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## The Spectral Connection Matrix for Any Change of Basis within the Classical Real Orthogonal Polynomials

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**Abstract:** The connection problem for orthogonal polynomials is, given a polynomial expressed in the basis of one set of orthogonal polynomials, computing the coefficients with respect to a different set of orthogonal polynomials. Expansions in terms of orthogonal polynomials are very common in many applications. While the connection problem may be solved by directly computing the change-of-basis matrix, this approach is computationally expensive. A recent approach to solving the connection problem involves the use of the spectral connection matrix, which is a matrix whose eigenvector matrix is the desired change-of-basis matrix. In Bella and Reis (2014), it is shown that for the connection problem between any two different classical real orthogonal polynomials of the Hermite, Laguerre, and Gegenbauer families, the related spectral connection matrix has quasiseparable structure. This result is limited to the case where both the source and target families are one of the Hermite, Laguerre, or Gegenbauer families, which are each defined by at most a single parameter. In particular, this excludes the large and common class of Jacobi polynomials, defined by two parameters, both as a source and as a target family. In this paper, we continue the study of the spectral connection matrix for connections between real orthogonal polynomial families. In particular, for the connection problem between any two families of the Hermite, Laguerre, or Jacobi type (including Chebyshev, Legendre, and Gegenbauer), we prove that the spectral connection matrix has quasiseparable structure. In addition, our results also show the quasiseparable structure of the spectral connection matrix from the Bessel polynomials, which are orthogonal on the unit circle, to any of the Hermite, Laguerre, and Jacobi types. Additionally, the generators of the spectral connection matrix

are provided explicitly for each of these cases, allowing a fast algorithm to be implemented following that in Bella and Reis (2014).

**Keywords:** orthogonal polynomials; connection problem; change of basis; quasiseparable matrices; semiseparable matrices; structured matrices; spectral connection matrix

## 1. Introduction

Let  $\{P_k(x)\}_{k=0}^{\infty}$  be a sequence of real-valued polynomials, with  $\deg(P_k(x)) = k$ , and let  $w(x)$  be a non-negative real-valued weight function on some interval  $[a, b]$ . Then  $\{P_k(x)\}_{k=0}^{\infty}$  is said to be orthogonal with respect to the weight function  $w(x)$  on  $[a, b]$  if for each  $j \neq k$ ,  $\langle P_k, P_j \rangle = \int_a^b P_k(x)P_j(x)w(x) dx = 0$  and  $\langle P_j, P_j \rangle \neq 0$ . The expansions of polynomials in bases of such orthogonal polynomials are of interest in mathematics, as are many other uses of orthogonal polynomials. The classical orthogonal polynomials are useful in applications too numerous to include a full list, but notably include Gaussian quadrature [1], random matrix theory [2], fluid dynamics [3], and computations in quantum mechanics [4]; for a longer attempt at listing applications, see [5].

We consider the problem of, given constants  $\{a_k\}$  and orthogonal polynomials  $\{P_k\}$  (the source family) and  $\{Q_k\}$  (the target family), computing constants  $\{b_k\}$  such that  $\sum a_k P_k = \sum b_k Q_k$ . We will refer to this as the connection problem, and in this paper we will consider the case where the source polynomial family  $\{P_k\}$  and the target polynomial family  $\{Q_k\}$  are Hermite, Laguerre, or Jacobi (including any of its special cases of Chebychev, Legendre, and Gegenbauer). At the conclusion will we also discuss a connection to the Bessel polynomials, which are often considered classical although they are orthogonal on the unit circle.

The connection problem appears in areas such as harmonic analysis [6], mathematical physics [7], combinatorics [8], *etc.* It is obvious that the connection problem can be solved by determining the entries of a suitable connection matrix. While the recurrence relations satisfied by real orthogonal polynomials may seem to reduce the computational complexity of this approach, inversion at a cost of  $\mathcal{O}(n^3)$  operations is still required. As such, the connection problem has been studied extensively in the literature, see for instance [9–15].

One of the most complete solutions to the connection problem currently is that in [16] (and continued in [17]). The authors propose an algorithm for producing recurrence relations for desired connection coefficients and provide some examples. While their method uses only the recurrence relations for the orthogonal families and covers a very wide range of cases, the authors readily acknowledge the shortcomings of their approach by stating that they “do not succeed in resolving it through a compact form.” They continue, explaining that “it is required to guess a closed form for the solution of the recurrence from enough data produced by a symbolic programming language [16].”

Another more recent approach to the connection problem has involved the use of rank structured matrices [18–20]. Although their traditional connection to orthogonal polynomials has been as moment matrices or recurrence matrices, alternative links have also been made recently, which have contributed

to the connection problem. In 1991 Alpert and Rokhlin [21] addressed the connection problem between Legendre and Chebyshev polynomials. Later Keiner made significant progress in solving the connection problem within the Gegenbauer family (that is, where both source and target families are Gegenbauer, but for different values of the parameter) in [22] by exploiting the rank structure of the spectral connection matrix. In [23] it was shown that the spectral connection matrix corresponding to a change of basis within the Laguerre or within the Jacobi families has semiseparable structure, which is a type of the quasiseparable structure considered here. In [24], the connection problem between different families chosen among Hermite, Laguerre, and Gegenbauer was solved. There, it was shown that the spectral connection matrix has quasiseparable rank structure, and explicit algorithms were given.

In this paper, we continue the work in [24] by proving that the spectral connection matrix has quasiseparable rank structure when the target family is the large and very useful family of Jacobi polynomials, including Chebychev polynomials as special cases. We also prove the same result for any change of basis within the Hermite, Laguerre, and Jacobi types. At the conclusion, we also address a change of basis into any of these types from the set of Bessel polynomials. Explicit formulas for the generators of these quasiseparable matrices are given, allowing a fast algorithm to be implemented as in [24].

