account of the connection with the spectral analysis of block operator matrices considered in [23,24] is also given. In case B is also a self-adjoint operator, the solution X of the operator equation above can be expressed as a double operator integral described in Section 4 using decoupled integrals. For spectral measures E and F acting on a Hilbert space $\mathcal{H}$, a double operator integral is an integral with respect to a finitely-additive spectral measure $(E \otimes F)_{\mathfrak{S}}$ acting on a symmetrically normed operator ideal $\mathfrak{S}$. The space $L^1((E \otimes F)_{\mathfrak{S}})$ of $(E \otimes F)_{\mathfrak{S}}$-integrable functions has been characterised by Peller in [25] for the cases $\mathfrak{S} = \mathcal{L}(\mathcal{H})$ and $\mathfrak{S} = \mathcal{C}_1(\mathcal{H})$. An elementary proof of Peller's characterisation is given in Theorem 16 of Section 5 by appealing to Pisier's recent account [26] of Grothendieck's theorem. Peller's representation facilitates an explicit formula given in [20] for the trace of the integral $\int_{\Lambda \times \Lambda} \varphi \, d(ETE)$ for

$\varphi \in L^1((E \otimes E)_{\mathfrak{S}})$ in the case $T \in \mathfrak{S} = \mathcal{C}_1(\mathcal{H})$. The existence of Krein's spectral shift function is established in Theorem 22 of Section 6 using double operator integrals and Fourier transforms.

2. Linear Operator Equations

The analysis of the equation $AX - XB = Y$ for linear operators A, B, X and Y acting in a Hilbert space $\mathcal{H}$ has many applications in operator theory, differential equations and quantum physics; see [27] for a relaxed discussion with numerous examples.

Starting with the case of scalars, the equation $ax - xb = y$ has a unique solution provided that $a \neq b$. For the case of diagonal matrices $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \text{diag}(\mu_1, \ldots, \mu_n)$, for any matrix $Y = \{y_{ij}\}_{i,j=1}^n$, there exists a unique solution X of the equation $AX - XB = Y$ if and only if $\lambda_i - \mu_j \neq 0$ for $i, j = 1, \ldots, n$, and then, the solution $X = \{x_{ij}\}_{i,j=1}^n$ is given by:

$$x_{ij} = \frac{y_{ij}}{\lambda_i - \mu_j}, \quad i, j = 1, \ldots, n.$$

The operator version is called the Sylvester–Rosenblum theorem in [27], although earlier versions are due to Krein and Daletskii ([27], p. 1). For a continuous linear operator A on a Banach space $\mathcal{X}$, the spectrum $\sigma(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible.

Theorem 1 (Sylvester–Rosenblum theorem). *Let $\mathcal{X}$ be a Banach space and let A and B be continuous linear operators on $\mathcal{X}$ for which $\sigma(A) \cap \sigma(B) = \emptyset$. Then, for each operator $Y \in \mathcal{L}(\mathcal{X})$, the equation $AX - XB = Y$ has a unique solution $X \in \mathcal{L}(\mathcal{X})$.*

As a taster for applications of the Sylvester–Rosenblum Theorem, suppose that A and B are bounded normal operators on a Hilbert space $\mathcal{H}$ with spectral measures P_A and P_B, respectively. Then, there exists $c > 0$, such that for any two Borel subsets S_1 and S_2 of $\mathbb{C}$ separated by a distance:

$$\delta = \inf\{|x - y| : x \in S_1, \ y \in S_2\}$$

the projections $E = P_A(S_1)$, $F = P_B(S_2)$, satisfy the norm estimate:

$$\|EF\| \leq \frac{c}{\delta}\|A - B\|.$$

The norm $\|EF\|$ represents the angle between the subspaces $\text{ran}(E)$ and $\text{ran}(F)$. Such estimates are useful in numerical computations. Even in finite dimensional Hilbert spaces, the Sylvester–Rosenblum theorem leads to eigenvalue estimates for matrix norms independent of dimension.

Theorem 2 ([28], Theorem 5.1a). *Let A and B be two normal $(n \times n)$ matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, counting multiplicity. With the same constant c mentioned above, if $\|A - B\| \leq \epsilon/c$, then there exists a permutation π of the index set $\{1, \ldots, n\}$, such that:*

$$|\alpha_i - \beta_{\pi i}| < \epsilon$$

for $i = 1, \ldots, n$.

The Sylvester–Rosenblum theorem also comes with a representation of the solution X of the equation $AX - XB = Y$ if $\sigma(A) \cap \sigma(B) = \emptyset$. Suppose that the contour Γ is the union of closed contours in the plane, with total windings one around $\sigma(A)$ and zero around $\sigma(B)$. Then:

$$X = \frac{1}{2\pi i} \int_\Gamma (\zeta I - A)^{-1} Y (\zeta I - B)^{-1}. \tag{3}$$

Other representations of the solution are possible by utilising the spectral properties of the operators A and B; see [27], Section 9.

In the present paper, we are concerned with solutions X of the operator equation $AX - XB = Y$ when A is an unbounded self-adjoint or normal operator acting in a Hilbert space $\mathcal{H}$ and B is a closed unbounded operator. If the spectra $\sigma(A)$ and $\sigma(B)$ are a positive distance apart, then we hope to construct the solution X of $AX - XB = Y$ by the formula:

$$X = \int_{\sigma(A)} dP_A(\zeta) Y (\zeta I - B)^{-1} \tag{4}$$

in place of (3) with respect to the spectral measure P_A of A. The operator-valued measure P_A acts on the values of the operator-valued function $\zeta \longmapsto Y(\zeta I - B)^{-1}$. As in the case of scattering theory considered in [12], for $h \in \mathcal{H}$, the vector $Xh \in \mathcal{H}$ often has the representation:

$$Xh = J \int_{\sigma(A)} P_A(d\zeta) \otimes \left(Y(\zeta I - B)^{-1} h \right)$$

where the $\mathcal{H}$-valued function $\zeta \longmapsto Y(\zeta I - B)^{-1}h$, $\sigma \in \sigma(A)$, is P_A-integrable in the tensor product space $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\tau \mathcal{H}$ and $J : \mathcal{L}(\mathcal{H}) \widehat{\otimes}_\tau \mathcal{H} \to \mathcal{H}$ is the continuous linear extension of the composition map $T \otimes h \mapsto Th$, $T \in \mathcal{L}(\mathcal{H})$, $h \in \mathcal{H}$.

If the operator B is itself a bounded linear operator, then the simpler representation (3) may be employed with the contour Γ winding once around $\sigma(B)$ and zero times around $\sigma(A)$.

Because we shall be dealing with unbounded operators A and B, we have to be careful about domains when interpreting the equation $AX - XB = Y$. We follow the treatment in [23], Section 2. Applications of Equation (4) to perturbation theory and the spectral shift function may also be found in [23] and at the end of the next section.

3. Integral Solutions of Operator Equations

Definition 1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $A : \mathcal{D}(A) \to \mathcal{K}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ are closed and densely-defined linear operators with domains $\mathcal{D}(A) \subset \mathcal{K}$ and $\mathcal{D}(B) \subset \mathcal{H}$. Given $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, a continuous linear operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be a weak solution of the equation:

$$AX - XB = Y \tag{5}$$

if for every $h \in \mathcal{D}(B)$ and $k \in \mathcal{D}(A^*)$, the equality:

$$(Xh, A^*k) - (XBh, k) = (Yh, k) \tag{6}$$

holds with respect to the inner product $(\cdot, \cdot)$ of $\mathcal{K}$.

The domain $\mathcal{D}(A^*)$ of the adjoint A^* of A is the set of all elements k of $\mathcal{K}$, such that the linear map $h \longmapsto (Ah, k)$, $h \in \mathcal{D}(A)$, is the restriction to $\mathcal{D}(A)$ of $h \longmapsto (h, y)$, $h \in \mathcal{H}$, for an element $y \in \mathcal{K}$ and then $y = A^*k$.

A strong solution $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ of (5) has the property that:

$$\mathrm{ran}(X \upharpoonright \mathcal{D}(B)) \subset \mathcal{D}(A) \tag{7}$$

and:

$$AXh - XBh = Yh, \quad h \in \mathcal{D}(B) \tag{8}$$

The existence of strong solutions of the operator Equation (5) is discussed in [29] under the assumption that A and $-B$ are the generators of C_0-semigroups, a situation that arises in delay or partial differential equations and control theory. Strong solutions of (5) may not exist in this setting, even when the spectra $\sigma(A)$ and $\sigma(B)$ are separated by a vertical strip ([29], Example 9).

In the case that A and B are both self-adjoint operators, the following result is a consequence of [28], Theorem 4.1; (see [23], Theorem 2.7).

Theorem 3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $A : \mathcal{D}(A) \to \mathcal{K}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ are self-adjoint operators whose spectra $\sigma(A)$ and $\sigma(B)$ are a distance $\delta > 0$ apart. Then, Equation (5) has a unique weak solution:*

$$X = \int_{\mathbb{R}} e^{-itA} Y e^{itB} f_\delta(t)\, dt$$

for any function $f_\delta \in L^1(\mathbb{R})$, continuous on $\mathbb{R} \setminus \{0\}$, such that:

$$\int_{\mathbb{R}} e^{-isx} f_\delta(s)\, ds = \frac{1}{x} \quad \text{for } |x| > \frac{1}{\delta}.$$

Moreover $\|X\| \leq \frac{\pi}{2\delta} \|Y\|$.

The integral representing the solution X is a Pettis integral for the strong operator topology.

We now turn to the tensor product topology τ mentioned above. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. For $y^* \in \mathcal{Y}^*$, we have:

$$\left| \left\langle \sum_{j=1}^{n} T_j x_j, y^* \right\rangle \right| = \left| \left\langle \sum_{j=1}^{n} x_j, T_j^* y^* \right\rangle \right|$$

$$\leq \sum_{j=1}^{n} \|x_j\|_{\mathcal{X}} . \|T_j^* y^*\|_{\mathcal{X}^*}$$

for all $T_j \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $x_j \in \mathcal{X}$, $j = 1, \ldots, n$ and all $n = 1, 2, \ldots$. Hence, if we let:

$$\|u\|_\tau = \sup_{\|y^*\| \leq 1} \inf \left\{ \sum_{j=1}^{n} \|x_j\|_{\mathcal{X}} . \|T_j^* y^*\|_{\mathcal{X}^*} : u = \sum_{j=1}^{n} T_j \otimes x_j \right\} \tag{9}$$

over all representations $u = \sum_{j=1}^{n} T_j \otimes x_j$, $n = 1, 2 \ldots$, of $u \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \otimes \mathcal{X}$, then the inequality $\|Ju\|_{\mathcal{Y}} \leq \|u\|_\tau$ holds for the product map $Ju = \sum_{j=1}^{n} T_j x_j$ by the Hahn–Banach theorem. The completion of the linear space $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \otimes \mathcal{X}$ with respect to the norm $\| \cdot \|_\tau$ is written as $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \widehat{\otimes}_\tau \mathcal{X}$.

For a self-adjoint operator A in a Hilbert space $\mathcal{K}$ and a closed, densely-defined operator B in a Hilbert space $\mathcal{H}$, the domains $\mathcal{D}(B)$ and $\mathcal{D}(A)$ are endowed with the respective graph norms associated with the closed operators B and A. Suppose also that τ is the topology on the tensor product $\mathcal{L}(\mathcal{X}) \otimes \mathcal{X}$ defined by Formula (9) with $\mathcal{X} = \mathcal{Y} = \mathcal{K}$, and let $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}$ be the completion of the tensor product with the norm topology τ. As in [12], Proposition B.11, the collection $\mathcal{K} \otimes \mathcal{K}^* \otimes \mathcal{K}^*$ of continuous linear functionals on the Banach space E separates points of E, and the composition map:

$$T \otimes k \longmapsto Tk, \quad T \in \mathcal{L}(\mathcal{K}), \quad k \in \mathcal{K}$$

has a continuous linear extension $J_E : E \rightarrow \mathcal{K}$. The following definition of a bilinear integral is suggested by [13].

Definition 2. *Let $\mathcal{K}$ be a Hilbert space. A function $f : \Omega \rightarrow \mathcal{K}$ is said to be m-integrable in $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}$ for an operator valued measure $m : \mathcal{S} \rightarrow \mathcal{L}(\mathcal{K})$, if for each $x, x', y' \in \mathcal{K}$, the scalar function (f, x') is integrable with respect to the scalar measure (mx, y') and for each $S \in \mathcal{S}$, there exists an element $(m \otimes f)(S)$ of E, such that:*

$$((m \otimes f)(S), x \otimes y' \otimes x') = \int_S (f, x') \, d(mx, y') \tag{10}$$

for every $x, x', y' \in \mathcal{K}$.

If f is m-integrable in E, then $mf(S) \in \mathcal{K}$ is defined for each $S \in \mathcal{S}$ by:

$$mf(S) = J_E\big((m \otimes f)(S)\big).$$

We also denote $mf(S)$ by $\int_S dm\, f$ or $\int_S dm(\omega)\, f(\omega)$.

In the present context, the representation of solutions of Equation (5) via bilinear integration is analogous to the treatment in [12], Section 3, for scattering theory.

Example 3. Suppose that A is a bounded self-adjoint operator defined on a Hilbert space $\mathcal{K}$, such that $\sigma(A) \subset (-\infty, -\delta)$ for some $\delta > 0$. Let $-B$ be the generator of a uniformly-bounded C_0-semigroup e^{-tB}, $t \geq 0$, on the Hilbert space $\mathcal{H}$.

We can employ (3) in this situation to represent the weak solution of Equation (5), but it is instructive to see how the integral (4) converges with the assumptions above.

Let $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$ be the projective tensor product of the Hilbert space $\mathcal{K}$ with the space $\mathcal{L}(\mathcal{K})$ of bounded linear operators on $\mathcal{K}$ with the uniform norm (see [30], Section III.6). Then, $e^{tA} \otimes (Ye^{-tB}h)$ belongs to the tensor product $\mathcal{L}(\mathcal{K}) \otimes \mathcal{K}$ for each $t \geq 0$ and $h \in \mathcal{H}$, and the function $t \longmapsto e^{tA} \otimes (Ye^{-tB}h)$, $t \geq 0$, is continuous in $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$, because A is assumed to be bounded, so:

$$e^{tA} \otimes (Ye^{-tB}h) = I \otimes (Ye^{-tB}h) + \sum_{n=1}^{\infty} \frac{t^n}{n!}(A^n \otimes (Ye^{-tB}h))$$

converges in $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$ uniformly for t in any bounded interval. The inequalities:

$$\int_0^\infty \left\| e^{tA} \otimes (Ye^{-tB}h) \right\|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}} \leq \int_0^\infty \|e^{tA}\|.\|(Ye^{-tB}h)\| \, dt$$

$$\leq \left(\int_0^\infty e^{-\delta t}\|e^{-tB}\| \, dt \right).\|Y\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}.\|h\|$$

ensure that $\int_0^\infty e^{tA} \otimes (Ye^{-tB}h)\, dt$ converges as a Bochner integral in the projective tensor product $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$ and:

$$
\begin{aligned}
\int_{\sigma(A)} P_A(d\zeta) \otimes \left(Y(\zeta I - B)^{-1}h \right) &= \int_{\sigma(A)} P_A(d\zeta) \otimes \left(Y \int_0^\infty e^{\zeta t} e^{-tB} h\, dt \right) \\
&= \int_0^\infty \left(\int_{\sigma(A)} e^{\zeta t} P_A(d\zeta) \right) \otimes \left(Y e^{-tB} h \right)\, dt \\
&= \int_0^\infty e^{tA} \otimes (Y e^{-tB} h)\, dt
\end{aligned}
$$

belongs to $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$, too. Then:

$$
\int_{\sigma(A)} P_A(d\zeta) \left(Y(\zeta I - B)^{-1}h \right) = J_E \int_{\sigma(A)} P_A(d\zeta) \otimes \left(Y(\zeta I - B)^{-1}h \right)
$$

defines a continuous linear operator:

$$
\int_{\sigma(A)} P_A(d\zeta) Y(\zeta I - B)^{-1} : h \longmapsto \int_{\sigma(A)} P_A(d\zeta) \left(Y(\zeta I - B)^{-1}h \right), \; h \in \mathcal{H}
$$

belonging to $\mathcal{L}(\mathcal{H}, \mathcal{K})$ with norm bounded by:

$$
\frac{\sup_{t \geq 0} \| e^{-tB} \|}{\delta} \| Y \|_{\mathcal{L}(\mathcal{H}, \mathcal{K})}
$$

In order to deal with unbounded operators, we replace the projective tensor product topology π by the topology τ defined by Formula (9).

