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Article

Some New Integral Identities for Solenoidal Fields and Applications

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Abstract: In spaces $R^n,\ n \geq 2$, it has been proved that a solenoidal vector field and its rotor satisfy the series of new integral identities which have covariant form. The interest in them is explained by hydrodynamics problems for an ideal fluid.

Keywords: rotor; solenoidal vector field; potential vector field; Euler equations; Navier-Stokes equations

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1. Introduction

In [1] , Dobrokhotov and A. Shafarevich found the interesting property of solutions in the Cauchy problem for the Navier-Stokes equations in space. If $u = (u_1, u_2, u_3)$ is a fluid velocity then

$$\int_{R^3} u_i u_k dx = \frac{\delta_{ki}}{3} \int_{R^3} |u|^2 dx$$

where δ_{ki} is the Kronecker symbol. In other words, we have a conservation of some things close to conformal properties. Later, in [2] , L. Brandolese confirmed this result for dimensions $n \geq 4$ (see also [3]). In fact, this is the property of solenoidal vector fields if a potential part of a mapping

$$u_i \frac{\partial u}{\partial x_i} := u_i u_{,\,i}$$

is summaable to a power $r > 1$ (see [4]). Here, the repeated index means summation.

Now, I would like to consider other things connected with integral identities. A part of them was studied by author in [4]. Every finite smooth solenoidal vector field u satisfies the following integral identity:

$$\int_{R^n} u_{i,j} u_{k,\,i} u_{k,j} dx = -\int_{R^n} u_i u_{k,\,i} \triangle u_k dx$$

(see [4] where some statements connected with this formula are given). This very simple formula implies some new identities and applications to the 2d Navier-Stokes and Euler equations. In particular, we obtain exact *a priori* estimates in Ladyzchenskaya's and Judovich's theorems.

1.1. Notations

Let $u : R^n \to R^n,\ u = (u_1, u_2, \cdots, u_n),\ n \geq 2$, be an arbitrary vector field. Symbols

$$u_{k,\,i} = \frac{\partial u_k}{\partial x_i},\ u_{k,\,ij} = \frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

and so forth mean a partial differentiation or differentiation in distributions. Naturally, Δ is the Laplace operator. Below, unless otherwise indicated, the repeated indices mean summation. For example,

$$u_i u_{j,\,i} = \sum_{i=1}^{n} u_i u_{j,\,i},\ u_{i,\,j} u_{j,\,i} = \sum_{i,j=1}^{n} u_{i,\,j} u_{j,\,i},\ u_i u_{j,\,i} \triangle u_j = \sum_{i,j=1}^{n} u_i u_{j,\,i} \triangle u_j$$

etc. Further, I consider rotor coordinates (for dimension $n = 2, 3$)

$$c_{ki}(u) = u_{k,i} - u_{i,k} \tag{1}$$

as elements of a skew-symmetric matrix C. The Jacobi matrix in distributions of the vector field u is denoted by ∇u . As in common practice, the modulus of a matrix A is defined by the equality

$$|A| = \Big(\sum_{i,\,j} a_{ij}^2\Big)^{\frac{1}{2}} \tag{2}$$

A symbol $W_p^l(R^n)$ denotes the Sobolev class of vector fields which have all derivatives in distributions until an order l and summable to a power $p \geq 1$. The norm in this space is given by formula:

$$\|v\|_{W_p^l(R^n)} = \sum_{|\alpha| \leq l} \|D^\alpha v\|_p$$

where a symbol $\|h\|_p$ is a norm in space $L_p(R^n)$.

Respectively, a class of infinitely smooth vector fields with a compact support is denoted by $C_0^\infty(R^n)$ and a closure of this set in the norm of the space $W_p^l(R^n)$ is written by $\overset{\circ}{W}{}_p^l(R^n)$.

2. Solenoidal Vector Fields and Integral Identities for Dimension $n \geq 2$

A classical integral identity for these fields goes on the Helmholtz-Weyl theorem about the decomposition of a smooth vector field by the sum of potential and solenoidal terms. Later, (see [5,6], p. 339) it was shown that the space $L_2(R^3)$ of vector fields has decomposition $L_2(R^3) = J \oplus G$ where subspaces J and G are closures in $L_2(R^3)$ of finite, smooth solenoidal and potential vector fields respectively.

The main results are described by theorems 1 and 2.