The structure of the paper is as follows. In Section 2, we collect some well-known results about the classes of orthogonal polynomials considered, for the reader’s convenience. Next, Section 3 introduces the class of quasiseparable matrices, the main computational tool of our algorithm. While the connection matrix is not itself quasiseparable, we show that it is a scaled eigenvector matrix of a known (and easily computable) quasiseparable matrix, known as the spectral connection matrix. This and implementation details are shown in Section 4, and explicit expressions for the generators are given in Section 5. A connection to the Bessel polynomials is made in Section 6, and some conclusions are offered in the final section.

**2. Orthogonal Polynomials and the Connection Problem**

Let  $P = \{P_k(x)\}_{k=0}^n$  and  $Q = \{Q_k(x)\}_{k=0}^n$  be two sequences of real orthogonal polynomials (orthogonal with respect to two inner products on  $\mathbb{P}_n$ ,  $\langle \cdot, \cdot \rangle_P$  and  $\langle \cdot, \cdot \rangle_Q$ , respectively) which we will refer to as the source family and the target family, respectively. Suppose that a polynomial  $p(x)$  is given in terms of the coefficients  $\{a_k\}_{k=0}^n$  such that

$$p(x) = \sum_{k=0}^n a_k P_k(x).$$

Then it is well-known that the coefficients  $\{b_k\}_{k=0}^n$  such that

$$p(x) = \sum_{k=0}^n b_k Q_k(x)$$

that we wish to compute are related via the change of basis matrix  $\Phi$  by  $[b_0 \cdots b_n]^T = \Phi[a_0 \cdots a_n]^T$ , and that

$$\Phi = (\phi_{i,j})_{i,j=0}^n \text{ such that } \phi_{i,j} = \frac{\langle P_j, Q_i \rangle_Q}{\langle Q_i, Q_i \rangle_Q}.$$

Computing this change of basis matrix directly from this relation, or in the standard way involving matrix inversion, is computationally expensive. In the case where the target family  $Q$  is Hermite, Laguerre, or Jacobi, we provide in this paper an alternative process. For reference, next we collect some useful information about these classical orthogonal polynomial families, noting that the source family  $P$  need not be among these families.

**Definition 1.** The (monic) Hermite polynomials  $\{H_k(x)\}_{k=0}^\infty$  form a sequence of polynomials orthogonal with respect to  $w(x) = e^{-x^2}$  on the interval  $(-\infty, \infty)$ . They are given by the following recurrence relation:

$$\begin{aligned}
 H_{-1}(x) &:= 0, & H_0(x) &:= 1, \text{ and} \\
 H_{k+1}(x) &:= xH_k(x) - \frac{k}{2}H_{k-1}(x) \text{ for } k = 0, 1, 2, \dots
 \end{aligned}
 \tag{1}$$

Each  $H_k$  is an eigenfunction of the differential operator

$$\mathcal{D}^H = -\frac{d^2}{dx^2} + 2x \frac{d}{dx}
 \tag{2}$$

corresponding to eigenvalue  $2k$ .

**Definition 2.** The (monic) Laguerre polynomials corresponding to fixed parameter  $\gamma > -1$ , denoted  $\{L_k^{(\gamma)}(x)\}_{k=0}^\infty$ , form a sequence of polynomials orthogonal with respect to  $w(x) = x^\gamma e^{-x}$  on the interval  $[0, \infty)$ . They are given by the following recurrence relation:

$$\begin{aligned}
 L_{-1}^{(\gamma)}(x) &:= 0, & L_0^{(\gamma)}(x) &:= 1, \text{ and} \\
 L_{k+1}^{(\gamma)}(x) &:= xL_k^{(\gamma)}(x) - (2k + \gamma + 1)L_k^{(\gamma)}(x) - k(k + \gamma)L_{k-1}^{(\gamma)}(x) \text{ for } k = 0, 1, 2, \dots
 \end{aligned}
 \tag{3}$$

Each  $L_k^{(\gamma)}$  is an eigenfunction of the differential operator

$$\mathcal{D}^\gamma = -x \frac{d^2}{dx^2} - (1 + \gamma - x) \frac{d}{dx}
 \tag{4}$$

corresponding to eigenvalue  $k$ .

**Definition 3.** The (monic) Jacobi polynomials corresponding to fixed parameters  $\alpha, \beta > -1$ , denoted  $\{J_k^{(\alpha, \beta)}(x)\}_{k=0}^\infty$ , form a sequence of polynomials orthogonal with respect to  $w(x) = (1 - x)^\alpha (1 + x)^\beta$  on the interval  $[-1, 1]$ . They are given by the following recurrence relation:

$$\begin{aligned}
 J_{-1}^{(\alpha, \beta)}(x) &:= 0 & J_0^{(\alpha, \beta)}(x) &:= 1 \\
 J_{k+1}^{(\alpha, \beta)}(x) &:= \left( x + \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2k)(\alpha + \beta + 2k + 2)} \right) J_k^{(\alpha, \beta)}(x) \\
 &\quad - \frac{4k(k + \alpha)(k + \beta)(k + \alpha + \beta)}{(\alpha + \beta + 2k)^2(\alpha + \beta + 2k + 1)(\alpha + \beta + 2k - 1)} J_{k-1}^{(\alpha, \beta)}(x) \text{ for } k = 0, 1, 2, \dots
 \end{aligned}
 \tag{5}$$

Each  $J_k^{(\alpha, \beta)}$  is an eigenfunction of the differential operator

$$\mathcal{D}^{\alpha, \beta} = (x^2 - 1) \frac{d^2}{dx^2} + ((\alpha + \beta + 2)x + \alpha - \beta) \frac{d}{dx}
 \tag{6}$$

corresponding to eigenvalue  $k(k + \alpha + \beta + 1)$ . Note that letting  $\alpha = \beta = 0$  generates the family of monic Legendre polynomials, and  $\alpha = \beta = -1/2$  generates the monic Chebyshev polynomials of the first kind, after some accommodations for the normalization. The Gegenbauer class is a subset of Jacobi where  $\alpha = \beta$ .