Lemma 4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $A : \mathcal{D}(A) \to \mathcal{K}$ is a self-adjoint operator with spectral measure P_A and $B : \mathcal{D}(B) \to \mathcal{H}$ is a densely-defined, closed linear operator, such that $\sigma(A) \cap \sigma(B) = \emptyset$.*

Let $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. For each $h \in \mathcal{H}$, the $\mathcal{K}$-valued function:

$$
\Phi_h : \zeta \longmapsto Y(\zeta I - B)^{-1}h, \quad \zeta \in \sigma(A) \tag{11}
$$

is P_A-integrable in $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}$ on every compact subset of $\sigma(A)$.

Furthermore, there exist $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued $\mathcal{B}(\sigma(A))$-simple functions:

$$
s_n : \sigma(A) \to \mathcal{L}(\mathcal{H}, \mathcal{K}), \qquad n = 1, 2, \ldots
$$

such that for each $h \in \mathcal{H}$, $s_n(\omega)h \to \Phi_h(\omega)$ in $\mathcal{K}$ as $n \to \infty$ for P_A-almost all $\omega \in \sigma(A)$ and for each compact subset of K of $\sigma(A)$,

$$
\sup_{S \in \mathcal{B}(K)} \| (P_A \otimes \Phi_h)(S) - (P_A \otimes (s_n h))(S) \|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}} \to 0
$$

as $n \to \infty$.

Proof. For a closed and densely-defined operator T, the resolvent $(\lambda I - T)^{-1}$ is defined for all complex numbers λ belonging to the resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Suppose that $\rho(T)$ is non-empty. Then, the resolvent equation:

$$(\lambda I - T)^{-1} - (\mu I - T)^{-1} = (\mu - \lambda)(\lambda I - T)^{-1}(\mu I - T)^{-1}$$

for $\lambda, \mu \in \rho(T)$ ensures that $\lambda \longmapsto (\lambda I - T)^{-1}$, $\lambda \in \rho(T)$, is a holomorphic operator-valued function for the uniform operator topology. It follows that for each $h \in \mathcal{H}$, the function:

$$\lambda \longmapsto (\lambda I - A)^{-1} \otimes (Y(\lambda I - B)^{-1}h)$$

is continuous in the projective tensor product $\mathcal{L}(\mathcal{K}) \widehat{\otimes}_\pi \mathcal{K}$ for the uniform norm on $\mathcal{L}(\mathcal{K})$. For a compact subset K of $\sigma(A)$, let $A_K = P_A(K)A$ be the part of A on K. Then, for a contour Γ_K with a winding of one around K and zero around the closed set $\sigma(B)$, the integral:

$$\int_{\Gamma_K} \|(\lambda I - A_K)^{-1}\|.\|Y(\lambda I - B)^{-1}h\|_{\mathcal{K}} \, |d\lambda|$$

is bounded by $(|\Gamma_K|.\sup_{\lambda \in \Gamma_K} \|(\lambda I - A_K)^{-1}\|.\|(\lambda I - B)^{-1}\|).\|Y\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}.\|h\|_{\mathcal{H}}$, so the function:

$$\int_{\Gamma_K} (\lambda I - A_K)^{-1} \otimes (Y(\lambda I - B)^{-1}h) \, d\lambda$$

converges as a Bochner integral in $\mathcal{L}(\mathcal{K}) \widehat{\otimes}_\pi \mathcal{K}$. For every Borel subset S of the set K and $x, x', y' \in \mathcal{K}$, an application of Cauchy's integral formula yields:

$$\int_S (P_A x, x')(d\zeta)(Y(\zeta I - B)^{-1}h, y')$$
$$= \frac{1}{2\pi i} \int_S (P_A x, x')(d\zeta) \int_{\Gamma_K} \frac{(Y(\lambda I - B)^{-1}h, y')}{\lambda - \zeta} \, d\lambda$$
$$= \frac{1}{2\pi i} \int_{\Gamma_K} ((\lambda I - P_A(S)A_K)^{-1}x, x')(Y(\lambda I - B)^{-1}h, y') \, d\lambda$$

so according to Definition 2 (replacing the topology τ by the stronger projective topology π), the function $\zeta \longmapsto Y(\zeta I - B)^{-1}h$, $\zeta \in \sigma(A)$, is P_A-integrable in $\mathcal{L}(\mathcal{K}) \widehat{\otimes}_\pi \mathcal{K}$ on the set K and:

$$\int_S dP_A(\zeta) \otimes (Y(\zeta I - B)^{-1}h)$$
$$= \frac{1}{2\pi i} \int_{\Gamma_K} (\lambda I - P(S)A_K)^{-1} \otimes (Y(\lambda I - B)^{-1}h) \, d\lambda \tag{12}$$

as an element of the projective tensor product $\mathcal{L}(\mathcal{K}) \widehat{\otimes}_\pi \mathcal{K}$ for each Borel subset S of K.

Because the operator-valued function $\lambda \longmapsto (\lambda I - B)^{-1}$, $\lambda \in \sigma(A)$, is uniformly continuous on the compact set K, for each $\epsilon > 0$, there exists an $\mathcal{L}(\mathcal{H})$-valued $\mathcal{B}(\sigma(A))$-simple function φ_ϵ, such that:

$$\sup_{\lambda \in K} \|(\lambda I - B)^{-1} - \varphi_\epsilon(\lambda)\|_{\mathcal{L}(\mathcal{H})} < \epsilon$$

so that:

$$\sup_{S \in \mathcal{B}(K)} \int_{\Gamma_K} \|(\lambda I - P(S)A_K)^{-1}\|.\|Y(\lambda I - B)^{-1}h - Y\varphi_\epsilon(\lambda)h\|_{\mathcal{K}} \, |d\lambda| \to 0$$

as $\epsilon \to 0+$ for each $h \in \mathcal{H}$. According to the identity (12), it follows that:

$$\sup_{S \in \mathcal{B}(K)} \|(P_A \otimes \Phi_h)(S) - (P_A \otimes (Y\varphi_\epsilon h))(S)\|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}} \to 0$$

as $\epsilon \to 0+$. Because the spectral measure P_A is inner regular on compact sets, the simple functions s_n, $n = 1, 2, \ldots$, can be pieced together from the simple functions $\varphi_{1/n}$, $n = 1, 2, \ldots$, on each compact set K. $\square$

If both operators A and B are self-adjoint, then Theorem 3 ensures that a weak solution X of Equation (5) exists and gives a norm estimate for X. If just one operator is self-adjoint, the following result is applicable.

Theorem 5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $A : \mathcal{D}(A) \to \mathcal{K}$ is a self-adjoint operator with spectral measure P_A, and $B : \mathcal{D}(B) \to \mathcal{H}$ is a densely-defined, closed linear operator, such that $\sigma(A) \cap \sigma(B) = \emptyset$. Let $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$.*

(i) *Equation (5) has a strong solution if and only if there exists an operator valued measure $M : \mathcal{B}(\sigma(A)) \to \mathcal{L}(\mathcal{H}, \mathcal{K})$, such that:*

$$M(K)h = \int_K dP_A(\zeta)(Y(\zeta I - B)^{-1}h), \quad h \in \mathcal{H}$$

for each compact subset K of $\sigma(A)$. The operator valued measure M exists if and only if:

$$\sup_K \left\| \int_K dP_A(\zeta)(Y(\zeta I - B)^{-1}h) \right\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} < \infty \tag{13}$$

for every $h \in \mathcal{H}$. Then, $X = M(\sigma(A))$ is the unique strong solution of Equation (5).

(ii) *If for each $h \in \mathcal{H}$, the function Φ_h given by Formula (11) is P_A-integrable in $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}$ on $\sigma(A)$, then the map $h \longmapsto J_E \int_{\sigma(A)} dP_A \otimes \Phi_h$, $h \in \mathcal{H}$, defines a continuous linear operator $\int_{\sigma(A)} dP_A(\zeta)Y(\zeta I - B)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, and the operator:*

$$X = \int_{\sigma(A)} dP_A(\zeta)Y(\zeta I - B)^{-1}$$

is the unique strong solution of Equation (5).

Let $h \in \mathcal{H}$. The function Φ_h is P_A-integrable in $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}$ on $\sigma(A)$ if and only if:

$$\sup_K \left\| \int_K dP_A(\zeta) \otimes (Y(\zeta I - B)^{-1}h) \right\|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}} < \infty \tag{14}$$

Proof. The proof of (i) is similar to the proof of (ii), which we now give. Suppose that for each $h \in \mathcal{H}$, the function Φ_h is P_A-integrable in $E = \mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau \mathcal{K}$ on $\sigma(A)$. Then, for $h \in \mathcal{D}(B)$, we have:

$$AP_A(K) \int_{\sigma(A)} P(d\zeta)\Phi_h(\zeta) - P_A(K) \int_{\sigma(A)} P(d\zeta)\Phi_{Bh}(\zeta) = P_A(K)Yh$$

because $P_A(K) \int_{\sigma(A)} P(d\zeta)\Phi_u(\zeta) = \int_K P_A(d\zeta)P_A(K)\Phi_u(\zeta)$ for all $u \in \mathcal{H}$, and by Formula (3), the operator $X_K = \int_K dP_A(\zeta)P_A(K)Y(\zeta I - B)^{-1}$ is the unique solution of the equation:

$$(P_A(K)A)X_Kh - X_KBh = P_A(K)Yh, \quad h \in \mathcal{D}(B)$$

The case of unbounded B is mentioned in [23], Lemma 2.5. Because $P_A(K)X_K = X_K$ and A and $P_A(K)$ commute, we have $AX_Kh - X_KBh = P_A(K)Yh$ for all $h \in \mathcal{D}(B)$. Now, $X_Ku = P_A(K)\int_{\sigma(A)} P(d\zeta)\Phi_u(\zeta)$ converges in $\mathcal{K}$ as $K \uparrow \sigma(A)$ for each $u \in \mathcal{H}$, so $X = \lim_K X_K$ belongs to $\mathcal{L}(\mathcal{H}, \mathcal{K})$ by the uniform boundedness principle. Suppose that $h \in \mathcal{D}(B)$. Then, $\lim_K AX_Kh = XBh + Yh$, so Xh belongs to the closure of A restricted to the subspace:

$$\{P_A(K)u : u \in \mathcal{K}, \ K \subset \sigma(A) \text{ compact }\}$$

Hence, $Xh \in \mathcal{D}(A)$, and X is therefore a strong solution of Equation (5). On the other hand, if (5) does have a strong solution X, it can be written as $X = \lim_K X_K$ with $X_K = P_A(K)X$ uniformly bounded over compact sets $K \subset \sigma(A)$.

Conversely, suppose that the bound (14) holds for every $h \in \mathcal{H}$. There exists an increasing sequence of compact subsets $K_j, j = 1, 2, \ldots$, of $\sigma(A)$, such that:

$$\|P_A((\sigma(A) \setminus K_j) \cap S)\| < 1/j$$

for every $j = 1, 2, \ldots$ and $S \in \sigma(A)$, because the spectral measure P_A is a regular operator valued Borel measure. Let $\Omega_j = K_j \setminus (\cup_{i<j} K_i)$. Then, $\sigma(A) \setminus \cup_j \Omega_j$ is P_A-null, and $\Omega_1, \Omega_2, \ldots$ are pairwise disjoint.

For each $y' \in \mathcal{K}$, $S \in \mathcal{B}(\sigma(A))$ and $j = 1, 2, \ldots$:

$$\int_{\Omega_j \cap S} (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y') \in \mathcal{K}\widehat{\otimes}_\pi\mathcal{K}$$

If the bound (14) holds, then:

$$C_h = \sup_{n,S,\|y'\|\leq 1} \left\| \int_{(\cup_{j=1}^n \Omega_j)\cap S} (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y') \right\|_{\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}} < \infty$$

The projective tensor product $\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}$ is associated with the trace class operators on $\mathcal{K}$ via the embedding $u : \mathcal{K}\widehat{\otimes}_\pi\mathcal{K} \to \mathcal{L}(\mathcal{K})$ defined by $u(x \otimes y)k = (k, y)x$. Then:

$$u\left(\int_{(\cup_{j=1}^n \Omega_j)\cap S} (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y')\right) k$$
$$= \sum_{j=1}^n \int_{\Omega_j \cap S} (Y(\zeta I - B)^{-1}h)(k, P_A(d\zeta)y')$$

for $x, y, k \in \mathcal{K}$ and the bound:

$$\sum_{j=1}^n \int_{\Omega_j \cap S} |(Y(\zeta I - B)^{-1}h, x')|.|(k, P_Ay')|(d\zeta) \leq 4C_h\|x'\|.\|y'\|.\|k\|$$

holds for each $x', y', k \in \mathcal{K}$ and $S \in \mathcal{B}(\sigma(A))$ by [31], Proposition I.1.11. It follows from the weak sequential completeness of the Hilbert space $\mathcal{K}$ and the Orlicz–Pettis theorem ([31], Corollary I.4.4) that the sum $\sum_{j=1}^\infty \int_{\Omega_j \cap S} (Y(\zeta I - B)^{-1}h)(k, P_A(d\zeta)y')$ converges unconditionally in $\mathcal{K}$ for each $S \in \mathcal{B}(\sigma(A))$ and:

$$k \longmapsto \sum_{j=1}^\infty \int_{\Omega_j \cap S} (Y(\zeta I - B)^{-1}h)(k, P_A(d\zeta)y'), \quad k \in \mathcal{K}$$

is a bounded linear operator whose norm is bounded by $4C_h\|y'\|$. According to the non-commutative Fatou lemma (see Section 4),

$$\int_S (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y') = \sum_{j=1}^{\infty} \int_{\Omega_j \cap S} (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y')$$

belongs to $\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}$ and:

$$\left\|\int_S (Y(\zeta I - B)^{-1}h) \otimes (P_A(d\zeta)y')\right\|_{\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}} \leq 4C_h\|y'\|$$

for each $S \in \mathcal{B}(\sigma(A))$. Hence, the function Φ_h is P_A-integrable in $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau\mathcal{K}$ on $\sigma(A)$ and $\left\|\int_{\sigma(A)} dP_A \otimes \Phi_h\right\|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau\mathcal{K}} \leq 4C_h$. The uniform boundedness principle and the Vitali–Hahn–Saks theorem ensures that the formula $M(S) = \int_S P_A(d\zeta)(Y(\zeta I - B)^{-1})$ defines an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued measure M for the strong operator topology, so that (i) applies. □

Remark 4. The operator-valued measure $M : \mathcal{B}(\sigma(A)) \to \mathcal{L}(\mathcal{H},\mathcal{K})$ is called a strong operator-valued Stieltjes integral in [23,24]. According to Lemma 4, for each compact subset K of $\sigma(A)$, the operator $M(K) \in \mathcal{L}(\mathcal{H})$ can be written as a Stieltjes integral:

$$M(K)h = \lim_{n\to\infty} \int_K P_A(d\zeta)Y s_n(\zeta)h$$

for $\mathcal{B}(\sigma(A))$-simple function $s_n : \sigma(A) \to \mathcal{L}(\mathcal{H})$, $n = 1, 2, \ldots$, which may be chosen to be step functions based on finite intervals, restricted to the spectrum $\sigma(A)$ of A.