Theorem 1. *Suppose, that solenoidal vector field* $u \in \overset{\circ}{W}{}_2^3(R^n),\ n \geq 2$. *Then integral identities are true:*

$$\int_{R^n} \Delta u_{i,j} c_{ki}(u) c_{kj}(u) dx = 0 \tag{3}$$

$$\int_{R^n} (c_{im,j}(u) + c_{jm,i}(u)) c_{ki}(u) c_{kj,m}(u) dx = 0 \tag{4}$$

$$\int_{R^n} u_{i,jm} u_{k,im} u_{k,j} dx = 0 \tag{5}$$

$$\int_{R^n} (\Delta u_{i,j} u_{k,i} u_{k,j} + u_{i,j} \Delta u_{k,i} u_{k,j}) dx = \int_{R^n} u_{i,j} u_{k,i} \Delta u_{k,j} dx \tag{6}$$

$$\int_{R^n} u_{i,j} c_{ki}(\Delta u) c_{kj}(u) dx = \int_{R^n} u_{i,j} c_{ki}(u) c_{kj}(\Delta u) dx \tag{7}$$

If $u \in \overset{\circ}{W}{}_2^4(R^n),\ n \geq 2$, *then*

$$\int_{R^n} u_i u_{k,i} \Delta^2 u_k dx = -\int_{R^n} (u_{i,j} \Delta u_{k,i} u_{k,j} + u_{i,j} u_{k,i} \Delta u_{k,j}) dx \tag{8}$$

The proof of this theorem relies on the following statement.

Lemma 1. *If a vector field* $u \in W_p^2(R^n),\ n \geq 2,\ p \geq 1$, *then vector field* $g = (g_1, g_2, \cdots, g_n)$ *where* $g_i = c_{ki,j} c_{kj} |C|^{p-2}$, c_{kj} *from formula* (1), *the matrix* C *is defined above, is potential* i.e., *for every smooth solenoidal vector field* $v \in C_0^\infty(R^n)$ *the integral identity is fulfilled:*

$$\int_{R^n} v_i c_{ki,j}(u)\ c_{kj}(u) |C|^{p-2} dx = 0$$

Proof of Lemma 1. It is sufficient to see the equality:

$$c_{ki,j}\ c_{kj} = \frac{1}{4} \sum_{k,j=1}^{n} \frac{\partial}{\partial x_i} \Big(u_{k,j} - u_{j,k} \Big)^2$$

which follows from relations of the type: $u_{i,kj} u_{j,k} = u_{i,jk} u_{k,j}$. □

Proof of Theorem 1. Without any restrictions we assume that a vector field $u \in C_0^\infty(R^n)$ (see [7], Theorem 1). In the equality of lemma 1 we take: $p = 2$, $v = \Delta u$. Further, we integrate by parts with respect to variable x_j. Since $c_{kj,j}(u) = \Delta u_k$ and $c_{ki,jj}(u) = c_{ki}(\Delta u)$ then we obtain equation (3). Applying $\Delta u_{i,j} = c_{im,mj}(u)$ for substitution to equation (3) and integrating by parts with respect to variable x_m we exchange summation indices i and j in the first product. Then it follows equation (4).

For the proof of equation (5) we note sums equality: $u_{i,jm} u_{k,im} u_{k,j} = u_{i,mj} u_{k,ij} u_{k,m}$. Then

$$2 \int_{R^n} u_{i,jm} u_{k,im} u_{k,j} dx = \int_{R^n} u_{i,jm} (u_{k,m} u_{k,j})_{,i} dx = 0$$

since $div\ u_{,jm} = 0$.

Now, we write equation (3) by expanded form:

$$\int_{R^n} (\Delta u_{i,j} u_{k,i} u_{k,j} - \Delta u_{i,j} u_{k,i} u_{j,k} - \Delta u_{i,j} u_{i,k} u_{k,j} + \Delta u_{i,j} u_{i,k} u_{j,k}) dx = 0$$

In the third term we exchange summation indices i and k. In the forth term we make it twice. By the first step we exchange i and k, after that i and j. The second term vanishes because

$$\int_{R^n} \Delta u_{i,j} u_{k,i} u_{j,k} dx = -\int_{R^n} \Delta u_i u_{k,ij} u_{j,k} dx =$$

$$= -\frac{1}{2} \int_{R^n} \Delta u_i \frac{\partial}{\partial x_i} (u_{k,j} u_{j,k}) dx = 0$$

Therefore, the previous equality reduces to formula (6).