It is convenient in what follows to be able to represent the first and second derivatives of these families in terms of the families themselves, so we collect these results next. The results, some of which are given in [24], follow in a similar way as those given in [23].

**Lemma 4.** *The (monic) Hermite polynomials  $\{H_k(x)\}_k$  satisfy*

$$\frac{d}{dx}(H_k(x)) = kH_{k-1}(x)$$

$$\text{and } \frac{d^2}{dx^2}(H_k(x)) = k(k-1)H_{k-2}(x).$$

*The (monic) Laguerre polynomials  $\{L_k^{(\gamma)}(x)\}_k$  satisfy*

$$\frac{d}{dx}(L_k^{(\gamma)}(x)) = \sum_{i=0}^{k-1} (-1)^{k-i+1} \frac{k!}{i!} L_i(x)$$

$$\text{and } \frac{d^2}{dx^2}(L_k^{(\gamma)}(x)) = \sum_{i=0}^{k-2} (-1)^{k-i} (k-i-1) \frac{k!}{i!} L_i(x).$$

*The (monic) Jacobi polynomials  $\{J_k^{(\alpha,\beta)}(x)\}_k$  satisfy*

$$\frac{d}{dx}(J_n^{(\alpha,\beta)}(x)) = A_n \sum_{i=0}^{n-1} B_i J_i^{(\alpha,\beta)}(x) + C_n \sum_{i=0}^{n-1} D_i J_i^{(\alpha,\beta)}(x)$$

$$\text{and } \frac{d^2}{dx^2}(J_k^{(\alpha,\beta)}(x)) = \sum_{i=0}^{k-2} \left( \sum_{l=1}^{k-i-1} (A_k B_{k-l} + C_k D_{k-l})(A_{k-l} B_i + C_{k-l} D_i) \right) J_i^{(\alpha,\beta)}(x)$$

where

$$A_k = \frac{(-1)^{k+1} \Gamma(k + \alpha + 1) 2^{k-1} \Gamma(k + 1)}{\Gamma(2k + \alpha + \beta + 1)} \quad B_i = \frac{(-1)^i \Gamma(2i + \alpha + \beta + 2)}{\Gamma(i + \alpha + 1) 2^i \Gamma(i + 1)}$$

$$C_k = \frac{\Gamma(k + \beta + 1) 2^{k-1} \Gamma(k + 1)}{\Gamma(2k + \alpha + \beta + 1)} \quad D_i = \frac{\Gamma(2i + \alpha + \beta + 2)}{\Gamma(i + \beta + 1) 2^i \Gamma(i + 1)}.$$

### 3. Quasiseparable Matrices

The main result of the paper is that the spectral connection matrix for a wide variety of connection problems exhibits a rank structure known as quasiseparability. The class of quasiseparable matrices has received a lot of attention in recent years, and many interesting relationships between quasiseparability and orthogonal polynomials are well-studied, such as recurrent matrices, where the matrix captures the recurrence relations, and the low-rank structure corresponds to sparse recurrence relations (see, for instance, [25]).

In terms of implementation, the low-rank structure means that the  $n^2$  entries of the matrix can be represented by a smaller  $\mathcal{O}(n)$  number of parameters, called the generators of the quasiseparable matrix. This sparse representation powers many of the fast and accurate algorithms available for quasiseparable matrices. For the purposes of this paper, we will only need upper triangular matrices which have rank structure in the upper triangular portion. We will also begin indexing at 0 instead of 1 as in the standard work in the area, to match indices on polynomials.



We conclude this section with a result from [24] that allows, given an upper triangular and quasiseparable matrix in terms of its generators, a fast algorithm to compute a scaled eigenvector matrix. As we will see, together with the later results in the paper, this will enable a fast algorithm for the connection problem described above.

**Theorem 7.** Let  $G$  be an  $(n + 1) \times (n + 1)$  upper triangular quasiseparable matrix with generators  $\{g_i\}_{i=0}^{n-1}$ ,  $\{h_j\}_{j=1}^n$ ,  $\{b_m\}_{m=1}^{n-1}$ , and  $\{d_l\}_{l=0}^n$ . Let

$$\tilde{g}_i^{(k)} = \frac{g_i}{d_k - d_i} \quad \text{and} \quad \tilde{b}_m^{(k)} = b_m + \frac{h_m g_m}{d_k - d_m}$$

for  $i = 0, \dots, n - 1$ ,  $m = 1, \dots, n - 1$  and  $k = 1, \dots, n + 1$ . Let  $x_0 = e_1$  and

$$\mathbf{x}_k = \begin{bmatrix} \tilde{g}_0^{(k)} \tilde{b}_1^{(k)} \dots \tilde{b}_{k-1}^{(k)} h_k \\ \tilde{g}_1^{(k)} \tilde{b}_2^{(k)} \dots \tilde{b}_{k-1}^{(k)} h_k \\ \vdots \\ \tilde{g}_{k-2}^{(k)} \tilde{b}_{k-1}^{(k)} h_k \\ \tilde{g}_{k-1}^{(k)} h_k \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for  $k = 1, \dots, n$ . Note that  $\mathbf{x}_k$  is column  $k$  of the upper triangular quasiseparable matrix defined by the parameters  $\{\tilde{d}_l, \tilde{g}_i^{(k)}, \tilde{b}_m^{(k)}, h_j\}$ , where each  $\tilde{d}_l = 1$ . Then

$$X = \left[ \mathbf{x}_0 \mid \dots \mid \mathbf{x}_n \right]$$

is an eigenvector matrix of  $G$ .

The following  $\mathcal{O}(n^2)$  algorithm can then be used to compute the desired scaled eigenvector matrix:

Set  $H = h_k$ ,  $x_k(k, 1) = 1$ ,  $x_k(j, 1) = 0$  for  $j = k + 1, \dots, n$

For  $j = k - 1 : -1 : 0$

    Compute  $x_k(j, 1) = \tilde{g}_j^{(k)} H$

    Update  $H = \tilde{b}_j^{(k)} H$

End.