Example 5. The solution X in Theorem 3 is actually a strong solution. If A and B are self-adjoint and $d(\sigma(A), \sigma(B)) = \delta > 0$, then:

$$\int_S dP_A(\zeta) \otimes (Y(\zeta I - B)^{-1}h) = \int_{\mathbb{R}} (P_A(S)e^{itA}) \otimes (Ye^{-itB}h)f_\delta(t)\, dt$$

belongs to $\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau\mathcal{K}$ for each $S \in \mathcal{B}(\sigma(A))$ and $h \in \mathcal{H}$. To see this, let $k \in \mathcal{K}$. Then, the integral:

$$\int_{\mathbb{R}} (Ye^{-itB}h) \otimes (P_A(S)e^{-itA}k)f_\delta(t)\, dt$$

converges in $\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}$ because $t \longmapsto (Ye^{-itB}h) \otimes (P_A(S)e^{-itA}k)$, $t \in \mathbb{R}$, is continuous in $\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}$ and $f_\delta \in L^1(\mathbb{R})$, so:

$$\int_{\mathbb{R}} \|Ye^{-itB}h) \otimes (P_A(S)e^{-itA}k)f_\delta(t)\|_{\mathcal{K}\widehat{\otimes}_\pi\mathcal{K}}\, dt$$
$$\leq \int_{\mathbb{R}} \|Ye^{-itB}h\|.\|e^{-itA}k\|.|f_\delta(t)|\, dt$$
$$\leq \|Y\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}\|h\|_{\mathcal{H}}\|k\|_{\mathcal{K}}\|f_\delta\|_1$$

and $\|\int_S dP_A(\zeta) \otimes (Y(\zeta I - B)^{-1}h)\|_{\mathcal{L}(\mathcal{K})\widehat{\otimes}_\tau\mathcal{K}} \leq \|Y\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}\|h\|_{\mathcal{H}}\|f_\delta\|_1$. Then, by an appeal to Theorem 5 (ii), the operator:

$$X = \int_{\sigma(A)} dP_A(\zeta)Y(\zeta I - B)^{-1} = \int_{\mathbb{R}} e^{-itA}Ye^{itB}f_\delta(t)\, dt$$

is the unique strong solution of Equation (5). It is shown in [24], Lemma 4.2, that there is actually no distinction between weak and strong solutions of the Sylvester–Rosenblum Equation (5) because the bound (13) follows from the boundedness in the weak operator topology.

Example 6. If A is self-adjoint, B is densely defined and closed, $\sup \sigma(A) \leq 0$ and there exists $0 < \omega < \pi/2$ and a sector:

$$S_{\omega-} = \{-z : z \in \mathbb{C} \setminus \{0\}, \arg |z| < \omega\} \cup \{0\}$$

that is contained in $\rho(B)$, then according to [29], Theorem 15:

$$\int_S dP_A(\zeta) \otimes (Y(\zeta I - B)^{-1}k) \in \mathcal{L}(\mathcal{K})\widehat{\otimes}_\pi \mathcal{K}, \quad S \in \mathcal{B}(\sigma(A)), \quad k \in \mathcal{K}$$

The application of the integral representation of solutions of the Sylvester–Rosenblum Equation (5) to the spectral analysis of block operator matrices is discussed in detail in [23,24]. It is worthwhile to mention the background concerning self-adjoint operator block matrices:

$$H = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix}$$

acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. Then, H can also be written as $H = A + B$ for the operator matrices:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{01} \\ B_{10} & 0 \end{pmatrix}$$

with A self-adjoint and B bounded. A strong solution Q of the equation:

$$QA - AQ + QBQ = B \tag{15}$$

having the form:

$$Q = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix}, \quad Q_{10} = -Q_{01}^*$$

determines a reducing subspace for the original block operator matrix operator H, so that:

$$(I + Q)^{-1}H(I + Q) = A + BQ = \begin{pmatrix} A_0 + B_{01}Q_{10} & 0 \\ 0 & A_1 B_{10}Q_{01} \end{pmatrix}$$

Consequently, if U is the unitary operator associated with the polar decomposition $I + Q = U|I+Q|$, then $U^*HU = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}$ is the block diagonalization of H for self-adjoint operators H_0, H_1 similar to the operators $A_0 + B_{01}Q_{10}$ and $A_1 + B_{10}Q_{01}$, respectively.

The Equation (15) is called Riccati's equation. It also arises in optimal control theory when the operator entries may not be self-adjoint; see [23] for a list of references. Equation (15) is quadratic in Q, and provided that $0 < \sqrt{\|B\|\|D\|} \leq \delta/\pi$ with respect to the distance $\delta = d(\sigma(U), \sigma(V))$ between the spectra of U and V, a fixed point argument produces a unique strong solution Q of the associated

operator equation $QU - VQ + QBQ = D$ ([23], Theorem 3.6) in the case that U and V are bounded self-adjoint operators; see also [24], Section 5. When $\sigma(A_0)$ and $\sigma(A_1)$ are separated and B_{01}, B_{10} are small perturbations, solutions of (15) are constructed in [23], Theorems 7.4, 7.6 and 7.7. In the analysis of resonances between scattering channels, the situation where $\sigma(A_0) \cap \sigma(A_1) \neq \emptyset$ also arises [32].

Conditions for which the equation $\xi = \xi_0 + \xi_1 \pmod{\mathbb{Z}}$ is valid almost everywhere are given in [23], Theorem 6.1, for the spectral shift function ξ with respect to the pair (H, A) and the spectral shift function ξ_j with respect to the self-adjoint pair (H_j, A_j), $j = 0, 1$. Actually, the almost sure decomposition $\xi = \xi_0 + \xi_1$ can be deduced from [23], Lemma 7.10, and Equation (28) below, where the distinguished Birman–Solomyak representation is chosen for Krein's spectral shift function by employing double operator integrals.

4. Double Operator Integrals

As mentioned in Example 5 above, if $A : \mathcal{D}(\mathcal{H}) \to \mathcal{H}$ and $B : \mathcal{D}(\mathcal{H}) \to \mathcal{H}$ are self-adjoint operators, $d(\sigma(A), \sigma(B)) = \delta > 0$ and $Y \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, then for each $h \in \mathcal{H}$, the function $\zeta \mapsto Y(\zeta I - B)^{-1}h$, $\zeta \in \sigma(A)$, is P_A-integrable in $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$, and $X = \int_{\sigma(A)} dP_A(\zeta)Y(\zeta I - B)^{-1}$ is the unique strong solution of Equation (5). Because B is self-adjoint, we can rewrite the solution X as an iterated integral:

$$X = \int_{\sigma(A)} dP_A(\zeta)Y \left(\int_{\sigma(B)} \frac{dP_B(\mu)}{\zeta - \mu} \right)$$

with respect to the spectral measures P_A, P_B associated with A and B.

An application of the Fubini strategy sees the expression:

$$X = \int_{\sigma(A) \times \sigma(B)} \frac{dP_A(\zeta)Y\,dP_B(\mu)}{\zeta - \mu} \tag{16}$$

as a representation of the strong solution of the operator equation:

$$AX - XB = Y$$

in the case that both A and B are self-adjoint operators.

Integrals like (16) have been studied extensively in the case that $Y \in \mathcal{L}(\mathcal{H})$ is a Hilbert–Schmidt operator and, more generally, when Y belongs to the Schatten ideal $\mathcal{C}_p(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$ for some $1 \leq p < \infty$, where they are called double operator integrals [20].

Following [33], Section III.2, a subspace $\mathfrak{S}$ of the collection $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a separable Hilbert space $\mathcal{H}$ is called a symmetrically-normed ideal with norm $\| \cdot \|_{\mathfrak{S}}$ if $(\mathfrak{S}, \| \cdot \|_{\mathfrak{S}})$ is a Banach space and:

a. for $S \in \mathfrak{S}$, $L, K \in \mathcal{L}(\mathcal{H})$, we have $LSK \in \mathfrak{S}$ and $\|LSK\|_{\mathfrak{S}} \leq \|L\|\|S\|_{\mathfrak{S}}\|K\|$;

b. if S has rank one, then $\|S\|_{\mathfrak{S}} = \|S\|$; and

c. the closed unit ball of $(\mathfrak{S}, \| \cdot \|_{\mathfrak{S}})$ is sequentially closed in the weak operator topology of $\mathcal{L}(\mathcal{H})$, that is if $S_n \in \mathfrak{S}$ with $\sup_n \|S_n\|_{\mathfrak{S}} < \infty$ and if $S_n \to S$ in the weak operator topology of $\mathcal{L}(\mathcal{H})$, then $S \in \mathfrak{S}$ and $\|S\|_{\mathfrak{S}} \leq \limsup_n \|S_n\|_{\mathfrak{S}}$.

For $1 \leq p \leq \infty$, the Schatten ideal $\mathcal{C}_p(\mathcal{H})$ consists of all compact operators T whose singular values $\{\lambda_j\}_{j=1}^{\infty}$ belong to ℓ^p with the norm $\|T\|_{\mathcal{C}_p(\mathcal{H})}^p = \sum_{j=1}^{\infty} \lambda_j^p$ for $1 \leq p < \infty$ and $\|T\|_{\mathcal{C}_\infty(\mathcal{H})} = \|T\|$. The singular values $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the positive operator $(T^*T)^{\frac{1}{2}}$. For $1 \leq p < \infty$, $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$ is a symmetrically-normed ideal. Condition c. is often called the non-commutative Fatou lemma. It fails for the compact operators $\mathcal{C}_\infty(\mathcal{H})$, but $\mathfrak{S} = \mathcal{L}(\mathcal{H})$ is itself a symmetrically-normed (improper) ideal with the uniform norm. The symmetrically-normed ideal $\mathcal{C}_1(\mathcal{H})$ of trace class operators on $\mathcal{H}$ may be identified with the projective tensor product $\mathcal{H} \widehat{\otimes}_\pi \mathcal{H}$ ([30], III.7.1).

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the expression:

$$\mathcal{I}_\varphi(T) = \int_{\Lambda \times M} \varphi(\lambda, \mu)\, E(d\lambda) T F(d\mu)$$

is a double operator integral if E is an $\mathcal{L}(\mathcal{H})$-valued spectral measure on the measurable space $(\Lambda, \mathcal{E})$ and F is an $\mathcal{L}(\mathcal{H})$-valued spectral measure on the measurable space $(M, \mathcal{F})$. The function $\varphi : \Lambda \times M \to \mathbb{C}$ is taken to be uniformly bounded on $\Lambda \times M$. In Formula (16), $\varphi(\lambda, \mu) = (\lambda - \mu)^{-1}$, so that $|\varphi(\lambda, \mu)|$ is bounded by $1/\delta$ for $(\lambda, \mu) \in \sigma(A) \times \sigma(B)$ when the spectra $\sigma(A)$ and $\sigma(B)$ are a positive distance δ apart.

The map $T \longmapsto \mathcal{I}_\varphi(T)$, $T \in \mathcal{C}_2(\mathcal{H})$, is continuous into the space $\mathcal{C}_2(\mathcal{H})$ of Hilbert–Schmidt operators and:

$$\|\mathcal{I}_\varphi\|_{\mathcal{C}_2(\mathcal{H})} = \|\varphi\|_{L^\infty(\Lambda \times M)}$$

so that the map $(E \otimes F)_{\mathcal{C}_2(\mathcal{H})} : U \longmapsto \mathcal{I}_{\chi_U}$, $U \in \mathcal{E} \otimes \mathcal{F}$, is actually a spectral measure acting on $\mathcal{C}_2(\mathcal{H})$, and the equality:

$$\mathcal{I}_\varphi = \int_{\Lambda \times M} \varphi\, d(E \otimes F)_{\mathcal{C}_2(\mathcal{H})}$$

holds for all bounded measurable functions $\varphi : \Lambda \times M \to \mathbb{C}$ ([20], Section 3.1).

The situation is more complicated if the space $\mathcal{C}_2(\mathcal{H})$ of Hilbert–Schmidt operators (with the Hilbert–Schmidt norm) is replaced by the Schatten ideal $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$ for some $1 \leq p < \infty$ not equal to two or as in the case of Formula (16), by $\mathfrak{S} = \mathcal{L}(\mathcal{H})$ itself, because the map $U \times V \mapsto \mathcal{I}_{\chi_{U \times V}}$, $U \in \mathcal{E}$, $V \in \mathcal{F}$, only defines a finitely-additive set function $(E \otimes F)_{\mathfrak{S}}$ acting on elements $T \in \mathfrak{S}$, so that:

$$(E \otimes F)_{\mathfrak{S}}(U \times V)T = E(U) T F(V), \quad U \in \mathcal{E}, V \in \mathcal{F}.$$

For a bounded function $\varphi : \Lambda \times M \to \mathbb{C}$, the double operator integral $\mathcal{I}_\varphi$ may be viewed as a continuous generalisation of a classical Schur multiplier:

$$T_\mu : x \longmapsto \sum_{i,j} \mu_{ij} \alpha_{ij} e_{ij}, \quad x = \sum_{i,j} \alpha_{ij} e_{ij} \tag{17}$$

for an infinite matrix $\mu = \{\mu_{ij}\} \in \mathcal{M}$, with respect to the matrix units e_{ij} corresponding to an orthonormal basis $\{h_j\}$ of $\mathcal{H}$. If P_j denotes the orthogonal projection onto the linear space $\mathrm{span}\{h_j\}$ for each $j = 1, 2, \ldots$, then:

$$T_\mu = \sum_{i,j} \mu_{ij} (P_i \otimes P_j)_\mathcal{M}$$

for the operators $(P_i \otimes P_j)_\mathcal{M} : x \longmapsto P_i x P_j$ acting on the infinite matrix $x \in \mathcal{M}$ for $i, j = 1, 2 \ldots$.

To be more precise, let $\mathfrak{S}$ be a symmetrically-normed ideal in $\mathcal{L}(\mathcal{H})$. The linear map $\mathcal{J}_{\mathfrak{S}} : \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathfrak{S})$ is defined by $\mathcal{J}_{\mathfrak{S}}(A \otimes B)T = ATB$ for $T \in \mathfrak{S}$ and $A, B \in \mathcal{L}(\mathcal{H})$. In the language of [20], Section 4, the element $\mathcal{J}_{\mathfrak{S}}(A \otimes B)$ of $\mathcal{L}(\mathfrak{S})$ is the transformer on $\mathfrak{S}$ associated with $A \otimes B$. The tensor product $W = \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$ is defined by completion with respect to the norm (9).

Definition 7. Let $(\Lambda, \mathcal{E})$ and $(M, \mathcal{F})$ be measurable spaces and $\mathcal{H}$ a separable Hilbert space. Let $m : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$ be an operator valued measure for the strong operator topology and $n : \mathcal{F} \to \mathcal{H}$ be a $\mathcal{H}$-valued measure.

An $(\mathcal{E} \otimes \mathcal{F})$-measurable function $\varphi : \Lambda \times M \to \mathbb{C}$ is said to be $(m \otimes n)$-integrable in $W = \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$ if for every $x, x', y' \in \mathcal{H}$, the function φ is integrable with respect to the scalar measure $(mx, x') \otimes (n, y')$ and for every $A \in \mathcal{E} \otimes \mathcal{F}$, there exists $\varphi.(m \otimes n)(A) \in \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$, such that:

$$\langle \varphi.(m \otimes n)(A), x \otimes x' \otimes y' \rangle = \int_A \varphi \, d((mx, x') \otimes (n, y'))$$

for every $x, x', y' \in \mathcal{H}$.

If φ is $(m \otimes n)$-integrable in $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$ and $J_W : \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H} \to \mathcal{H}$ is the multiplication map, then:

$$\int_A \varphi \, d(mn) = J_W(\varphi.(m \otimes n)(A)), \quad A \in \mathcal{E} \otimes \mathcal{F}$$

The following observation is useful for treating double operator integrals.