For verification (8) we use identity:

$$\int_{R^n} u_i u_{k,jmi} u_{k,jm} dx = 0$$

where we integrate by parts. Then

$$\int_{R^n} u_{i,m} u_{k,ji} u_{k,jm} dx + \int_{R^n} u_i u_{k,ji} \Delta u_{k,j} dx = 0$$

Here, for every integral we apply the integration by parts with respect to variable x_j. As the result we get:

$$\int_{R^n} u_i u_{k,i} \Delta^2 u_k dx = -\int_{R^n} u_{i,j} \Delta u_{k,i} u_{k,j} dx - \int_{R^n} u_{i,mj} u_{k,im} u_{k,j} dx -$$

$$- \int_{R^n} u_{i,m} u_{k,i} \Delta u_{k,m} dx$$

On the right hand side, the middle integral vanishes by formula (5). In the third integral in the same place we exchange index m by index j. Hence, we have equation (8). □

Corollary 1. *Let* $u \in \overset{\circ}{W}{}_2^6(R^n),\ n \geq 2$, *be a solenoidal vector field. Then the integral identity is true:*

$$\int_{R^n} (u_{k,i} \Delta^2 u_{i,j} + 2\Delta u_{i,j} \Delta u_{k,i} + u_{i,j} \Delta^2 u_{k,i}) u_{k,j} dx = \int_{R^n} u_{i,j} u_{k,i} \Delta^2 u_{k,j} dx$$

Proof of Corollary 1. Let $T = T(t,x)$ be a solution of equation $\dot{T} = \nu \Delta T$ with an initial data $T(0,x) = u(x)$. Now, we rewrite equation (6) for the solenoidal vector field T and differentiate it with respect to t. A passage to the limit as $t \to 0$ gives the necessary equality. □

3. Solenoidal Vector Fields and Integral Identities for Dimension *n* = 2

In this case we can give more precise identities. Applying them we can obtain the exact estimates for solutions in the Cauchy problem for the Navier-Stokes equations and Euler equations.

Theorem 2. *Let* u, v *be solenoidal vector fields in* R^2.

(1) If $u, v \in W_2^1(R^2)$ *then almost everywhere there are fulfilled:*

$$u_{i,j} u_{k,i} v_{k,j} = 0,\ v_{i,j} c_{ki}(u) c_{kj}(u) = 0 \tag{9}$$

(2) If $u, v \in \overset{\circ}{W}{}_2^3(R^2)$ *then the following integral identities are true:*

$$\int_{R^2} v_i c_{ki}(u) \Delta u_k dx = 0 \tag{10}$$

$$\int_{R^2} (\Delta u_{i,j} u_{k,i} u_{k,j} + u_{i,j} \Delta u_{k,i} u_{k,j}) dx = 0 \tag{11}$$

(3) If $u \in \overset{\circ}{W}{}_2^5(R^2)$ *then*

$$\int_{R^2} (\Delta^2 u_{i,j} u_{k,i} u_{k,j} + u_{i,j} \Delta^2 u_{k,i} u_{k,j}) dx = 0 \tag{12}$$

$$\int_{R^2} u_i u_{k,i} \Delta^2 u_k dx = -\int_{R^2} u_{i,j} \Delta u_{k,i} u_{k,j} dx = \int_{R^2} \Delta u_{i,j} u_{k,i} u_{k,j} dx \tag{13}$$

Proof of Theorem 2. Direct calculations and equalities $div\ u = 0$, $div\ v = 0$ prove equation (9). Since

$$\int_{R^2} v_{i,j} c_{ki}(u) c_{kj}(u) dx = 0$$

then integrating by parts we get equation (10). Formula (11) follows from equations (9) and (6) where $v = \Delta u$. Identity (12) we obtain from corollary 1 and equation (9) where we must replace u by Δu, v by u. Finally, equation (13) we have from equation (8) and the first identity from equation (9). □

4. Applications

Let us consider the Cauchy problem for the Navier–Stokes equations

$$\dot{u} + u_i u_{,i} = \nu \Delta u - \nabla p \tag{14}$$

$$divu = 0,\ u(0, x) = \varphi(x) \tag{15}$$

if dimension $n = 2$ or $n = 3$. We also suppose that an initial data $\varphi \in C_0^\infty(R^n)$. An existence of a weak solutions for small time interval was proved in [8] (their regularity it was shown in [9]). There are some conditions of an existence of global regular solutions.