#### 4. The Spectral Connection Matrix

In contrast to the obvious algorithm of computing the change of basis matrix directly, this section shows how the spectral connection matrix, for which we will have an explicit description in terms of the quasiseparable generators, is related to this change of basis matrix in a manner that allows it to be computed using the results of the previous section.

**Definition 8.** Let  $n \in \mathbb{N}$ , and suppose that  $P = \{P_k(x)\}_{k=0}^n$  and  $Q = \{Q_k(x)\}_{k=0}^n$  are two finite families of real orthogonal polynomials with respect to inner products  $\langle \cdot, \cdot \rangle_P$  and  $\langle \cdot, \cdot \rangle_Q$  respectively. Let  $\mathcal{D}^P$  be the differential operator associated with  $P$  (that is, each  $P_k$  is an eigenfunction of  $\mathcal{D}^P$ ). Let

$$g_{ij} = \frac{\langle Q_i, \mathcal{D}^P(Q_j) \rangle_Q}{\langle Q_i, Q_i \rangle_Q}$$

Then the matrix  $G = (g_{ij})_{i,j=0}^n$  is called the spectral connection matrix from  $P$  to  $Q$ .

The next theorem reveals the relationship between the spectral connection matrix and the connection matrix itself. The proof comes from [23].

**Theorem 9.** Let  $P = \{P_k(x)\}_{k=0}^n$  and  $Q = \{Q_k(x)\}_{k=0}^n$  be two finite families of classical orthogonal polynomials. Let  $\Phi$  be the connection matrix from  $P$  to  $Q$ , and let  $G$  be the spectral connection matrix from  $P$  to  $Q$ . Then  $\Phi$  is the eigenvector matrix of  $G$  with each diagonal entry scaled to 1.

The next theorem gives an explicit expression of the entries of the spectral connection matrix for a connection problem given in terms of the parameters of the orthogonal polynomials involved. This generalizes a corresponding theorem from [24] which was proved only under the restriction that the source family  $P$  was among the Hermite, Laguerre, or Gegenbauer families. The following theorem is valid for any change of basis between orthogonal polynomials where the hypotheses are satisfied.

**Theorem 10.** Let  $P = \{P_k(x)\}_{k=0}^n$  and  $Q = \{Q_k(x)\}_{k=0}^n$  be two families of (monic) real orthogonal polynomials where  $Q$  is a classical family, and let  $G = (g_{i,j})_{i,j=0}^n$  be the spectral connection matrix from  $P$  to  $Q$ . Suppose that each  $Q_k(x)$  is an eigenfunction of the differential operator  $\mathcal{D}^Q$ , as provided in Definitions 1, 2, and 3, corresponding to eigenvalue  $\lambda_k$ . Similarly let each  $P_k(x)$  be an eigenfunction of the differential operator

$$\mathcal{D}^P = (\tilde{a}x^2 + \tilde{b}x + \tilde{c}) \frac{d^2}{dx^2} + (\tilde{d}x + \tilde{e}) \frac{d}{dx}$$

Suppose that there exist constants  $a, b, c, d, e, \{\{n_k^j\}_{k=0}^{n-1}\}_{j=0}^n, \{\{m_k^j\}_{k=0}^{n-2}\}_{j=0}^n, \{\{\gamma_k\}_{k=0}^n, \{\delta_k\}_{k=0}^n$  and  $\{\epsilon_k\}_{k=0}^n$  satisfying

$$(\mathcal{D}^P - \mathcal{D}^Q)(y) = (ax^2 + bx + c) \frac{d^2y}{dx^2} + (dx + e) \frac{dy}{dx}$$

$$\frac{d}{dx}(Q_j(x)) = \sum_{k=0}^{j-1} n_k^j Q_k(x)$$

$$\frac{d^2}{dx^2}(Q_j(x)) = \sum_{k=0}^{j-2} m_k^j Q_k(x)$$

$$xQ_k(x) = \gamma_k Q_{k+1}(x) + \delta_k Q_k(x) + \epsilon_k Q_{k-1}(x).$$

Note that for any  $k < 0$ , we adopt the convention that  $n_k^j = m_k^j = \gamma_k = 0$  for any  $j$ . Then the entries of  $G$  are given by

$$\begin{aligned}
 g_{i,j} &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} a\gamma_{i-2}\gamma_{i-1}m_{i-2}^j + d\gamma_{i-1}n_{i-1}^j, & i \leq j \\ 0, & \text{otherwise} \end{cases} \\
 &+ \begin{cases} en_i^j + dn_i^j\delta_i + b\gamma_{i-1}m_{i-1}^j + a\gamma_{i-1}m_{i-1}^j(\delta_{i-1} + \delta_i), & i < j \\ 0, & \text{otherwise} \end{cases} \\
 &+ \begin{cases} cm_i^j + bm_i^j\delta_i + am_i^j(\gamma_i\epsilon_{i+1} + \delta_i^2 + \gamma_{i-1}\epsilon_i) + dn_{i+1}^j\epsilon_{i+1}, & i < j - 1 \\ 0, & \text{otherwise} \end{cases} \\
 &+ \begin{cases} bm_{i+1}^j\epsilon_{i+1} + am_{i+1}^j(\delta_{i+1}\epsilon_{i+1} + \epsilon_{i+1}\delta_i), & i < j - 2 \\ 0, & \text{otherwise} \end{cases} \\
 &+ \begin{cases} am_{i+2}^j\epsilon_{i+2}\epsilon_{i+1}, & i < j - 3 \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