Proposition 6. *Let $\mathcal{H}$, $m : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$ and $n : \mathcal{F} \to \mathcal{H}$ be as in Definition 7. If $T \in \mathcal{C}_1(\mathcal{H})$, then there exists a unique vector measure:*

$$m \otimes (Tn) : \mathcal{E} \otimes \mathcal{F} \to \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$$

such that $(m \otimes (Tn))(E \times F) = m(E) \otimes (Tn(F)) \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$ for each $E \in \mathcal{E}$ and $F \in \mathcal{F}$. Consequently, every bounded $(\mathcal{E} \otimes \mathcal{F})$-measurable function $\varphi : \Lambda \times M \to \mathbb{C}$ is $(m \otimes (Tn))$-integrable in $W = \mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$ and:

$$\int_A \varphi \, d(m(Tn)) = J_W(\varphi.(m \otimes (Tn))(A)), \quad A \in \mathcal{E} \otimes \mathcal{F}.$$

Proof. If T is a trace class operator on $\mathcal{H}$, then there exists orthonormal sets $\{\phi_j\}_j$, $\{\psi_j\}_j$ and a summable sequence $\{\lambda_j\}_j$ of scalars, such that $Th = \sum_{j=1}^{\infty} \lambda_j \phi_j(h, \psi_j)$ for every $h \in \mathcal{H}$. For each $j = 1, 2 \ldots$, the total variation of the product measure:

$$(mh, k) \otimes (n, \psi_j) : E \times F \longmapsto (m(E)h, k) \otimes (n(F), \psi_j), \quad E \in \mathcal{E}, \ F \in \mathcal{F}$$

is bounded by $\|m\|(\Lambda).\|n\|(M).\|h\|.\|k\|$ for every $h, k \in \mathcal{H}$. Here, $\|m\|$ and $\|n\|$ denote the semi-variation of m and n, respectively ([31], p. 2). It follows that $(n, \psi_j)(m \otimes \phi_j)$ admits a unique countably-additive extension $M_j : \mathcal{E} \otimes \mathcal{F} \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$ whose semi-variation with respect to the norm (9) is bounded by $\|m\|(\Lambda).\|n\|(M)$ and $m \otimes (Tn) = \sum_j \lambda_j M_j$ converges in $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\tau \mathcal{H}$ uniformly on $\mathcal{E} \otimes \mathcal{F}$. $\square$

Corollary 7. *Let $(\Lambda, \mathcal{E})$ and $(M, \mathcal{F})$ be measurable spaces and $\mathcal{H}$ a separable Hilbert space. Let $m : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$ and $n : \mathcal{F} \to \mathcal{L}_s(\mathcal{H})$ be operator valued measures for the strong operator topology. Then, there exists a unique operator valued measure:*

$$\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n) : \mathcal{E} \otimes \mathcal{F} \to \mathcal{L}_s(\mathcal{C}_1(\mathcal{H}), \mathcal{L}_s(\mathcal{H}))$$

such that:

$$\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)(E \times F) = \mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m(E) \otimes n(F)), \quad E \in \mathcal{E}, \ F \in \mathcal{F}.$$

Proof. It is easy to check that for $A \in \mathcal{E} \otimes \mathcal{F}$ and $T \in \mathcal{C}_1(\mathcal{H})$, the formula:

$$\left(\left[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)(A) \right] T \right) h = J_W((m \otimes (T(nh)))(A)), \quad h \in \mathcal{H}$$

defines a linear operator $\left[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)(A) \right] T$ on $\mathcal{H}$ whose operator norm is bounded by $\|m\|(\Lambda).\|n\|(M)\|T\|_{\mathcal{C}_1(\mathcal{H})}$, and $A \longmapsto \left[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)(A) \right] T$, $A \in \mathcal{E} \otimes \mathcal{F}$, is countably additive in $\mathcal{L}(\mathcal{H})$ for the strong operator topology for each $T \in \mathcal{C}_1(\mathcal{H})$. $\square$

Given $T \in \mathcal{C}_1(\mathcal{H})$, the expression $(mTn)(E \times F) = m(E)Tn(F)$, $E \in \mathcal{E}$ and $F \in \mathcal{F}$, is the restriction to product sets of the $\mathcal{L}(\mathcal{H})$-valued measure $mTn = \mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)T$.

The following notation gives an interpretation of Formula (16) in the case that the operator Y belongs to the symmetrically-normed ideal $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$ or $\mathfrak{S} = \mathcal{L}(\mathcal{H})$. The collection $\mathcal{C}_1(\mathcal{H})$ of trace class operators is a linear subspace of $\mathfrak{S}$ in each case.

Let $(m \otimes n)_{\mathfrak{S}}$ be the finitely-additive set function defined by:

$$(m \otimes n)_{\mathfrak{S}}(E \times F) = \mathcal{J}_{\mathfrak{S}}(m(E) \otimes n(F)), \quad E \in \mathcal{E}, \ F \in \mathcal{F}$$

that is $(m \otimes n)_{\mathfrak{S}} : \mathcal{A} \to \mathcal{L}(\mathfrak{S})$ is finitely additive on the algebra $\mathcal{A}$ of all finite unions of product sets $E \times F$ for $E \in \mathcal{E}$, $F \in \mathcal{F}$.

Suppose that the function $\varphi : \Lambda \times M \to \mathbb{C}$ is integrable with respect to the measure $\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)$ with values in $\mathcal{L}_s(\mathcal{C}_1(\mathcal{H}), \mathcal{L}_s(\mathcal{H}))$. If for $E \in \mathcal{E}$ and $F \in \mathcal{F}$, the linear map:

$$u \longmapsto \left(\int_{E \times F} \varphi \, d[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(m \otimes n)] \right) u, \quad u \in \mathcal{C}_1(\mathcal{H})$$

is the restriction to $\mathcal{C}_1(\mathcal{H})$ of a continuous linear map $T_\varphi \in \mathcal{L}(\mathfrak{S})$, then we write:

$$\int_{E \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}}$$

for for the continuous linear map T_φ and we say that φ is $(m \otimes n)_{\mathfrak{S}}$-*integrable* if:

$$\int_{E \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$$

for every $E \in \mathcal{E}$ and $F \in \mathcal{F}$.

To check that the operator $\int_{E \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$ is uniquely defined, observe that $\mathcal{C}_1(\mathcal{H})$ is norm dense in $\mathcal{C}_p(\mathcal{H})$ for $1 < p \leq \infty$. In the case $\mathfrak{S} = \mathcal{L}(\mathcal{H})$, the closure in the ultra-weak topology can be taken.

Although $(m \otimes n)_{\mathfrak{S}}$ is only a finitely-additive set function, the $\mathcal{L}(\mathfrak{S})$-valued set function:

$$E \times F \longmapsto \int_{E \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}}, \quad E \in \mathcal{E}, \ F \in \mathcal{F}$$

of an $(m \otimes n)_{\mathfrak{S}}$-integrable function φ defines a finitely-additive $\mathcal{L}(\mathfrak{S})$-valued set function on the algebra generated by all product sets $E \times F$ for $E \in \mathcal{E}$ and $F \in \mathcal{F}$.

Corollary 7 tells us that for an $(m \otimes n)_{\mathcal{C}_1(\mathcal{H})}$-integrable function $\varphi : \Lambda \times M \to \mathbb{C}$, the $\mathcal{L}(\mathcal{H})$-valued set function:

$$A \longmapsto \left(\int_A \varphi \, d(m \otimes n)_{\mathcal{C}_1(\mathcal{H})} \right) T, \quad A \in \mathcal{E} \otimes \mathcal{F}$$

is countably additive in the strong operator topology for each $T \in \mathcal{C}_1(\mathcal{H})$. The following simple observation describes the situation for other operator ideals $\mathfrak{S}$.

Proposition 8. *Suppose that $\varphi : \Lambda \times M \to \mathbb{C}$ is an $(m \otimes n)_{\mathfrak{S}}$-integrable function. For each $T \in \mathfrak{S}$, the set function:*

$$E \times F \longmapsto \left(\int_{E \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}} \right) T, \quad E \in \mathcal{E}, \ F \in \mathcal{F}$$

is separately σ-additive in the strong operator topology of $\mathcal{L}(\mathcal{H})$, that is,

$$\left(\int_{(\cup_{j=1}^{\infty} E_j) \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}} \right) T = \sum_{j=1}^{\infty} \left(\int_{E_j \times F} \varphi \, d(m \otimes n)_{\mathfrak{S}} \right) T, \quad F \in \mathcal{F}$$

$$\left(\int_{E \times (\cup_{j=1}^{\infty} F_j)} \varphi \, d(m \otimes n)_{\mathfrak{S}} \right) T = \sum_{j=1}^{\infty} \left(\int_{E \times F_j} \varphi \, d(m \otimes n)_{\mathfrak{S}} \right) T, \quad E \in \mathcal{E}$$

for all pairwise disjoint $E_j \in \mathcal{E}$, $j = 1, 2, \ldots$ and all pairwise disjoint $F_j \in \mathcal{F}$, $j = 1, 2, \ldots$.

The following result was proven by Birman and Solomyak ([20], Section 3.1).

Theorem 9. *Let $(\Lambda, \mathcal{E})$ and $(M, \mathcal{F})$ be measurable spaces and $\mathcal{H}$ a separable Hilbert space. Let $P : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$ and $Q : \mathcal{F} \to \mathcal{L}_s(\mathcal{H})$ be spectral measures. Then, there exists a unique spectral measure $(P \overline{\otimes} Q)_{\mathcal{C}_2(\mathcal{H})} : \mathcal{E} \otimes \mathcal{F} \to \mathcal{L}(\mathcal{C}_2(\mathcal{H}))$, such that $(P \overline{\otimes} Q)_{\mathcal{C}_2(\mathcal{H})}(A) = (P \otimes Q)_{\mathcal{C}_2(\mathcal{H})}(A)$ for all $A \in \mathcal{A}$ and:*

$$\int_A \varphi \, d(P \otimes Q)_{\mathcal{C}_2(\mathcal{H})} = \int_A \varphi \, d(P \overline{\otimes} Q)_{\mathcal{C}_2(\mathcal{H})} \in \mathcal{L}(\mathcal{C}_2(\mathcal{H})), \quad A \in \mathcal{E} \otimes \mathcal{F}$$

for every bounded $(\mathcal{E} \otimes \mathcal{F})$-measurable function $\varphi : \Lambda \times M \to \mathbb{C}$. Moreover,

$$\|(P \overline{\otimes} Q)_{\mathcal{C}_2(\mathcal{H})}(\varphi)\|_{\mathcal{L}(\mathcal{C}_2(\mathcal{H}))} = \|\varphi\|_{\infty}.$$

For spectral measures P and Q, the formula:

$$\left(\int_{E \times F} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \right) T = \left(\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \right) P(E) T Q(F)$$

holds for each $E \in \mathcal{E}$, $F \in \mathcal{F}$ and $T \in \mathfrak{S}$, so it is only necessary to verify that $\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$ in order to show that φ is $(P \otimes Q)_{\mathfrak{S}}$-integrable.

The following observation gives an interpretation of Formula (16) as a double operator integral. The Fourier transform of $f \in L^1(\mathbb{R})$ is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx$ for $\xi \in \mathbb{R}$.

Theorem 10. *Let $\mathcal{H}$ be a separable Hilbert space. Let $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ and $Q : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ be spectral measures on $\mathbb{R}$. Let $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$ for some $1 \le p < \infty$ or $\mathfrak{S} = \mathcal{L}(\mathcal{H})$. Suppose that $f \in L^1(\mathbb{R})$ and $\varphi(\lambda, \mu) = \hat{f}(\lambda - \mu)$ for all $\lambda, \mu \in \mathbb{R}$. Then, $\int_{\mathbb{R} \times \mathbb{R}} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$ and:*

$$\left\| \int_{\mathbb{R} \times \mathbb{R}} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \right\|_{\mathcal{L}(\mathfrak{S})} \le \|f\|_1 \tag{18}$$

Proof. For $T \in \mathcal{C}_1(\mathcal{H})$ and Borel subsets E, F of $\mathbb{R}$, by Fubini's theorem, we have:

$$\left(\int_{E \times F} \varphi \, d[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(P \otimes Q)] \right) T = \int_{\mathbb{R}} \left(\int_E e^{-i\lambda\xi} \, dP(\lambda) \right) T \left(\int_F e^{i\mu\xi} \, dQ(\mu) \right) f(\xi) \, d\xi$$

The right-hand side is a Bochner integral in the strong operator topology of $\mathcal{L}(\mathcal{H})$ because:

$$\xi \longmapsto \int_{\mathbb{R}} e^{-i\lambda\xi} \, dP(\lambda), \quad \xi \longmapsto \int_{\mathbb{R}} e^{i\mu\xi} \, dQ(\mu), \quad \xi \in \mathbb{R}$$

are continuous unitary groups in the strong operator topology. Moreover,

$$\xi \longmapsto \left(\int_E e^{-i\lambda\xi} \, dP(\lambda) \right) T \left(\int_F e^{i\mu\xi} \, dQ(\mu) \right), \quad \xi \in \mathbb{R}$$

is continuous in the norm of $\mathfrak{S}$ for compact subsets E, F of $\mathbb{R}$, because $\mathfrak{S}$ is a symmetrically-normed ideal in $\mathcal{L}(\mathcal{H})$; so, the Bochner integral converges in $\mathfrak{S}$ itself, and we obtain:

$$\left\| \left(\int_{E \times F} \varphi \, d[\mathcal{J}_{\mathcal{C}_1(\mathcal{H})}(P \otimes Q)] \right) T \right\|_{\mathfrak{S}} \le \|f\|_1 \|T\|_{\mathfrak{S}}$$

For E, F increasing to $\mathbb{R}$, the inclusion $\int_{\mathbb{R} \times \mathbb{R}} \varphi \, d(P \otimes Q)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$ and the bound (18) is now a consequence of the non-commutative Fatou lemma. $\square$

Corollary 11. *Let $\mathcal{H}$ be a separable Hilbert space, and let A, B be self-adjoint operators with spectral measures $P_A : \mathcal{B}(\sigma(A)) \to \mathcal{L}_s(\mathcal{H})$ and $P_B : \mathcal{B}(\sigma(B)) \to \mathcal{L}_s(\mathcal{H})$, respectively. Let $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$ for some $1 \le p < \infty$ or $\mathfrak{S} = \mathcal{L}(\mathcal{H})$. If the spectra of A and B are separated by a distance $d(\sigma(A), \sigma(B)) = \delta > 0$, then $\int_{\sigma(A) \times \sigma(B)} (\lambda - \mu)^{-1} (P_A \otimes P_B)_{\mathfrak{S}}(d\lambda, d\mu) \in \mathcal{L}(\mathfrak{S})$ and:*

$$\left\| \int_{\sigma(A) \times \sigma(B)} \frac{(P_A \otimes P_B)_{\mathfrak{S}}(d\lambda, d\mu)}{\lambda - \mu} \right\|_{\mathcal{L}(\mathfrak{S})} \le \frac{\pi}{2\delta}$$

In particular, Equation (5) has a unique strong solution for $Y \in \mathfrak{S}$ given by the double operator integral:

$$X = \int_{\sigma(A) \times \sigma(B)} \frac{dP_A(\lambda) Y \, dP_B(\mu)}{\lambda - \mu} := \left(\int_{\sigma(A) \times \sigma(B)} \frac{(P_A \otimes P_B)_{\mathfrak{S}}(d\lambda, d\mu)}{\lambda - \mu} \right) Y$$

so that $\|X\|_{\mathfrak{S}} \le \frac{\pi}{2\delta} \|Y\|_{\mathfrak{S}}$.

Although the Heaviside function $\chi_{(0,\infty)}$ is not the Fourier transform of an L^1-function, the following result of Gohberg and Krein ([34], Section III.6) holds, in case $P = Q$. The general case is outlined in [20], Theorem 7.2.

Theorem 12. *Let $\mathcal{H}$ be a separable Hilbert space. Let $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ and $Q : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ be spectral measures on $\mathbb{R}$. Then:*

$$\int_{\mathbb{R} \times \mathbb{R}} \chi_{\{\lambda > \mu\}} \, d(P \otimes Q)_{\mathcal{C}_p(\mathcal{H})} \in \mathcal{L}(\mathcal{C}_p(\mathcal{H}))$$

for every $1 < p < \infty$.

The following recent result of Sukochev and Potapov [35] settled a long outstanding conjecture of Krein for the index p in the range $1 < p < \infty$.

Theorem 13. *Let $\mathcal{H}$ be a separable Hilbert space. Let $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ and $Q : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ be spectral measures on $\mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which the difference quotient:*

$$\varphi_f(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} & , \quad \lambda \neq \mu \\ 0 & , \quad \lambda = \mu \end{cases}$$

is uniformly bounded. Then, for every $1 < p < \infty$,

$$\int_{\mathbb{R} \times \mathbb{R}} \varphi_f \, d(P \otimes Q)_{\mathcal{C}_p(\mathcal{H})} \in \mathcal{L}(\mathcal{C}_p(\mathcal{H}))$$

and there exists $C_p > 0$, such that:

$$\left\| \int_{\mathbb{R} \times \mathbb{R}} \varphi_f \, d(P \otimes Q)_{\mathcal{C}_p(\mathcal{H})} \right\|_{\mathcal{C}_p(\mathcal{H})} \leq C_p \|\varphi_f\|_\infty$$

Such a function f is said to be uniformly Lipschitz on $\mathbb{R}$ and $\|f\|_{\mathrm{Lip}_1} := \|\varphi_f\|_\infty$.

Corollary 14. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a uniformly Lipschitz function. Then, for every $1 < p < \infty$, there exists $C_p > 0$, such that:*

$$\|f(A) - f(B)\|_{\mathcal{C}_p(\mathcal{H})} \leq C_p \|f\|_{\mathrm{Lip}_1} \|A - B\|_{\mathcal{C}_p(\mathcal{H})}$$

for any self-adjoint operators A and B on a separable Hilbert space $\mathcal{H}$.