Now, we note only monotonicity properties of regular solutions. Every regular solution satisfies the conditions:

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = \int_{R^2} u_i u_{k,\,i} \Delta u_k dx - \nu \|\Delta u\|_2^2 \tag{16}$$

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 = -\int_{R^3} u_i u_{k,\,i} \Delta^2 u_k dx - \nu \|\nabla \Delta u\|_2^2 \tag{17}$$

Integrals finiteness in these formulas follows from [10].

Theorem 3. *Let be an initial data* $\varphi \in C_0^\infty(R^n)$. *Suppose, that* u *is a regular solution of the problem* (14) *and* (15). *If* $n = 2$ *then a function*

$$\eta(t) = \|\nabla u\|_2^2$$

is a decreasing function.

If $n = 3$ *and*

$$\|\varphi\|_2^3 \|\Delta \varphi\|_2 \le \frac{4\sqrt{3}}{3} \nu^4$$

then a function

$$\omega(t) = \|u\|_2^3 \|\Delta u\|_2$$

is a decreasing function.

Proof. We take $v = u$ in formula (10) then from (16) it can be deduced the inequality

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = -\nu\|\Delta u\|_2^2$$

Hence, we have the first statement.

To prove the second part we combine (16) and (13). Then

$$\int_{R^3} u_i u_{k,\,i}\Delta^2 u_k dx = -2\int_{R^3} u_{i,j}u_{k,\,im}u_{k,jm}dx$$

Further, we apply Hölder's inequality. Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 \le 2\int_{R^3}|\nabla u||\nabla(\nabla u)|^2 dx - \nu\|\nabla\Delta u\|_2^2 \le$$

$$\le 2\|\nabla u\|_2\|\nabla(\nabla u)\|_4^2 - \nu\|\nabla\Delta u\|_2^2$$

Use the estimate from [9] (see p. 74). Then

$$\|v\|_4^2 \le \sqrt{2}\cdot 3^{-3/4}\|v\|_2^{1/2}\|\nabla v\|_2^{3/2}$$

where $v = \nabla(\nabla u)$. (For $n = 3$ the factor 2 is omitted among intermediate calculations there.) Since $\|\nabla\nabla u\|_2 = \|\Delta u\|_2$, $\|\nabla(\nabla(\nabla u))\|_2 = \|\nabla\Delta u\|_2$, then finding a function maximum $f(z) = bz^{3/2} - \nu z^2$, where $z = \|\nabla\Delta u\|_2$, we get:

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 \le \frac{3\sqrt{3}}{4}\|\nabla u\|_2^4\|\Delta u\|_2^2 \le \frac{3\sqrt{3}}{4}\|u\|_2\|\nabla u\|_2^2\|\Delta u\|_2^3$$

Then for the function $\omega(t) = \|u\|_2^3\|\Delta u\|_2$ we have:

$$\frac{d\omega}{dt} \le 3\|u\|_2\|\nabla u\|_2\|\Delta u\|_2\Big(-\nu + \frac{\sqrt{3}}{4\nu}\omega(t)\Big)$$

The inequality follows from a nonpositiveness of the derivative ω'. □

Remark 1. *Monotonicity properties are very important for the exact a priory estimates for solutions of Navier-Stokes equations. For example, if dimension $n = 2$, we get*

$$\|\nabla u\|_2^2 \le \|\nabla\varphi\|_2^2$$

in the problem (14) *and* (15).

Remark 2. *Obvious and exact estimates may be obtained for weak solutions of Euler equations (see [4]) and Navier-Stokes equations with an outer force.*

Remark 3. *If dimension $n = 3$ then from Gagliardo's and Nirenberg's inequalities (see [11] and [12]) with some constant C we get for solution in the problem* (14) *and* (15) *the following uniform estimate*

$$\|\nabla u\|_2^2 \le C\|\varphi\|_2\|\Delta\varphi\|_2$$

Theorem 4. *Let be dimension* $n = 2$ *and a solenoidal vector field* $u \in \overset{\circ}{W}{}_2^2(R^2)$. *Then a vector field*

$$w = \Delta u_i u_{,i} + u_{i,j} u_{,ij}$$

is a potential field in distributions.

Proof. It follows from the first equality (9) where a solenoidal vector field $v \in C_0^\infty(R^2)$. Since

$$\int_{R^2} u_{i,j} u_{k,i} v_{k,j} dx = 0$$

then

$$\int_{R^2} w_k v_k dx = 0.$$

Therefore, the statement is proved. □

Conflicts of Interest

The author declares no conflict of interest.

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