**Proof.** First, note that through direct substitution,  $xQ_k(x) = \gamma_k Q_{k+1}(x) + \delta_k Q_k(x) + \epsilon_k Q_{k-1}(x)$  gives that  $x^2 Q_k(x) = \gamma_k \gamma_{k+1} Q_{k+2} + (\gamma_k \delta_k + \gamma_k \delta_{k+1}) Q_{k+1} + (\gamma_k \epsilon_{k+1} + \delta_k^2 + \gamma_{k-1} \epsilon_k) Q_k + (\delta_k \epsilon_k + \epsilon_k \delta_{k-1}) Q_{k-1} + \epsilon_k \epsilon_{k-1} Q_{k-2}$ . By definition,  $g_{ij} = \frac{\langle Q_i, \mathcal{D}^P(Q_j) \rangle_Q}{\langle Q_i, Q_i \rangle_Q}$ , and applying identities we have

$$\begin{aligned}
 g_{ij} &= \frac{1}{\langle Q_i, Q_i \rangle_Q} (\langle Q_i, \mathcal{D}^Q(Q_j) \rangle_Q + \langle Q_i, (\mathcal{D}^P - \mathcal{D}^Q)(Q_j) \rangle_Q) \\
 &= \frac{1}{\langle Q_i, Q_i \rangle_Q} (\langle Q_i, \lambda_j Q_j \rangle_Q + \langle Q_i, (\mathcal{D}^P - \mathcal{D}^Q)(Q_j) \rangle_Q) \\
 &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \frac{1}{\langle Q_i, Q_i \rangle_Q} (\langle Q_i, (\mathcal{D}^P - \mathcal{D}^Q)(Q_j) \rangle_Q) \\
 &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \frac{1}{\langle Q_i, Q_i \rangle_Q} (\langle Q_i, (ax^2 + bx + c)Q_j'' \rangle_Q + \langle Q_i, (dx + e)Q_j' \rangle_Q) \\
 &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \frac{1}{\langle Q_i, Q_i \rangle_Q} \left( \langle Q_i, a \sum_{k=0}^{j-2} m_k^j x^2 Q_k \rangle_Q + \langle Q_i, b \sum_{k=0}^{j-2} m_k^j x Q_k \rangle_Q \right. \\
 &\quad \left. + \langle Q_i, c \sum_{k=0}^{j-2} m_k^j Q_k \rangle_Q + \langle Q_i, d \sum_{k=0}^{j-1} n_k^j x Q_k \rangle_Q + \langle Q_i, e \sum_{k=0}^{j-1} n_k^j Q_k \rangle_Q \right)
 \end{aligned}$$

Simplifying, we have

$$\begin{aligned}
 &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} cm_i^j, & i < j - 1, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} en_i^j, & i < j, \\ 0, & \text{otherwise} \end{cases} \\
 &+ \frac{1}{\langle Q_i, Q_i \rangle_Q} \left( \langle Q_i, a \sum_{k=0}^{j-2} m_k^j (\gamma_k \gamma_{k+1} Q_{k+2} + (\gamma_k \delta_k + \gamma_k \delta_{k+1}) Q_{k+1} \right. \\
 &+ (\gamma_k \epsilon_{k+1} + \delta_k^2 + \gamma_{k-1} \epsilon_k) Q_k + (\delta_k \epsilon_k + \epsilon_k \delta_{k-1}) Q_{k-1} + \epsilon_k \epsilon_{k-1} Q_{k-2} \rangle_Q \\
 &+ \langle Q_i, b \sum_{k=0}^{j-2} m_k^j (\gamma_k Q_{k+1} + \delta_k Q_k + \epsilon_k Q_{k-1}) \rangle_Q \\
 &+ \left. \langle Q_i, d \sum_{k=0}^{j-1} n_k^j (\gamma_k Q_{k+1} + \delta_k Q_k + \epsilon_k Q_{k-1}) \rangle_Q \right) \\
 &= \begin{cases} \lambda_i, & i = j, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} cm_i^j, & i < j - 1, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} en_i^j, & i < j, \\ 0, & \text{otherwise} \end{cases} \\
 &+ \frac{1}{\langle Q_i, Q_i \rangle_Q} \left( a \sum_{k=0}^{j-2} m_k^j \langle Q_i, \gamma_k \gamma_{k+1} Q_{k+2} + (\gamma_k \delta_k + \gamma_k \delta_{k+1}) Q_{k+1} \right. \\
 &+ (\gamma_k \epsilon_{k+1} + \delta_k^2 + \gamma_{k-1} \epsilon_k) Q_k + (\delta_k \epsilon_k + \epsilon_k \delta_{k-1}) Q_{k-1} + \epsilon_k \epsilon_{k-1} Q_{k-2} \rangle_Q \\
 &+ b \sum_{k=0}^{j-2} m_k^j \langle Q_i, \gamma_k Q_{k+1} + \delta_k Q_k + \epsilon_k Q_{k-1} \rangle_Q + \left. d \sum_{k=0}^{j-1} n_k^j \langle Q_i, \gamma_k Q_{k+1} + \delta_k Q_k + \epsilon_k Q_{k-1} \rangle_Q \right)
 \end{aligned}$$

Then, taking advantage of the linearity of the inner product and orthogonality, we arrive at the desired result. □

### 5. Structure of Spectral Connection Matrices

In [23], it is proven that the spectral connection matrices within the Jacobi family and within the Laguerre family (that is, when the source and target families are of the same type, defined by different parameters) have diagonal-plus-upper-semiseparable structure, which is a subclass of quasiseparable. In [24], it is proven that between different classical families defined by at most a single parameter (that is, when both source and target families are from the Hermite, Laguerre, and Gegenbauer types), the spectral connection matrix has quasiseparable structure (but not always diagonal-plus-upper-semiseparable structure).

In this section, we further extend this work to include source and target families of Jacobi polynomials. That is, we show that for the connection problem from any real orthogonal polynomial family satisfying an appropriate differential operator into any family of the Hermite, Laguerre, or Jacobi types, the spectral connection matrix has quasiseparable structure.

Furthermore, the generators of the spectral connection matrix are given explicitly, allowing the results of Section 3 to be used in the fast eigenvector algorithm provided. To produce a desired connection matrix within the classical orthogonal polynomials, one would simply use the appropriate generators

provided below in Theorem 7. The output of this algorithm, the eigenvector matrix of the prescribed spectral connection matrix, is the desired connection matrix.