Proof. Let P_A and P_B be the spectral measures of A and B, respectively, and suppose that $\|A - B\|_{\mathcal{C}_p(\mathcal{H})} < \infty$. Then, according to [20], Theorem 8.1 (see also [21], Corollary 7.2), the equality:

$$f(A) - f(B) = \left(\int_{\mathbb{R} \times \mathbb{R}} \varphi_f \, d(P_A \otimes P_B)_{\mathcal{C}_p(\mathcal{H})} \right) (A - B)$$

holds, and the norm estimate follows from Theorem 13. $\square$

5. Traces of Double Operator Integrals

In this section, let $(\Lambda, \mathcal{E})$ and $(M, \mathcal{F})$ be given measurable spaces, $\mathcal{H}$ a separable Hilbert space and $P : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$, $Q : \mathcal{F} \to \mathcal{L}_s(\mathcal{H})$ spectral measures. Let $\mathfrak{S} = \mathcal{C}_p(\mathcal{H})$ for some $1 \leq p < \infty$ or $\mathfrak{S} = \mathcal{L}(\mathcal{H})$. The Banach space $L^1(P)$ of P-integrable functions is isomorphic to the C*-algebra $L^\infty(P)$ of P-essentially bounded functions. The analogous result for $(P \otimes Q)_\mathfrak{S}$-integrable functions follows.

Proposition 15. *For an $(\mathcal{E} \otimes \mathcal{F})$-measurable function $\varphi : \Lambda \times M \to \mathbb{C}$, let $[\varphi]$ be the equivalence class of all functions equal to φ $(P \otimes Q)$-almost everywhere. Let:*

$$L^1((P \otimes Q)_\mathfrak{S}) = \{[\varphi] : \varphi \text{ is } (P \otimes Q)_\mathfrak{S}\text{-integrable}\}$$

with the pointwise operations of addition and scalar multiplication with the norm:

$$\|[\varphi]\|_\mathfrak{S} = \left\| \int_{\Lambda \times \Lambda} \varphi \, d(P \otimes Q)_\mathfrak{S} \right\|_{\mathcal{L}(\mathfrak{S})}$$

Then, $\|[\varphi]\|_\infty \leq \|[\varphi]\|_\mathfrak{S}$, and $L^1((P \otimes Q)_\mathfrak{S})$ is a commutative Banach *-algebra under pointwise multiplication. If $\mathfrak{S} = \mathcal{C}_2(\mathcal{H})$, then:

$$L^1((P \otimes Q)_\mathfrak{S}) = L^\infty(P \otimes Q)$$

is a commutative C^*-algebra. Furthermore, the Banach *-algebras:

$$L^1((P \otimes Q)_{\mathcal{C}_1(\mathcal{H})}) = L^1((P \otimes Q)_{\mathcal{C}_\infty(\mathcal{H})}) = L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$$

are isometric, where $\mathcal{C}_\infty(\mathcal{H})$ is the uniformly-closed subspace of $\mathcal{L}(\mathcal{H})$ consisting of compact operators on $\mathcal{H}$.

Remark 8. The analogy of double operator integrals with multiplier theory in harmonic analysis is fleshed out in [21], Example 2.13, as follows.

If Λ is a locally-compact abelian group with Fourier transform $\mathcal{F}$, the spectral measure Q is defined by multiplication by characteristic functions on $L^2(\Lambda)$ and $P = \mathcal{F}^{-1}Q\mathcal{F}$ is the spectral measure of the "momentum operator" on Λ, then for $1 < p < \infty$, the space $\mathcal{M}_p(\Lambda)$ of Fourier multipliers on $L^p(\Lambda)$ coincides with the commutative Banach *-algebra $L^1(P_p)$ for the finitely-additive set function $P_p : \mathcal{A} \to \mathcal{L}(L^p(\Lambda))$ defined as in [21], Example 2.13, by the spectral measure P acting on on $L^2(\Lambda)$. For example, when $\Lambda = \mathbb{R}$, the operator $\int_\mathbb{R} \text{sgn} \, dP_p \in \mathcal{L}(L^p(\Lambda))$ is the Hilbert transform for $1 < p < \infty$.

It is only in the case $p = 2$ that $L^1(P_2) = L^\infty(P)$. One might argue that multiplier theory in commutative harmonic analysis is devoted to the study of the commutative Banach *-algebra $L^1(P_p)$ for $1 < p < \infty$. The analysis of the commutative Banach *-algebra $L^1((E \otimes F)_\mathfrak{S}))$ for general spectral measures E and F and symmetric operator ideal $\mathfrak{S}$ has many applications to scattering theory and quantum physics [20].

The commutative Banach *-algebra $L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ is characterised by a result of Peller [25].

Theorem 16. *Let $\varphi : \Lambda \times M \to \mathbb{C}$ be a uniformly-bounded function. Then, $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ if and only if there exist a finite measure space $(T, \mathcal{S}, \nu)$ and measurable functions $\alpha : \Lambda \times T \to \mathbb{C}$ and $\beta : M \times T \to \mathbb{C}$, such that $\int_T \|\alpha(\cdot, t)\|_{L^\infty(P)} \|\beta(\cdot, t)\|_{L^\infty(Q)} \, d\nu(t) < \infty$ and:*

$$\varphi(\lambda, \mu) = \int_T \alpha(\lambda, t)\beta(\mu, t) \, d\nu(t), \quad \lambda \in \Lambda, \ \mu \in M \tag{19}$$

The norm of $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ with the representation (19) satisfies:

$$K_G^{-1} \int_T \|\alpha(\cdot, t)\|_{L^\infty(P)} \|\beta(\cdot, t)\|_{L^\infty(Q)} \, d\nu(t) \leq \|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}$$

$$\leq \left\| \left(\int_T |\alpha(\cdot, t)|^2 \, d\nu(t) \right)^{\frac{1}{2}} \right\|_{L^\infty(P)} \left\| \left(\int_T |\beta(\cdot, t)|^2 \, d\nu(t) \right)^{\frac{1}{2}} \right\|_{L^\infty(Q)} \tag{20}$$

for Grothendieck's constant K_G. Moreover, $\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}$ is the infimum of all numbers on the right-hand side of the inequality (20) for which there exists a finite measure ν, such that the representation (19) holds for φ.

Formula (19) is to be interpreted in the sense that φ is a special representative of the equivalence class $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$. It is worthwhile to make a few remarks on the significance of Formula (19) in order to motivate its proof below.

If the functions α and β in the representation (19) have the property that $t \longmapsto \alpha(\cdot, t)$, $t \in T$ and $t \longmapsto \beta(\cdot, t)$, $t \in T$, are strongly ν-measurable in $L^\infty(P)$ and $L^\infty(Q)$, respectively, then the function $t \longmapsto \alpha(\cdot, t) \otimes \beta(\cdot, t)$, $t \in T$, is strongly measurable in the projective tensor product $L^\infty(P) \widehat{\otimes}_\pi L^\infty(Q)$, and:

$$\int_T \|\alpha(\cdot, t)\|_{L^\infty(P)} \|\beta(\cdot, t)\|_{L^\infty(Q)} \, d\nu(t) < \infty$$

Hence, the function $t \longmapsto \alpha(\cdot, t) \otimes \beta(\cdot, t)$, $t \in T$, is Bochner integrable in $L^\infty(P) \widehat{\otimes}_\pi L^\infty(Q)$, that is $[\varphi] \in L^\infty(P) \widehat{\otimes}_\pi L^\infty(Q)$. However, it is only assumed α is $(\mathcal{E} \otimes \mathcal{S})$-measurable and β is $(\mathcal{F} \otimes \mathcal{S})$-measurable, so this conclusion is unavailable.

Let $\nu_P : \mathcal{E} \to [0, \infty)$ be a finite measure, such that $\nu_P(E) \leq \|P\|(E)$ for $E \in \mathcal{E}$ and $\lim_{\nu_P(E) \to 0} \|Ph\|(E) = 0$ for all $h \in \mathcal{H}$ with $\|h\| \leq 1$. Such a measure exists by the Bartle–Dunford–Schwartz Theorem ([31], Corollary I.2.6) or, more simply, $\nu_P = \sum_{n=1}^\infty 2^{-n}(Pe_n, e_n)$ for some orthonormal basis $\{e_n\}_n$ of $\mathcal{H}$. Let $\nu_Q : \mathcal{F} \to [0, \infty)$ be a finite measure corresponding to Q. Then, $L^\infty(P) = L^\infty(\nu_P)$ and $L^\infty(Q) = L^\infty(\nu_Q)$.

There is a bijective correspondence between elements $[k]$ of the projective tensor product $L^\infty(\nu_P) \widehat{\otimes}_\pi L^\infty(\nu_Q) \subset L^\infty(\nu_P \otimes \nu_Q)$ and nuclear operators $T_k : L^1(\nu_Q) \to L^\infty(\nu_P)$, such that for each $f \in L^1(\nu_Q)$,

$$(T_k f)(\lambda) = \int_M k(\lambda, \mu) f(\mu) \, d\nu_Q(\mu)$$

for ν_P almost all $\lambda \in \Lambda$, in the sense that for functions with:

$$\sum_{j=1}^\infty \|\phi_j\|_{L^\infty(\nu_P)} \|\psi_j\|_{L^\infty(\nu_Q)} < \infty$$

the kernel $[k] = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j$ corresponds to the nuclear operator:

$$(T_k f) = \sum_{j=1}^{\infty} \phi_j \int_M f \psi_j \, d\nu_Q, \quad f \in L^1(\nu_Q)$$

Nuclear operators between Banach space are discussed in [30], Section III.7.

In the case that $\mathcal{H} = \ell^2$ and $P = Q$ are projections onto the standard basis vectors, then $\int_{\mathbb{N} \times \mathbb{N}} \varphi \, d(P \otimes Q)_{\mathcal{L}(\ell^2)}$ is the classical Schur multiplier operator (17) and Grothendieck's inequality ensures that $L^1((P \otimes Q)_{\mathcal{L}(\ell^2)}) = \ell^\infty \widehat{\otimes}_\pi \ell^\infty$; see Proposition 18 below and [26], Theorem 3.2. In this case, the measure ν in Formula (19) is the counting measure on $\mathbb{N}$, and there is no difficulty with strong ν-measurability in an L^∞-space.

The passage from the discrete case to the case of general spectral measures P and Q sees the nuclear operators from $L^1(\nu_Q)$ to $L^\infty(\nu_P)$ replaced by one-integral operators from $L^1(\nu_Q)$ to $L^\infty(\nu_P)$, which leads to the Peller representation (19).

5.1. Schur Multipliers and Grothendieck's Inequality

If E is any $\mathcal{L}(\mathcal{H})$-valued spectral measure and $h \in \mathcal{H}$, the identity:

$$\sum_{n=1}^{\infty} \|E(f_n)h\|_{\mathcal{H}}^2 = \left(E \left(\sum_{n=1}^{\infty} |f_n|^2 \right) h, h \right)$$

ensures that the $\mathcal{H}$-valued measure Eh has bounded ℓ^2-semi-variation in $\ell^2(\mathcal{H})$, the Hilbert space tensor product $\mathcal{H} \widehat{\otimes} \ell^2 = \oplus_{j=1}^{\infty} \mathcal{H}$ with norm $\|u\|_{\ell^2(\mathcal{H})}^2 = \sum_{j=1}^{\infty} \|u_j\|_{\mathcal{H}}^2$. It follows from [5] that for any essentially bounded functions $f : \Lambda \to \ell^2$ and $g : M \to \ell^2$ and $h \in \mathcal{H}$, the ℓ^2-valued function f is (Ph)-integrable in $\ell^2(\mathcal{H})$, and the ℓ^2-valued function g is (Qh)-integrable in $\ell^2(\mathcal{H})$. Then, there exist operator-valued measures $f \otimes P : \mathcal{E} \to \mathcal{L}(\mathcal{H}, \ell^2(\mathcal{H}))$ and $g \otimes Q : \mathcal{F} \to \mathcal{L}(\mathcal{H}, \ell^2(\mathcal{H}))$, such that:

$$(f \otimes P)(E)h = (f \otimes (Ph))(E), \quad E \in \mathcal{E}, \ h \in \mathcal{H} \text{ and}$$
$$(g \otimes Q)(F)h = (g \otimes (Qh))(F), \quad F \in \mathcal{F}, \ h \in \mathcal{H}$$

There is a simple sufficient condition for $\varphi \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$. Observe first that the linear map $J : \ell^2(\mathcal{H}) \otimes \ell^2(\mathcal{H}) \to \mathcal{H} \widehat{\otimes}_\pi \mathcal{H}$ defined by:

$$J((\{\phi_n\}_n) \otimes (\{\psi_m\}_m)) = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j$$

has a continuous linear extension to a contraction $\overline{J} : \mathcal{C}_1(\ell^2(\mathcal{H})) \to \mathcal{C}_1(\mathcal{H})$ corresponding to taking the trace in the discrete index. The formula:

$$(((f \otimes P) \otimes (g \otimes Q))_{\mathcal{C}_1(\mathcal{H})}(E \times F))(h \otimes k) := (((f \otimes (Ph))(E)) \otimes (g \otimes (Qk))(F))$$

for $h, k \in \mathcal{H}, E \in \mathcal{E}$ and $F \in \mathcal{F}$ defines a finitely-additive set function:

$$((f \otimes P) \otimes (g \otimes Q))_{\mathcal{C}_1(\mathcal{H})}$$

with values in $\mathcal{L}(\mathcal{C}_1(\mathcal{H}), \mathcal{C}_1(\ell^2(\mathcal{H})))$, because $\mathcal{C}_1(\mathcal{K})$ can be identified with $\mathcal{K}\widehat{\otimes}_\pi \mathcal{K}$ for any Hilbert space $\mathcal{K}$. Moreover,

$$\|((f \otimes P) \otimes (g \otimes Q))_{\mathcal{C}_1(\mathcal{H})}(\Lambda \times M)\|_{\mathcal{L}(\mathcal{C}_1(\mathcal{H}), \mathcal{C}_1(\ell^2(\mathcal{H})))} \leq \|f\|_\infty \cdot \|g\|_\infty$$

Then, the operator:

$$\int_{\Lambda \times M} (f, \overline{g}) \, d(P \otimes Q)_{\mathcal{C}_1(\mathcal{H})} = \overline{J}[((f \otimes P) \otimes (g \otimes Q))_{\mathcal{C}_1(\mathcal{H})}(\Lambda \times M)]$$

is an element of $\mathcal{L}(\mathcal{C}_1(\mathcal{H}))$, that is $\varphi = (f, \overline{g})$ belongs to $L^1((P \otimes Q)_{\mathcal{C}_1(\mathcal{H})}) = L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$, and we have the representation:

$$\varphi(\lambda, \mu) = \sum_{n=1}^\infty f_n(\lambda) g_n(\mu) \tag{21}$$

where P-ess.sup$\{\sum_{n=1}^\infty |f_n|^2\} < \infty$ and Q-ess.sup$\{\sum_{n=1}^\infty |g_n|^2\} < \infty$. Moreover, the bound:

$$\left\| \int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{L}(\mathcal{H})} \right\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq \left\| \left(\sum_{n=1}^\infty |f_n|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(P)} \cdot \left\| \left(\sum_{n=1}^\infty |g_n|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(Q)} \tag{22}$$

holds. The same argument works if the spectral measures P, Q are replaced by any two operator valued measures $m : \mathcal{E} \to \mathcal{L}_s(\mathcal{H})$ and $n : \mathcal{F} \to \mathcal{L}_s(\mathcal{H})$ by appealing to the metric form ([26], Theorem 2.4) of Grothendieck's inequality, so that:

$$\left\| \int_{\Lambda \times M} \varphi \, d(m \otimes n)_{\mathcal{L}(\mathcal{H})} \right\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq K_G^2 \|m\|(\Lambda) \|n\|(M) \left\| \left(\sum_{n=1}^\infty |f_n|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(m)} \cdot \left\| \left(\sum_{n=1}^\infty |g_n|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(n)}.$$

Alternatively, for each $T \in \mathcal{L}(\mathcal{H})$, the linear operator:

$$\left(\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{L}(\mathcal{H})} \right) T \in \mathcal{L}(\mathcal{H})$$

can be realised as the operator associated with the bounded sesquilinear form:

$$(h, k) \longmapsto \sum_{n=1}^\infty (TQ(g_n)h, P(\overline{f_n})k).$$

See ([20], Theorem 4.1).

A remarkable consequence of Grothendick's inequality is that for $\varphi \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$, Peller's representation (19) is necessary $(\nu_P \otimes \nu_Q)$ almost everywhere. The analysis of Pisier [26] leads the way.

The projective tensor product $\ell^\infty \widehat{\otimes}_\pi \ell^\infty$ is the completion of the tensor product $\ell^\infty \otimes \ell^\infty$ with respect to the norm:

$$\|u\|_\pi = \inf \left\{ \sum_{j=1}^n \|x_j\|_\infty \|y_j\|_\infty : u = \sum_{j=1}^n x_j \otimes y_j, \ x_j, y_j \in \ell^\infty \right\}$$

Another distinguished norm on $\ell^\infty \otimes \ell^\infty$ is given by:

$$\gamma_2(u) = \inf \left\{ \sup_{\xi \in \ell^1} \left(\sum_{j=1}^n |\xi(x_j)|^2 \right)^{\frac{1}{2}} \cdot \sup_{\eta \in \ell^1} \left(\sum_{j=1}^n |\eta(y_j)|^2 \right)^{\frac{1}{2}} \right\}$$

where the infimum runs over all possible representation $u = \sum_{j=1}^{n} x_j \otimes y_j$ for $x_j, y_j \in \ell^\infty$, $j = 1, \ldots, n$ and $n = 1, 2, \ldots$. Then, γ_2 may also be viewed as the norm of factorisation through a Hilbert space:

$$\gamma_2(u) = \inf\{\sup_i \|x_i\| \cdot \sup_j \|y_j\|\}$$

where the infimum runs over all Hilbert spaces $\mathcal{H}$ and all $x_j, y_j \in \mathcal{H}$ for which $u \in \ell^\infty \otimes \ell^\infty$ has the finite representation $u = \sum_{i,j} (x_i, x_j) e_i \otimes e_j$ with respect to the standard basis $\{e_j\}_j$ of ℓ^∞. Another way of viewing $\gamma_2(u)$ is:

$$\gamma_2(u) = \inf \left\{ \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \right\|_\infty \cdot \left\| \left(\sum_{j=1}^{n} |y_j|^2 \right)^{\frac{1}{2}} \right\|_\infty \right\}$$

over representations $u = \sum_{j=1}^{n} x_j \otimes y_j$, $x_j, y_j \in \ell^\infty$, because:

$$\sup_{\xi \in \ell^1} \left(\sum_{j=1}^{n} |\xi(x_j)|^2 \right)^{\frac{1}{2}} = \sup_{\sum_j |\alpha_j|^2 \leq 1} \left\| \sum_{j=1}^{n} \alpha_j x_j \right\|_\infty$$

$$= \sup_k \left(\sum_{j=1}^{n} |x_j(k)|^2 \right)^{\frac{1}{2}}$$

$$= \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \right\|_\infty$$

Proposition 17. *Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be a function that defines a Schur multiplier $M_\varphi : \mathcal{L}(\ell^2) \to \mathcal{L}(\ell^2)$, that is in matrix notation $M_\varphi(\{a_{ij}\}_{i,j \in \mathbb{N}}) = \{\varphi(i,j) a_{ij}\}_{i,j \in \mathbb{N}}$. The following conditions are equivalent.*

(i) *$\|M_\varphi\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \leq 1$.*
(ii) *There exists a Hilbert space $\mathcal{H}$ and functions $x : \mathbb{N} \to B_1(\mathcal{H})$, $y : \mathbb{N} \to B_1(\mathcal{H})$ with values in the closed unit ball $B_1(\mathcal{H})$ of $\mathcal{H}$, such that $\varphi(n,m) = (x(n), x(m))$, $n, m \in \mathbb{N}$.*
(iii) *For all finite subsets E, F of $\mathbb{N}$, the bound:*

$$\left\| \sum_{i \in E, j \in F} \varphi(i,j) e_i \otimes e_j \right\|_{\ell^\infty \otimes_{\gamma_2} \ell^\infty} \leq 1$$

holds.

Proof. Suppose first that φ is zero off a finite set $E \times F$. Then, the bound (i) is equivalent to the condition that:

$$\left| \sum_{i \in E, j \in F} \varphi(i,j) a_{ij} \alpha(i) \beta(j) \right| \leq 1$$

for all linear maps $a : \ell^2(E) \to \ell^2(F)$ with norm $\|a\| \leq 1$ and matrix $\{a_{ij}\}$ with respect to the standard basis and all $\alpha \in B_1(\ell^2(E))$, $\beta \in B_1(\ell^2(F))$, that is φ belongs to the polar C_1° of the set C_1 of all matrices $\{\alpha(i) a_{ij} \beta(j)\}_{(i,j) \in E \times F}$ with a, α, β as described. According to [26], Remark 23.4, the set C_1 is itself the polar C_2° of the set C_2 of all matrices $\{\psi_{ij}\}_{(i,j) \in E \times F}$ with $\left\| \sum_{i \in E, m \in F} \psi(i,j) e_i \otimes e_j \right\|_{\ell^\infty \otimes_{\gamma_2} \ell^\infty} \leq 1$. Then (i) holds if and only if φ belongs to $C_2^{\circ\circ} = C_2$, which is exactly Condition (iii). Conditions (ii) and (iii) are equivalent by the the definition of the norm γ_2. The passage to all of $\mathbb{N} \times \mathbb{N}$ follows from a compactness argument. $\square$

Remark 9. (a) The argument above uses the factorisation of the norm γ_2^* dual to γ_2 described in [26], Proposition 3.3 and Remark 23.4; this only relies on the Hahn–Banach Theorem.

(b) The representation (21) is the measure space version of the implication (ii) $\implies$ (i) above. The necessity of the condition (21) in the general measure space setting is proven using complete boundedness arguments in [36], Theorem 3.3; see also [37,38].

One version of Grothendieck's inequality from [26] is that the norm γ_2 and the projective tensor product norm are equivalent on $\ell^\infty \otimes \ell^\infty$ with:

$$\gamma_2(u) \le \|u\|_\pi \le K_G \gamma_2(u), \quad u \in \ell^\infty \otimes \ell^\infty$$

The constant K_G is Grothendieck's constant. The projective tensor product version of Proposition 17 follows, with the same notation.

Proposition 18. *Let E, F be finite subsets of $\mathbb{N}$, and let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be a function vanishing off $E \times F$. Then:*

$$\frac{1}{K_G} \left\| \sum_{i \in E, j \in F} \varphi(i,j) e_i \otimes e_j \right\|_{\ell^\infty \otimes_\pi \ell^\infty} \le \|M_\varphi\|_{\mathcal{L}(\mathcal{L}(\ell^2))}$$

$$= \left\| \sum_{i \in E, j \in F} \varphi(i,j) e_n \otimes e_j \right\|_{\ell^\infty \otimes_{\gamma_2} \ell^\infty}$$

Passing to infinite sets, a bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ with $\|M_\varphi\|_{\mathcal{L}(\mathcal{L}(\ell^2))} < \infty$ necessarily has a representation:

$$\varphi(i,j) = \sum_{k=1}^\infty a(i,k)\beta(j,k), \quad i, j \in \mathbb{N}$$

with $\sum_{k=1}^\infty \|a(\cdot,k)\|_\infty \|\beta(\cdot,k)\|_\infty < \infty$, as in Peller's representation (19).

5.2. Schur Multipliers on Measure Spaces

We first note that for any choice of finite measures ν_P, ν_Q equivalent to P and Q, respectively, the Banach algebra $L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ is isometrically isomorphic to the set of multipliers of the projective tensor product $L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$, that is $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ if and only if for every $[h] \in L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$, the function $\varphi.h$ is equal $(\nu_P \otimes \nu_Q)$-a.e. to an element of $L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$ and $\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}$ is equal to the norm of the linear map:

$$[h] \longmapsto [\varphi.h], \quad [h] \in L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$$

If ν_P' and ν_Q' are another pair of such equivalent measures, then the operator of multiplication by $\sqrt{d\nu_P'/d\nu_P}$ is a unitary map from $L^2(\nu_P)$ to $L^2(\nu_P')$ and similarly for ν_Q, so that multiplication by $\sqrt{d\nu_P'/d\nu_P} \otimes \sqrt{d\nu_Q'/d\nu_Q}$ is an isometric isomorphism from the space $L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$ onto $L^2(\nu_P')\widehat{\otimes}_\pi L^2(\nu_Q')$.

Proposition 19. *Let ν_P, ν_Q be finite measures equivalent to the spectral measures P, Q, respectively. Then, $L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$ is isometrically isomorphic to the set of multipliers of the projective tensor product $L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)$ and the identity:*

$$\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})} = \sup_{\|h\|_{\mathcal{H}} \leq 1, \|g\|_{\mathcal{H}} \leq 1} \|[\varphi]\|_{L^2((Ph,h)) \widehat{\otimes}_\pi L^2((Qg,g))} \tag{23}$$

holds.

Proof. Let $\{h_n\}_n$ be a sequence of vectors in $\mathcal{H}$ with $\sum_n \|h_n\|^2 < \infty$, such that $\{P(E)h_n : n = 1, 2, \dots\}$ is an orthogonal set of vectors in $\mathcal{H}$ for each $E \in \mathcal{E}$. Such a sequence of vectors can always be manufactured by taking any vectors $\xi_n \in \mathcal{H}$ with $\sum_n \|\xi_n\|^2 < \infty$ and for a measure ν_P equivalent to P, the sets Λ_n where $d(P\xi_n, \xi_n)/d\nu_P > 0$. Then, $h_n = P(\Lambda_n \setminus \bigcup_{m<n} \Lambda_n)\xi_n$, $n = 1, 2, \dots$, will do the job. Let $\{g_n\}_n$ be the corresponding vectors for Q.

As noted above, the norm of $L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)$ is invariant under a change of equivalent measures, so we may as well assume that:

$$\nu_P = \sum_{n=1}^{\infty} (Ph_n, h_n) \text{ and } \nu_Q = \sum_{n=1}^{\infty} (Qg_n, g_n)$$

so that the mappings $\chi_E \to \sum_{n=1}^{\infty} P(E)h_n$, $E \in \mathcal{E}$, and $\chi_F \to \sum_{n=1}^{\infty} Q(F)h_n$, $F \in \mathcal{F}$, define a unitary equivalences U_P, U_Q between $L^2(\nu_P)$ and $L^2(\nu_Q)$ and $\mathcal{H}$, respectively.

The map $T_k : L^2(\nu_Q) \to L^2(\nu_P)$ with integral kernel $k \in L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)$ is the trace class. Let $\tilde{T}_k \in \mathcal{C}_1(\mathcal{H})$ be the corresponding trace class operator $\mathcal{H}$. Then:

$$\sum_{n,m=1}^{\infty} (\tilde{T}_k Q(F)g_m, P(E)h_n) = \int_{E \times F} k \, d(\nu_P \otimes \nu_Q)$$

Let $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})}) = L^1((P \otimes Q)_{\mathcal{C}_1(\mathcal{H})})$. Then, $\tilde{T}_k \in \mathcal{C}_2(\mathcal{H})$ and:

$$\left(\left(\int\!\!\!\!\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{C}_1(\mathcal{H})} \right) \tilde{T}_k \left(\sum_{m=1}^{\infty} Q(F)g_m \right), \left(\sum_{n=1}^{\infty} P(E)h_n \right) \right)$$

$$= \mathrm{tr} \left(\left(\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{C}_2(\mathcal{H})} \right) \tilde{T}_k \left(\sum_{m=1}^{\infty} Q(F)g_m \right) \otimes \left(\sum_{n=1}^{\infty} P(E)h_n \right)^* \right)$$

$$= \sum_{n,m=1}^{\infty} \int_{E \times F} \varphi \, d((P\tilde{T}_k Q)g_m, h_n)$$

$$= \int_{E \times F} \varphi . k \, d(\nu_P \otimes \nu_Q)$$

It follows that $\varphi . k$ is the kernel of the trace class operator $T_{\varphi . k} : L^2(\nu_Q) \to L^2(\nu_P)$, such that:

$$U_P T_{\varphi . k} U_Q^* = \left(\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{C}_1(\mathcal{H})} \right) \tilde{T}_k \in \mathcal{C}_1(\mathcal{H})$$

the equality $\|\varphi . k\|_{L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)} = \left\| \left(\int_{\Lambda \times M} \varphi \, d(P \otimes Q)_{\mathcal{C}_1(\mathcal{H})} \right) \tilde{T}_k \right\|_{\mathcal{C}_1(\mathcal{H})}$ holds and:

$$\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})} = \sup\{\|\varphi . k\|_{L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)} : \|k\|_{L^2(\nu_P) \widehat{\otimes}_\pi L^2(\nu_Q)} \leq 1\} \tag{24}$$

According to the identities above,

$$\left\| \int_{\Lambda \times M} \varphi \, d(P\tilde{T}_k Q) \right\|_{\mathcal{C}_1(\mathcal{H})} = \|\varphi.k\|_{L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)}$$

for all $k \in L^2(\nu_P) \otimes L^2(\nu_Q)$, so if:

$$\|\varphi.k\|_{L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)} \leq C$$

for all $k \in L^2(\nu_P) \otimes L^2(\nu_Q)$ satisfying $\|k\|_{L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)} \leq 1$, then $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$, the identity (24) holds and:

$$\|[\varphi]\|_{L^1((P\otimes Q)_{\mathcal{L}(\mathcal{H})})} = \sup_{\sup_{\|u\|_2 \leq 1, \|v\|_2 \leq 1}} \|\varphi.(u \otimes v)\|_{L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)}$$

The equality (23) follow from the identities:

$$\int_\Lambda |\psi_1|^2 |\psi_2|^2 \, d\nu_P = \left\| \sum_{n=1}^\infty P(\psi_1.\psi_2)h_n \right\|_{\mathcal{H}}^2$$
$$= (P(|\psi_1|^2)(U_P\psi_2), (U_P\psi_2))$$

for $\psi_1 \in L^\infty(\nu_P)$, $\psi_2 \in L^2(\nu_P)$ and the unitary equivalence U_P defined above. The analogous identities hold for the spectral measure Q. $\square$

Proof of Theorem 16. We proceed by reduction to the ℓ^2-case considered in Proposition 18. A Lusin μ-filtration of a σ-finite measure space $(\Sigma, \mathcal{E}, \mu)$ is an increasing family $\mathcal{F} = \langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ of σ-algebras, such that for a set Σ_0 of full measure, $\mathcal{E} \cap \Sigma_0 = \bigvee(\mathcal{F} \cap \Sigma_0)$, and each element of $\mathcal{F}_n$ is the countable union of sets belonging to a countable partition $\mathcal{G}_n$ of Σ_0 into sets of finite positive μ-measure and such that each set in $\mathcal{G}_{n+1}$ is contained in an element of $\mathcal{G}_n$, for $n = 1, 2 \ldots$.

Let ν_P, ν_Q be finite measures equivalent to P, Q, respectively. Because both $L^2(\nu_P)$ and $L^2(\nu_Q)$ are isomorphic to the separable Hilbert space $\mathcal{H}$, for the purpose of obtaining the representation (19), we may suppose that the underlying σ-algebras are countably generated.

Let $\mathcal{P} = \{\mathcal{P}_n\}_n$ be a Lusin ν_P-filtration, and let $\mathcal{Q} = \{\mathcal{Q}_n\}_n$ be a Lusin ν_Q-filtration. Suppose that $n = 1, 2, \ldots$, $\{A_i\}_{i=1}^\infty$ is the n-th partition associated with $\mathcal{P}$ and $\{B_j\}_{j=1}^\infty$ is the n-th partition associated with $\mathcal{Q}$. The corresponding projection operators $P_n : L^1(\nu_P) \to \ell^1$ and $Q_n : L^1(\nu_Q) \to \ell^1$ are defined by:

$$P_n : f \longmapsto \left\{ \int_{A_i} f \, d\nu_P \right\}_{i=1}^\infty, \; f \in L^1(\nu_P)$$

$$Q_n : g \longmapsto \left\{ \int_{B_j} f \, d\nu_Q \right\}_{j=1}^\infty, \; g \in L^1(\nu_Q)$$

The conditional expectation $(\mathbb{E}_n \otimes \mathbb{F}_n)(f) = \mathbb{E}(f|\mathcal{P}_n \otimes \mathcal{Q}_n)$ is defined for any measurable function $f : \Lambda \times M \to \mathbb{C}$ that is integrable over any set $A_i \times B_j$, $i, j \in \mathbb{N}$.

It is easy to verify that $P_n^* T_{\varphi_n} Q_n = T_{(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi}$ for the matrix:

$$\varphi_n = \left\{ \frac{\int_{A_i \times B_j} \varphi \, d(\nu_P \otimes \nu_Q)}{\nu_P(A_i)\nu_Q(B_j)} \right\}_{i,j=1}^\infty$$

and the operator $T_{\varphi_n} : \ell^1 \to \ell^\infty$ with kernel φ_n.