**Theorem 11.** Let  $P = \{P_k(x)\}_{k=0}^\infty$  be a family of real-valued polynomials orthogonal on some  $[a, b]$  with respect to a weight function  $w(x)$  such that each  $P_k(x)$  is an eigenfunction of

$$\mathcal{D} = (\tilde{a}x^2 + \tilde{b}x + \tilde{c})\frac{d^2}{dx^2} + (\tilde{d}x + \tilde{e})\frac{d}{dx}$$

Then the spectral connection matrix from  $P$  to the monic Hermite family  $H = \{H_k(x)\}_{k=0}^n$  is  $(0, 4)$ -quasiseparable, with generators

$$d_k = 2k + \tilde{a}k(k - 1) + (\tilde{d} - 2)k$$

$$g_i = \left[ (1/4)\tilde{a} \quad (1/2)\tilde{b} \quad \tilde{c} + (1/2)\tilde{a}(2i + 1) + (1/2)\tilde{d} \quad \tilde{e} + \tilde{b}i \right]$$

$$h_j = \left[ 0 \quad 0 \quad 0 \quad j \right]^T \quad b_k = \begin{bmatrix} 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Theorem 12.** Let  $P = \{P_k(x)\}_{k=0}^\infty$  be a family of real-valued polynomials orthogonal on some  $[a, b]$  with respect to a weight function  $w(x)$  such that each  $P_k(x)$  is an eigenfunction of

$$\mathcal{D} = (\tilde{a}x^2 + \tilde{b}x + \tilde{c})\frac{d^2}{dx^2} + (\tilde{d}x + \tilde{e})\frac{d}{dx}.$$

Then the spectral connection matrix from  $P$  to the Laguerre family  $H = \{L_k^\gamma(x)\}_{k=0}^n$  is  $(0, 4)$ -quasiseparable, with generators

$$d_k = k(\tilde{a}k - \tilde{a} + \tilde{d})$$

$$g_i = \left[ (-1)^i \tilde{c}/i! \quad (-1)^i(-i\tilde{c} + \tilde{b} - \tilde{c} - \tilde{e} + \tilde{b}\gamma)/i! \quad 1/2 \quad 1/2 \right]$$

$$h_j = \left[ (-1)^j j! j \quad (-1)^j j! \quad a(j) \quad a(j) \right]^T$$

$$b_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & B(k) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B(k) \end{bmatrix}$$

for  $k$  even or  $k$  odd, respectively, where

$$a(j) = j(\tilde{b}j + \tilde{d}j + 2\tilde{a}j^2 - 2\tilde{a}j + \tilde{d}\gamma + 2\tilde{a}j\gamma - 2\tilde{a}\gamma + \tilde{b}\gamma) \text{ and}$$

$$B(k) = (2k\tilde{a}/a)(k + 1)(k + \gamma)(k + \gamma + 1).$$

**Theorem 13.** Let  $P = \{P_k(x)\}_{k=0}^\infty$  be a family of real-valued polynomials orthogonal on some  $[a, b]$  with respect to a weight function  $w(x)$  such that each  $P_k(x)$  is an eigenfunction of

$$\mathcal{D} = (\tilde{a}x^2 + \tilde{b}x + \tilde{c})\frac{d^2}{dx^2} + (\tilde{d}x + \tilde{e})\frac{d}{dx}.$$

The spectral connection matrix from  $P$  to the Jacobi family  $J = \{J_k^{(\alpha, \beta)}(x)\}_{k=0}^n$  is  $(0, 8)$ -quasiseparable, with generators

$$d_k = k(k + \alpha + \beta + 1) + (\tilde{a} - 1)f(k, k - 1)f(k - 1, k - 2) + (\tilde{d} - \alpha - \beta - 2)f(k, k - 1)$$

$$g_i = \begin{bmatrix} \eta_1(i) & H(i) & \eta_2(i) & 0 & \eta_1(i) & J(i) & \eta_2(i) & 0 \end{bmatrix}$$

$$h_j = \begin{bmatrix} 0 & A_j & 0 & A_j & 0 & C_j & 0 & C_j \end{bmatrix}^T$$

$$b_k = \begin{bmatrix} 1 & A_k B_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & B_k C_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & A_k D_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & C_k D_k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where  $A_k, B_k, C_k,$  and  $D_k$  are as defined in the Jacobi derivative identity,  $\delta_i$  and  $\epsilon_i$  are as defined by the Jacobi recurrence relation, and

$$f(l, m) = A_l B_m + C_l D_m$$

$$\begin{aligned} \eta_1(i) &= B_{i-2}(\tilde{a} - 1) + B_{i-1} \left( (\tilde{a} - 1)(\delta_{i-1} + \delta_i) + \tilde{b} \right) + B_i \left( (\tilde{a} - 1)(\epsilon_i + \epsilon_{i+1} + \delta_i^2) + \tilde{c} + 1 + \tilde{b}\delta_i \right) \\ &\quad + B_{i+1}\epsilon_{i+1} \left( (\tilde{a} - 1)(\delta_{i+1} + \delta_i) + \tilde{b} \right) + B_{i+2}\epsilon_{i+2}\epsilon_{i+1}(\tilde{a} - 1) \end{aligned}$$

$$\begin{aligned} \eta_2(i) &= D_{i-2}(\tilde{a} - 1) + D_{i-1} \left( (\tilde{a} - 1)(\delta_{i-1} + \delta_i) + \tilde{b} \right) + D_i \left( (\tilde{a} - 1)(\epsilon_i + \epsilon_{i+1} + \delta_i^2) + \tilde{c} + 1 + \tilde{b}\delta_i \right) \\ &\quad + D_{i+1}\epsilon_{i+1} \left( (\tilde{a} - 1)(\delta_{i+1} + \delta_i) + \tilde{b} \right) + D_{i+2}\epsilon_{i+2}\epsilon_{i+1}(\tilde{a} - 1) \end{aligned}$$