Moreover, for every finite rank operator $U : L^\infty(\nu_P) \to L^1(\nu_Q)$, the bound:

$$|\text{tr}(P_n^* T_{\varphi_n} Q_n U)| = |\text{tr}(T_{\varphi_n} Q_n U P_n^*)| \le \|\varphi_n\|_{\ell^\infty \widehat{\otimes}_\pi \ell^\infty} \|Q_n U P_n^*\|$$

Suppose that there exists $C > 0$, such that $\|\varphi_n\|_{\ell^\infty \widehat{\otimes}_\pi \ell^\infty} \le C$ for all $n = 1, 2, \dots$

Then, $\text{tr}(P_n^* T_{\varphi_n} Q_n U) = \text{tr}(T_{(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi} U) = \text{tr}(T_\varphi \mathbb{E}_n U \mathbb{F}_n)$, and taking $n \to \infty$, the martingale convergence theorem shows that the bound:

$$|\text{tr}(T_\varphi U)| \le C \|U\|_{\mathcal{L}(L^\infty, L^1)}$$

holds for every finite rank operator $U : L^\infty(\nu_P) \to L^1(\nu_Q)$. It follows from [39], Theorem 6.16, that T_φ belongs to the Banach ideal $\mathcal{I}_1(L^1(\nu_Q), L^\infty(\nu_P))$ of one-integral operators from $L^1(\nu_Q)$ to $L^\infty(\nu_P)$. Because $L^\infty(\nu_P)$ is a dual space, [39], Corollary 5.4, ensures that T_φ enjoys the factorisation:

$$
\begin{array}{ccc}
L^1(\nu_Q) & \xrightarrow{T_\varphi} & L^\infty(\nu_P) \\
T_1 \downarrow & & \uparrow T_2 \\
L^\infty(\nu) & \xrightarrow[j]{} & L^1(\nu)
\end{array}
$$

for some bounded linear operators T_1 and T_2 and finite measure space $(T, \mathcal{S}, \nu)$. The given factorisation also follows by the original 1954 Grothendieck argument with the choice $E = L^1(\nu_Q)$, $F = L^1(\nu_P)$ in [30], Section IV.9.2.

Every bounded linear operator u from $L^1(\eta_1)$ to $L^\infty(\eta_2)$ is an integral operator with a bounded kernel, because $f \otimes g \mapsto \langle uf, g \rangle$ defines a continuous linear functional on $L^1(\eta_1) \widehat{\otimes}_\pi L^1(\eta_2) \equiv L^1(\eta_1 \otimes \eta_2)$ (see [40], Lemma 2.2, for a compactness argument), so there exist bounded measurable functions $\alpha : \Lambda \times T \to \mathbb{C}$ and $\beta : M \times T \to \mathbb{C}$, such that:

$$(T_1 f)(t) = \int_M \beta(\mu, t) f(\lambda) d\nu_Q(\lambda), \quad f \in L^1(\nu_Q)$$

$$(T_2 g)(\lambda) = \int_T \alpha(\lambda, t) g(t) d\nu(t), \quad g \in L^1(\nu)$$

The representation (19) and the associated bounds follow if we can take:

$$C = K_G \|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}$$

We know from the bounds (18) that:

$$\|\varphi_n\|_{\ell^\infty \widehat{\otimes}_\pi \ell^\infty} \le K_G \|M_{\varphi_n}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} = K_G \|\varphi_n\|_{\ell^\infty \widehat{\otimes}_{\gamma_2} \ell^\infty}$$

The norm γ_2 defined on $\ell^\infty \otimes \ell^\infty$ is the norm of factorisation through a Hilbert space. For any bounded linear operator $u : X \to Y$ between Banach spaces X and Y, $\gamma_2(u) = \inf\{\|u_1\|, \|u_2\|\}$ where the infimum runs over all Hilbert spaces $\mathcal{H}$ and all possible factorisations:

$$u : X \xrightarrow{u_2} \mathcal{H} \xrightarrow{u_1} Y$$

of u through $\mathcal{H}$ with $u = u_1 \circ u_2$. Taking $X = L^1(\nu_Q)$ and $Y = L^\infty(\nu_P)$, the bound (22) says that:

$$\|\varphi\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})} \leq \|\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)}$$

with respect to the completion $L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)$ of $L^\infty(\nu_P) \otimes L^\infty(\nu_Q)$ in the norm $\phi \mapsto \gamma_2(T_\phi)$, $\phi \in L^\infty(\nu_P) \otimes L^\infty(\nu_Q)$.

The norm estimates:

$$\|\varphi_n\|_{\ell^\infty\widehat{\otimes}_{\gamma_2}\ell^\infty} = \|(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)} \leq \|\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)}$$

follow from the definition of γ_2 and the contractivity of the conditional expectation operators $\mathbb{E}_n, \mathbb{F}_n$.

According to Proposition 19, the norm of the linear operator:

$$M_\varphi : \mathcal{C}_1(L^2(\nu_Q), L^2(\nu_P)) \to \mathcal{C}_1(L^2(\nu_Q), L^2(\nu_P))$$

associated with multiplication by φ on $L^2(\nu_P)\widehat{\otimes}_\pi L^2(\nu_Q)$ is equal to:

$$\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})} = \|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{C}_1(\mathcal{H})})}$$

The equality $\|\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)} = \|M_\varphi\|_{\mathcal{L}(\mathcal{L}(L^2(\nu_Q),L^2(\nu_P))}$ is proven in [36], Theorem 3.3, using complete boundedness techniques, but this can be established in a more elementary way by noting that if $[\varphi] \in L^1((P \otimes Q)_{\mathcal{C}_1(\mathcal{H})})$, then the martingale convergence theorem ensures that $M_{(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi} \to M_\varphi$ in the strong operator topology of:

$$\mathcal{L}(\mathcal{C}_1(L^2(\nu_Q), L^2(\nu_P)), \mathcal{C}_1(L^2(\nu_Q), L^2(\nu_P)))$$

as $n \to \infty$ and also:

$$\|(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)} \longrightarrow \|\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)}$$

as $n \to \infty$. Then, $\|M_\varphi\| = \sup_n \|M_{(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi}\|$ by duality. The equality:

$$\|(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)} = \|M_{(\mathbb{E}_n \otimes \mathbb{F}_n)\varphi}\|_{\mathcal{L}(\mathcal{L}(L^2(\nu_Q),L^2(\nu_P)))}$$

follows for each $n = 1, 2, \ldots$ from Proposition 17 by replacing $e_i \otimes e_j$ in (iii) by $\chi_{A_i \times B_j}$ for $i, j = 1, 2, \ldots$. The final assertion of Theorem 16 follows from the equalities:

$$\|\varphi\|_{L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2}L^\infty(\nu_Q)} = \|M_\varphi\|_{\mathcal{L}(\mathcal{L}(L^2(\nu_Q),L^2(\nu_P)))} = \|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}$$

$\square$

Remark 10. (a) The original proof of Peller [25,40], Theorem 2.2, factorises the finite rank operator $U : L^\infty(\nu_P) \to L^1(\nu_Q)$ instead, so the constant K_G^2 appears in place of K_G in the bound associated with (19).

(b) Let $L^1(\nu_P)\tilde{\otimes}L^1(\nu_Q)$ be the closure of the linear space of all $k \in L^1(\nu_P) \otimes L^1(\nu_Q)$ in the uniform norm of the space of operators $T_k \in \mathcal{L}(L^\infty(\nu_Q), L^1(\nu_P))$ corresponding to the compact linear operators

from $L^\infty(\nu_Q)$ to $L^1(\nu_P)$. By [30], Section IV.9.2, the function $\alpha \otimes \beta$ in Formula (19) is ν-integrable in the space of one-integral operators:

$$\mathcal{I}_1(L^1(\nu_Q), L^\infty(\nu_P)) \equiv (L^1(\nu_P)\tilde{\otimes}L^1(\nu_Q))'$$

and $\varphi = \int_T \alpha \otimes \beta \, d\nu$.

(c) The proof above shows that operator $T_\varphi : L^1(\nu_Q) \to L^\infty(\nu_P)$ is (strictly) one-integral in the sense of [39], p. 97, and [30], Section IV.9.2, if and only if $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$. The reason that we may have $\|[\varphi]\|_{L^\infty(\nu_P)\widehat{\otimes}_\pi L^\infty(\nu_Q)} = \infty$ for some $[\varphi] \in L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})$, that is the function $\alpha \otimes \beta$ associated with the representation (19) fails to be ν-integrable in $L^\infty(\nu_P)\widehat{\otimes}_\pi L^\infty(\nu_Q)$, so that $T_\varphi : L^1(\nu_Q) \to L^\infty(\nu_P)$ thereby fails to be a nuclear operator, is that the vector measure $E \longmapsto u\chi_E$ associated with a continuous linear map u from L^1 to L^∞ has a weak*-density, but not necessarily a strongly-measurable density in L^∞.

For any $u \in \mathcal{C}_1(\mathcal{H})$ and $\varphi \in L^1((P \otimes P)_{\mathcal{L}(\mathcal{H})})$, the operator:

$$M_\varphi u = \left(\int_{\Lambda \times \Lambda} \varphi \, d(P \otimes P)_{\mathcal{C}_1(\mathcal{H})} \right) u$$

is the trace class. Moreover, the expression $E \longmapsto \text{tr}(uP(E))$, $E \in \mathcal{E}$, is a complex measure μ_u on the σ-algebra $\mathcal{E}$, such that $|\mu_u| << \nu_P$. As indicated in [20], Section 9.1, the identity:

$$\text{tr}(M_\varphi u) = \int_\Lambda \varphi(\lambda, \lambda) \, d\mu_u(\lambda) \tag{25}$$

holds. In the case that $u : \mathcal{H} \to \mathcal{H}$ is a finite rank operator, together with the polarisation, the bound (23) shows that the operator $T_\varphi : L^2(\mu_u) \to L^2(\mu_u)$ with integral kernel φ is the trace class and:

$$\|T_\varphi\|_{\mathcal{C}_1(L^2(\mu_u))} \leq 16\|[\varphi]\|_{L^1((P \otimes Q)_{\mathcal{L}(\mathcal{H})})}\|u\|_{\mathcal{C}_1(\mathcal{H})}.$$

The same bound holds for all $u \in \mathcal{C}_1(\mathcal{H})$. The identity:

$$|\psi|^2.\nu_P = (P(U_P\psi), (U_P\psi)), \quad \psi \in L^2(\nu_P)$$

ensures that $\text{tr}(M_{\phi_1 \otimes \phi_2} u) = \text{tr}(T_{\phi_1 \otimes \phi_2})$ for $T_{\phi_1 \otimes \phi_2} \in \mathcal{C}_1(L^2(\mu_u))$ with $u \in \mathcal{C}_1(\mathcal{H})$ and ϕ_1, ϕ_2 bounded on Λ. Then, the equality:

$$\text{tr}(M_\varphi u) = \text{tr}(T_\varphi)$$

holds, because both sides are continuous for $\varphi \in L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2} L^\infty(\nu_P)$.

The representation (21) converges in $L^\infty(\nu_P)\widehat{\otimes}_{\gamma_2} L^\infty(\nu_P)$, and there exists a set Λ_0 of full ν_P-measure, such that:

$$\varphi(\lambda, \mu) = \sum_{n=1}^\infty f_n(\lambda)g_n(\mu)$$

for all $\lambda, \mu \in \Lambda_0$, where the right-hand sum converges absolutely. The expression above constitutes a distinguished element of the equivalence class $[\varphi]$. Consequently, Formula (25) is valid because:

$$\operatorname{tr}(T_\varphi) = \sum_{n=1}^\infty \operatorname{tr}(T_{f_n \otimes g_n})$$
$$= \sum_{n=1}^\infty \int_\Lambda f_n(\lambda) g_n(\lambda) \, d\mu_u(\lambda)$$
$$= \int_\Lambda \varphi(\lambda, \lambda) \, d\mu_u(\lambda)$$

As in the proof of Theorem 16, for any Lusin ν_P-filtration $\mathcal{F} = \langle \mathcal{E}_k \rangle_k$ of Λ, for each $k = 1, 2, \ldots$, the conditional expectation operators $\mathbb{E}_k : f \longmapsto \mathbb{E}(f|\mathcal{E}_k)$ with respect to the σ-algebra $\mathcal{E}_k$ and the finite measure ν_P have the property that:

$$\sum_{n=1}^\infty |f_n.g_n - \mathbb{E}_k(f_n).\mathbb{E}_k(g_n)|$$
$$\leq \sum_{n=1}^\infty |(f_n - \mathbb{E}_k(f_n)).g_n| + \sum_{n=1}^\infty |\mathbb{E}_k(f_n).(g_n - \mathbb{E}_k(g_n)|$$
$$\leq \left(\sum_{n=1}^\infty |(f_n - \mathbb{E}_k(f_n))|^2 \right)^{\frac{1}{2}} . \left(\sum_{n=1}^\infty |g_n|^2 \right)^{\frac{1}{2}} +$$
$$\left(\sum_{n=1}^\infty |\mathbb{E}_k(f_n)|^2 \right)^{\frac{1}{2}} . \left(\sum_{n=1}^\infty |(g_n - \mathbb{E}_k(g_n)|^2 \right)^{\frac{1}{2}}$$
$$\to 0 \quad \nu_P\text{-almost everywhere as } k \to \infty$$

by the martingale convergence theorem. Consequently, setting:

$$\tilde{\varphi} = \lim_{k \to \infty} (\mathbb{E}_k \otimes \mathbb{E}_k)\varphi$$

wherever the limit exists, the equality $\tilde{\varphi}(\lambda, \lambda) = \varphi(\lambda, \lambda)$ holds for ν_P-almost all $\lambda \in \Lambda$.

Remark 11. There is a representative function φ of the equivalence class $[\varphi]$ that is continuous for the so-called ω-topology of [37], Proposition 9.1, so Formula (25) may also be derived from the trace formula for a trace class operator with a continuous integral kernel. In fact, Peller's representation (19) can be deduced directly from Proposition 18 by employing the ω-continuity of φ rather than the martingale convergence theorem; see [37], Remark p. 139.

6. The Spectral Shift Function

The following perturbation formula of Birman and Solomyak ([20], Theorem 8.1) was mentioned in the proof of Corollary 14. The operator ideal $\mathfrak{S}$ is taken to be $\mathcal{C}_p(\mathcal{H})$ for $1 \leq p < \infty$ or $\mathcal{L}(\mathcal{H})$ for a given Hilbert space $\mathcal{H}$.

Theorem 20. *Let $\mathcal{H}$ be a separable Hilbert space, and let A and B be self-adjoint operators with the same domain, such that $A - B \in \mathfrak{S}$. Let $P_A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ and $P_B : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$ be the spectral*

measures on $\mathbb{R}$ associated with A and B, respectively. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which the difference quotient:

$$\varphi_f(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} & , \quad \lambda \neq \mu \\ 0 & , \quad \lambda = \mu \end{cases}$$

is uniformly bounded and $\varphi_f \in L^1((P_A \otimes Q_B)_{\mathfrak{S}})$. Then:

$$\int_{\mathbb{R} \times \mathbb{R}} \varphi_f \, d(P_A \otimes P_B)_{\mathfrak{S}} \in \mathcal{L}(\mathfrak{S})$$

and:

$$f(A) - f(B) = \left(\int_{\mathbb{R} \times \mathbb{R}} \varphi_f \, d(P_A \otimes P_B)_{\mathfrak{S}} \right)(A - B)$$

If $\mathfrak{S} = \mathcal{C}_1(\mathcal{H})$, then we would like to calculate the trace of $f(A) - f(B)$. The method of the preceding section is unavailable with different spectral measures P_A, P_B, so we can try to invoke the Daletskii–Krein formula ([20], Equations (9) and (10)). For a sufficiently smooth function f, this takes the form:

$$f(A) - f(B) = \int_0^1 \left(\int_{\mathbb{R} \times \mathbb{R}} \varphi_f(\lambda, \mu) \, d(P_{A(t)} \otimes P_{A(t)})_{\mathcal{C}_1(\mathcal{H})} \right)(A - B) \, dt$$

with $A(t) = B + t(A - B)$, $0 \leq t \leq 1$ and $\varphi_f(\lambda, \lambda) = f'(\lambda)$, $\lambda \in \mathbb{R}$. At each point $0 \leq t \leq 1$, the same spectral measure $P_{A(t)}$ is involved, so from Formula (25), we can expect that:

$$\mathrm{tr}(f(A) - f(B)) = \int_{\mathbb{R}} f'(\lambda) \, d\Xi(\lambda)$$

for the complex measure $\Xi : E \longmapsto \int_0^1 \mathrm{tr}(V P_{A(t)}(E)) \, dt$, $E \in \mathcal{B}(\mathbb{R})$, with $V = (A - B) \in \mathcal{C}_1(\mathcal{H})$. It turns out that Ξ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ from which the formula:

$$\mathrm{tr}(f(A) - f(B)) = \int_{\mathbb{R}} f'(\lambda) \xi(\lambda) \, d\lambda \tag{26}$$

is obtained. The function $\xi : \mathbb{R} \to \mathbb{C}$ is Krein's spectral shift function.