$$\begin{aligned} H(i) &= B_{i-1} \left( \tilde{d} - \alpha - \beta - 2 \right) + B_i \left( \tilde{e} - \alpha + \beta + \delta_i(\tilde{d} - \alpha - \beta - 2) \right) \\ &\quad + B_{i+1}\epsilon_{i+1} \left( \tilde{d} - \alpha - \beta - 2 \right) + (\tilde{a} - 1) [B_{i-2}(A_i B_i + A_{i-1} B_{i-1}) + D_{i-2}(B_i C_i + B_{i-1} C_{i-1})] \\ &\quad + B_i \left( (\tilde{a} - 1)(\delta_i + \delta_{i-1}) + \tilde{b} \right) f(i, i - 1) - B_{i+1}(\tilde{a} - 1)\epsilon_{i+2}\epsilon_{i+1}f(i + 1, i + 2) \end{aligned}$$

$$\begin{aligned} J(i) &= D_{i-1}(\tilde{d} - \alpha - \beta - 2) + D_i \left( \tilde{e} - \alpha + \beta + \delta_i(\tilde{d} - \alpha - \beta - 2) \right) + D_{i+1}\epsilon_{i+1}(\tilde{d} - \alpha - \beta - 2) \\ &\quad + (\tilde{a} - 1) [B_{i-2}(A_i D_i + A_{i-1} D_{i-1}) + D_{i-2}(D_i C_i + D_{i-1} C_{i-1})] \\ &\quad + D_i \left( (\tilde{a} - 1)(\delta_i + \delta_{i-1}) + \tilde{b} \right) f(i, i - 1) - D_{i+1}(\tilde{a} - 1)\epsilon_{i+2}\epsilon_{i+1}f(i + 1, i + 2). \end{aligned}$$

The preceding three theorems can be proven by direct verification. From Equations (2), (4), and (6), it is clear that the Hermite, Laguerre, and Jacobi families all satisfy the differential operator condition in the hypotheses of the preceding three theorems, so overall we come to the following statement.

**Theorem 14.** *Let  $G$  be the spectral connection matrix corresponding to a change of basis among the Hermite, Laguerre, or Jacobi types. Then  $G$  is quasiseparable, with generators as provided above.*

### 6. The Bessel Polynomials

In this section we will briefly explore the connection of the preceding work to the Bessel polynomials, which are often considered a classical type. They have many properties similar to those of the Hermite, Laguerre, and Jacobi types, but they are orthogonal on the unit circle and not on any real interval.

**Definition 15.** *The (monic) Bessel polynomials  $\{B_k(z)\}_{k=0}^\infty$  form a sequence of polynomials orthogonal with respect to  $w(z) = e^{-2/z}$  on the unit circle. They are given by the following recurrence relation:*

$$B_{-1}(z) := 0, \quad B_0(z) := 1, \quad \text{and} \\ B_{k+1}(z) := zB_k(z) + \frac{1}{(2k+1)(2k-1)}B_{k-1}(z) \text{ for } k = 0, 1, 2, \dots \tag{7}$$

Each  $B_k$  is an eigenfunction of the differential operator

$$\mathcal{D} = z^2 \frac{d^2}{dz^2} + (2z + 2) \frac{d}{dz} \tag{8}$$

corresponding to eigenvalue  $k(k + 1)$ .

The generalized Bessel polynomials are a larger parameterized type that contains Bessel as a special case. Each generalized Bessel polynomial of degree  $k$  corresponding to parameter  $a$  is an eigenfunction of the differential operator

$$\mathcal{D}^{a,b} = z^2 \frac{d^2}{dz^2} + (az + 2) \frac{d}{dz} \tag{9}$$

corresponding to eigenvalue  $k(k + a - 1)$ . The traditional Bessel polynomials correspond to the special case where  $a = 2$ .

It is clear that the generalized Bessel polynomials, and in particular the traditional Bessel polynomials, satisfy a differential operator whose form satisfies the conditions on the source family in Theorems 11, 12, and 13. We therefore have also shown that the spectral connection matrix corresponding to a change of basis from the Bessel polynomials to any of the other classical types has quasiseparable structure, with generators provided in the previous section. This allows us to enjoy the advantages of the efficient spectral connection matrix approach when changing coordinates from a family of polynomials orthogonal on the unit circle to a family of polynomials orthogonal on the real line.

In addition, the following theorem from [28] (and further developed in [29]) reveals the total scope of the source families in Theorems 11, 12, and 13.

**Theorem 16.** *Any orthogonal sequence  $\{P_k\}_{k=0}^\infty$  such that for each  $k$*

$$(\tilde{a}x^2 + \tilde{b}x + \tilde{c}) \frac{d^2}{dx^2} P_k + (\tilde{d}x + \tilde{e}) \frac{d}{dx} P_k = \lambda_k P_k$$

with  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e} \in \mathbb{C}$  is of the Hermite, Laguerre, Jacobi, or Bessel type. This is known as Bochner’s property.

So we may conclude that the Bessel, Hermite, Laguerre, and Jacobi families are the only orthogonal polynomial types that satisfy the differential operator required for Theorems 11, 12, and 13.

## 7. Conclusions

This paper contains an improvement on the recent work in using the rank structured spectral connection matrix to solve the connection problem. Previous work required that the source and target orthogonal polynomial families each be one of the single-parameter classical real orthogonal polynomial families. It also excluded the very important case of Jacobi polynomials (save for the Gegenbauer subclass), both as a source and a target family. The work presented here has greatly relaxed these restrictions. The target family is allowed to be any classical one, which alone broadens the result dramatically to include changes of basis into any of the Jacobi families, including Chebyshev and Legendre. The restrictions on the source family have been lessened to include all of the classical types, including Jacobi, as well as the Bessel polynomials. Although the Bessel polynomials have much in common with the classical types and are sometimes considered classical themselves, they differ in that they are orthogonal on the unit circle. Thus the progress here has extended the research to directly address a change of basis from a set of polynomials orthogonal on the unit circle to a set orthogonal on a real interval. We have shown that all of the spectral connection matrices in this much larger set of connections have quasiseparable structure. Additionally we have provided the specific generators for each of the three types, allowing for a fast algorithm for the connection problem in this more general case.