We now turn to establishing the validity of Formula (26) for a restricted class of functions f. Better results are known, for example, from [25,41–43], but our purpose is to describe applications of singular bilinear integrals, such as double operator integrals to problems in the perturbation theory of linear operators. The approach of Boyadzhiev [44] best suits the purpose.

Setting $V = A - B \in \mathcal{C}_1(\mathcal{H})$, we first note that $e^{isA(t)} - e^{isB} \in \mathcal{C}_1(\mathcal{H})$ for each $s \in \mathbb{R}$ and $0 \leq t \leq 1$, because the perturbation series:

$$e^{isA(t)} = e^{isB} +$$

$$\sum_{n=1}^{\infty} (is)^n \int_0^t \cdots \int_0^{s_2} e^{isB(s - s_n)} V \cdots e^{isB(s_2 - s_1)} V e^{isBs_1} \, ds_1 \cdots ds_n$$

converges in the norm of $\mathcal{C}_1(\mathcal{H})$ and $t \longmapsto e^{isA(t)} - e^{isB}$ is norm differentiable in $\mathcal{C}_1(\mathcal{H})$. Moreover,

$$\|e^{isA(t)} - e^{isB}\|_{\mathcal{C}_1(\mathcal{H})} \leq (e^{|s| \|V\|_{\mathcal{C}_1(\mathcal{H})}} - 1) \tag{27}$$

The following result is straightforward, but it depends on some measure theoretic facts. It establishes that Ξ is a complex measure.

Lemma 21. *The function* $t \longmapsto P_{A(t)}(E)h$, $t \in [0,1]$, *is strongly measurable in* $\mathcal{H}$ *for each* $h \in \mathcal{H}$ *and* $E \in \mathcal{B}(\mathbb{R})$. *There exists a unique operator-valued measure* $M : \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s(\mathcal{H})$, σ-*additive for the strong operator topology, such that the equality:*

$$(M(X \times Y)h, h) = \int_X (P_{A(t)}(Y)h, h)\, dt, \quad X \in \mathcal{B}([0,1]),\ Y \in \mathcal{B}(\mathbb{R})$$

holds for each $h \in \mathcal{H}$. *For each* $V \in \mathcal{C}_1(\mathcal{H})$, *the set function* $E \longmapsto \mathrm{tr}(VM(E))$, $E \in \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R})$, *is a complex measure, and we have:*

$$\mathrm{tr}(VM([0,1] \times Y)) = \int_0^1 \mathrm{tr}(VP_{A(t)}(Y))\, dt = \Xi(Y), \quad Y \in \mathcal{B}(\mathbb{R}) \tag{28}$$

Proof. If $f = \hat{\mu}$ is the Fourier transform of a finite measure μ, then:

$$(P_{A(t)}h)(f) = \int_{\mathbb{R}} e^{-i\xi A(t)} h\, d\mu(\xi)$$

as a Bochner integral and by dominated convergence $t \longmapsto (P_{A(t)}h)(f)$, $0 \le t \le 1$, is continuous in $\mathcal{H}$ for each $h \in \mathcal{H}$. By a monotone class argument, $t \longmapsto (P_{A(t)}h)(f)$, $0 \le t \le 1$, is strongly measurable for all bounded Borel measurable functions f.

For each $h \in \mathcal{H}$, the set function (Mh, h) is nonnegative and finitely additive, and the algebra $\mathcal{A}$ is generated by product sets $X \times Y$ for $X \in \mathcal{B}([0,1])$ and $Y \in \mathcal{B}(\mathbb{R})$, so $|(M(A)h, h)| \le \|h\|_1^2$, $A \in \mathcal{A}$. The set function $(Mh, h) : \mathcal{A} \to [0, \|h\|_2]$ is separately countably additive with respect to Borel sets, so it is inner regular with respect to compact product sets and, so, countably additive (countable additivity may fail without inner-regularity; see [45]).

Denoting the extended measure by the same symbol, $|(M(E)h, h)| \le \|h\|_1^2$ for all $E \in \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R})$. The $\mathcal{H}$-valued measure Mh is weakly countable additive by polarity and, so, norm countably additive by the Orlicz–Pettis theorem.

For each $V \in \mathcal{C}_1(\mathcal{H})$ and orthonormal basis $\{h_j\}_j$ of $\mathcal{H}$, the bound:

$$\sum_{j=1}^{\infty} |(VM(E)h_j, h_j)| \le 4\|V\|_{\mathcal{C}_1(\mathcal{H})}, \quad E \in \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R})$$

holds and:

$$\mathrm{tr}(VM([0,1] \times Y)) = \sum_{j=1}^{\infty} (VM([0,1] \times Y)h_j, h_j)$$
$$= \int_0^1 \sum_{j=1}^{\infty} (VP_{A(t)}(Y)h_j, h_j)\, dt$$

by the Beppo–Levi convergence theorem, because:

$$\sum_{j=1}^{\infty} |(VP_{A(t)}(Y)h_j, h_j)| \le 4\|V\|_{\mathcal{C}_1(\mathcal{H})}, \quad 0 \le t \le 1$$

so Equation (28) holds. $\square$

An application of Fubini's Theorem for disintegrations of measures shows that:

$$\int_{[0,1]\times\mathbb{R}} e^{-i\lambda s}\, d(Mh,h)(t,\lambda) = \int_{\mathbb{R}} e^{-i\lambda s}\left(\int_0^1 (P_{A(t)}h,h)\,dt\right)(d\lambda)$$
$$= \int_0^1 \int_{\mathbb{R}} e^{-i\lambda s}(P_{A(t)}h,h)(d\lambda)\,dt$$
$$= \int_0^1 (e^{-isA(t)}h,h)\,dt$$

for each $h \in \mathcal{H}$. The identity:

$$\int_{\mathbb{R}} e^{-i\lambda s}\left(\int_0^1 (VP_{A(t)}h,h)\,dt\right)(d\lambda) = \int_0^1 (Ve^{-isA(t)}h,h)\,dt$$

follows for each $h \in \mathcal{H}$ by polarisation. Because:

$$\Xi(E) = \int_0^1 \operatorname{tr}(VP_{A(t)}(E))\,dt = \sum_{j=1}^\infty \int_0^1 (VP_{A(t)}(E)h_j,h_j)\,dt, \quad E \in \mathcal{B}(\mathbb{R})$$

for any orthonormal basis $\{h_j\}_j$ of $\mathcal{H}$, the Fourier transform of the measure Ξ is:

$$\int_{\mathbb{R}} e^{-i\lambda s}\, d\Xi(\lambda) = \int_0^1 \operatorname{tr}(Ve^{-isA(t)})\,dt$$
$$= i\int_0^1 s^{-1}\frac{d}{dt}\operatorname{tr}(e^{-isA(t)})\,dt$$
$$= i\frac{\operatorname{tr}(e^{-isA} - e^{-isB})}{s}$$

We need to establish that the inverse Fourier transform $\check{\Phi}$ of the uniformly bounded, continuous function:

$$\Phi : s \longmapsto i\frac{\operatorname{tr}(e^{-isA} - e^{-isB})}{s}, \quad s \in \mathbb{R}\setminus\{0\}, \quad \Phi(0) = \operatorname{tr}(V)$$

belongs to $L^1(\mathbb{R})$. Then, $\xi = \check{\Phi}$ is the spectral shift function. Clearly, the value of Φ at zero is irrelevant.

It suffices to show that there exists $\xi \in L^1(\mathbb{R})$, such that:

$$\mu(\Phi) = 2\pi\int_{\mathbb{R}} \xi(t)\overline{\check{\mu}(t)}\,dt = \int_{\mathbb{R}} \xi(t)\hat{\mu}(t)\,dt$$

with $\check{\mu}(t) = (2\pi)^{-1}\int_{\mathbb{R}} e^{ist}\,d\mu(s)$ and $\hat{\mu}(t) = \int_{\mathbb{R}} e^{-ist}\,d\mu(s)$, $t \in \mathbb{R}$, for every finite positive measure μ, because then $\check{\Phi} = \xi$ as elements of the space $\mathcal{S}'$ of Schwartz distributions on $\mathbb{R}$. Therefore, we consider the class of functions $f : \mathbb{R} \to \mathbb{R}$ for which $f' = \hat{\mu}$ and $f(0) = 0$ and, consequently, $\operatorname{tr}(f(A) - f(B)) = (2\pi)^{-1}\mu(\Phi)$.

Theorem 22. *Let $\mathcal{H}$ be a separable Hilbert space, and let A and B be self-adjoint operators with the same domain, such that $A - B \in \mathcal{C}_1(\mathcal{H})$. Then, there exists a function $\xi \in L^1(\mathbb{R})$, such that:*

$$\operatorname{tr}(f(A) - f(B)) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda)\,d\lambda \tag{29}$$

for every function $f : \mathbb{R} \to \mathbb{C}$ for which there exists a finite positive Borel measure μ on $\mathbb{R}$, such that:

$$f(x) = i \int_{\mathbb{R}} \frac{e^{-isx} - 1}{s} \, d\mu(s), \quad x \in \mathbb{R}$$

(a) $\operatorname{tr}(A - B) = \int_{\mathbb{R}} \xi(\lambda) \, d\lambda$.
(b) $\|\xi\|_1 \leq \|A - B\|_{\mathcal{C}_1(\mathcal{H})}$.
(c) *If $B \leq A$, then $\xi \geq 0$ a.e.*
(d) *ξ is zero a.e. outside of the interval $(\inf(\sigma(A) \cup \sigma(B)), \sup(\sigma(A) \cup \sigma(B)))$.*

Proof. The proof is set out in considerable detail in [44]. Here, we review the salient points.

The estimate $\|f(A) - f(B)\|_{\mathcal{C}_1(\mathcal{H})} \leq \mu(\mathbb{R})\|A - B\|_{\mathcal{C}_1(\mathcal{H})}$ follows from the bound (27) and the calculation:

$$f(A) - f(B) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{e^{-isA} - e^{-isB}}{s} \, d\mu(s)$$

obtained from an application of Fubini's theorem with respect to $P_A \otimes \mu$ and $P_B \otimes \mu$ on $\mathbb{R} \times [\epsilon, \infty)$ for $\epsilon > 0$. Then:

$$\operatorname{tr}(f(A) - f(B)) = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi \, d\mu$$

An expression for the spectral shift function ξ may be obtained from Fatou's theorem ([46], Theorem 11.24). Suppose that ν is a finite measure on $\mathbb{R}$ and:

$$\phi_\nu(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\nu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

is the Cauchy transform of ν. Then, ν is absolutely continuous if:

$$\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} (\phi_\nu(x + i0+) - \phi_\nu(x + i0-)) \, dx, \quad \xi \in \mathbb{R}.$$

The jump function $x \longmapsto \phi_\nu(x + i0+) - \phi_\nu(x + i0-)$ defined for almost all $x \in \mathbb{R}$ is then the density of ν with respect to the Lebesgue measure. For $\nu = \Xi$, if the representation:

$$\begin{aligned}
\Phi(s) &= i \frac{\operatorname{tr}(e^{-isA} - e^{-isB})}{s} \\
&= \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-isx} \left(\lim_{\epsilon \to 0+} \int_0^1 \operatorname{tr}(V(A + tV - x - i\epsilon)^{-1} - \right. \\
&\qquad \left. V(A + tV - x + i\epsilon)^{-1}) \, dt \right) dx
\end{aligned}$$

were valid, then we would expect that $\xi = \check{\Phi}$ has the representation:

$$\begin{aligned}
\xi(s) &= \frac{1}{2\pi i} \lim_{\epsilon \to 0+} \int_{\mathbb{R}} e^{isx - \epsilon|x|} \frac{\operatorname{tr}(e^{-ixA} - e^{-ixB})}{x} dx, \quad s \in \mathbb{R}, \\
&= \lim_{\epsilon \to 0+} \frac{1}{\pi} \operatorname{tr} \left[\arctan\left(\frac{A - sI}{\epsilon} \right) - \arctan\left(\frac{B - sI}{\epsilon} \right) \right]
\end{aligned} \tag{30}$$

where the arctan function may be expressed as:

$$\arctan t = \frac{1}{2i} \int_{\mathbb{R}} \frac{e^{ist} - 1}{s} e^{-|s|} \, ds, \quad t \in \mathbb{R}$$

In the case that $A - B = \alpha(\cdot, w)w$ for $\alpha > 0$ and $w \in \mathcal{H}$, $\|w\| = 1$, a calculation, given explicitly in [44], shows that the function:

$$
\begin{aligned}
h(x, y) &= \frac{1}{\pi}\mathrm{tr}\left[\arctan\left(\frac{A - xI}{y}\right) - \arctan\left(\frac{B - xI}{y}\right)\right] \\
&= \frac{1}{2\pi i}\log(1 + 2iy\alpha((B - z)^{-1}w, (B - z)^{-1}w)), \quad z = x + iy, \; y > 0
\end{aligned}
$$

is harmonic and uniformly bounded in the upper half-plane. By Fatou's theorem ([46], Theorem 11.23), the boundary values $\xi(x) = \lim_{y \to 0+} h(x, y)$ are defined for almost all $x \in \mathbb{R}$ and satisfy:

$$
\lim_{y \to \infty} \pi y h(x, y) = \int_{\mathbb{R}} \xi(t)\, dt = \|\xi\|_1 \le \|A - B\|_{\mathcal{C}_1(\mathcal{H})}
$$

for every $x \in \mathbb{R}$, so in the case that $A - B$ has rank one, Formula (30) is valid.

For an arbitrary self-adjoint perturbation:

$$
V = \sum_{j=1}^{\infty} \alpha_j(\cdot, w_j)w_j
$$

with $\sum_{j=1}^{\infty}|\alpha_j| = \|A - B\|_{\mathcal{C}_1(\mathcal{H})} < \infty$, the function $\xi_n \in L^1(\mathbb{R})$ may be defined in a similar fashion for $A_n = B + \sum_{j=1}^{n}\alpha_j(\cdot, w_j)w_j$, $n = 1, 2, \ldots$, so that $\xi_n \to \xi$ in $L^1(\mathbb{R})$ as $n \to \infty$ from which it verified that $\xi = \check{\Phi}$. □

The representation $\xi = \check{\Phi}$ obtained above may be viewed as the Fourier transform approach. In the case of a rank one perturbation $V = \alpha(\cdot, w)w$, the Cauchy transform approach is developed by Simon [47] with the formula:

$$
\mathrm{tr}((A - zI)^{-1} - (B - zI)^{-1}) = -\int_{\mathbb{R}} \frac{\xi(\lambda)}{(\lambda - z)^2}\, d\lambda
$$

for $z \in \mathbb{C} \setminus [a, \infty)$ for some $a \in \mathbb{R}$, established in [47], Theorem 1.9, by computing a contour integral. Here, the boundary value $\xi(x) = \lim_{y \to 0+} h(x, y)$ is expressed as:

$$
\xi(x) = \frac{1}{\pi}\mathrm{Arg}(1 + \alpha F(\lambda + i0+))
$$

for almost all $x \in \mathbb{R}$ with respect to the Cauchy transform:

$$
F(z) = \int_{\mathbb{R}} \frac{d(P_B w, w)(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus (-\infty, a)
$$

The Cauchy transform approach is generalised to Type II von Neumann algebras in [41].

Many different proofs of Krein's Formula (29) are available for a wide class of functions f, especially in a form that translates into the setting of non-commutative integration [41–43]. As remarked in [20], p. 163, an ingredient additional to double operator integrals (such as complex function theory) is needed to show that the measure Ξ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Krein's original argument uses perturbation determinants from which follows the representation $\mathrm{Det}(S(\lambda)) = e^{-2\pi i \xi(\lambda)}$ for the scattering matrix $S(\lambda)$ for A and B ([22], Chapter 8).

Conflicts of Interest

The authors declare no conflict of interest.

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