## Author Contributions

Jenna Reis and Tom Bella contributed to the results and wrote the paper.

## Conflicts of Interest

The authors declare no conflict of interest.

## References

1. Gautschi, W. Orthogonal polynomials: Applications and computation. *Acta Numer.* **1996**, *5*, 45–119.
2. Fyodorov, Y.V. Introduction to the random matrix theory: Gaussian unitary ensemble and beyond. *Lond. Math. Soc. Lect. Note Ser.: Recent Persp. Random Matrix Theory Number Theory* **2005**, *322*, 31–78.
3. Paterson, A.R.A *First Course in Fluid Dynamics*; Cambridge University Press: Cambridge, UK, 1983.
4. Seaborn, J.B. *Hypergeometric Functions and Their Applications*; Springer: New York, NY, USA, 1991.

5. Koornwinder, T.H.; Wong, R.; Koekoek, R.; Swarttouw, R.F. Orthogonal Polynomials. In *NIST Handbook of Mathematical Functions*; Oliver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W., Eds.; Cambridge University Press: New York, NY, USA, 2010; pp. 478-479.
6. Szwarz, R. Orthogonal polynomials and special functions. In *Proceedings of the Regional Conference Series in Applied Mathematics, 21 Lecture 7 (SIAM)*, Philadelphia, PA, USA, 1975.
7. Askey, R. Orthogonal polynomials and special functions. In *Proceedings of the Regional Conference in Applied Mathematics, (SIAM)*, Philadelphia, PA, USA, 1975; Volume 21.
8. Szwarz, R. Connection coefficients of orthogonal polynomials. *Canad. Math. Bull.* **1992**, *35*, 548–556.
9. Maroni, P.; da Rocha, Z. Connection coefficients between orthogonal polynomials and the canonical sequence: An approach based on symbolic computation. *Numer. Algor.* **2008**, *47*, 291–314.
10. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions Vol. 2*; McGraw-Hill: New York, NY, USA, 1953.
11. Lewanowicz, S. Quick construction of recurrence relations for the Jacobi coefficients. *J. Comput. Appl. Math.* **1992**, *43*, 355–372.
12. Srivastava, H.S.; Manocha, H.L. *A Treatise on Generating Functions*; Ellis Horwood: Chichester, UK, 1984.
13. Stirling, J. *Methodus Differentialis: Sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*; London, 1730.
14. Godoy, E.; Ronveaux, A.; Zarzo, A.; Area, I. Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Continuous case. *J. Comput. Appl. Math.* **1997**, *84*, 257–275.
15. Ronveaux, A.; Zarzo, A.; Godoy, E. Recurrence relation for connection coefficients between two families of orthogonal polynomials. *J. Comput. Appl. Math.* **1995**, *62*, 67–73.
16. Maroni, P.; da Rocha, Z. Connection coefficients between orthogonal polynomials and the canonical sequence: an approach based on symbolic computation. *Numer. Algor.* **2008**, *47*, 291–314.
17. Maroni, P.; da Rocha, Z. Connection coefficients for orthogonal polynomials: Symbolic computations, verifications and demonstrations in the Mathematica language. *Numer. Algor.* **2013**, *63*, 507–520.
18. Higham, N.J. Fast solution of Vandermonde-like systems involving orthogonal polynomials. *IMA J. Numer. Anal.* **1988**, *8*, 473–486.
19. Olshevsky, V.; Pan, V. Polynomial and rational evaluation and interpolation (with structured matrices). In *Proceedings of the Automata, Languages and Programming-26th International Colloquium ICALP '99*, Prague, Czech Republic, 11–15 July 1999; *Lecture Notes in Comput. Sci. Vol. 1644*; Wiedermann, J., van Emde Boas, P., Nielsen, M., Eds.; Springer: Berlin, Germany, 1999; pp. 585–594.
20. Pan, V. Fast evaluation and interpolation at the Chebyshev set of points. *Appl. Math. Lett.* **1989**, *2*, 255–258.

21. Alpert, B.K.; Rokhlin, V. A fast algorithm for the evaluation of Legendre expansions. *SIAM J. Sci. Stat. Comput.* **1991**, *12*, 158–179.
22. Keiner, J. Gegenbauer polynomials and semiseparable matrices. *Elect. Trans. Numer. Anal.* **2008**, *30*, 26–53.
23. Keiner, J. *Fast Polynomial Transforms*; Logos Verlag: Berlin, Germany, 2011.
24. Bella, T.; Reis, J. The spectral connection matrix for classical orthogonal polynomials of a single parameter. *Linear Algebra Appl.* **2014**, *458*, 161–182.
25. Bella, T.; Eidelman, Y.; Gohberg, I.; Olshevsky, V. Computations with quasiseparable polynomials and matrices. *Theor. Comput. Sci.* **2008**, *409*, 158–179.
26. Bella, T.; Eidelman, Y.; Gohberg, I.; Olshevsky, V. Classifications of three-term and two-term recurrence relations via subclasses of quasiseparable matrices. *SIAM J. Matrix Anal.* **2008**.
27. Eidelman, Y.; Gohberg, I. On a new class of structured matrices. *Integr. Equat. Oper. Theory* **1999**, *34*, 293–324.
28. Bochner, S. Über Sturm-Liouvillesche Polynomsysteme. *Mathematische Zeitschrift* **1929**, *1*, 730–736.
29. Loureiro, A.F.; Maroni, P.; da Rocha, Z. The generalized Bochner condition about classical orthogonal polynomials revisited. *J. Math. Anal. Appl.* **2006**, *322*, 645–667.

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