

Editorial: S. N. Mergelyan's Dissertation "Best Approximations in the Complex Domain"

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Preface by Hovik A. Matevossian

In this Editorial, we present an authentic translation of the Dissertation written by Mergelyan, S. N. called *Best approximations in the complex domain*. (Ph.D. thesis, Steklov Mathematical Institute of the USSR Academy of Sciences, Moscow, 1948, 56 pages (In Russian)). It presents phenomenal results from the outstanding Soviet and Armenian mathematician Sergey N. Mergelyan (1928–2008) in connection with his 95th birthday.

S. Mergelyan's main scientific research included the theory of functions of complex variables, approximation theory, the theory of potentials, and harmonic functions.

S. Mergelyan carried out in-depth research and obtained valuable results in such areas as the best approximation by polynomials on an arbitrary continuum, weighted approximation by polynomials on the real axis, point approximation by polynomials on closed sets of the complex plane, uniform approximation by harmonic functions on compact sets, and entire functions on an unbounded continuum.

For the exceptional results obtained in the field of approximation theory, the Scientific Council of the Steklov Mathematical Institute of the USSR Academy of Sciences awarded the 20-year-old genius Sergey Mergelyan a degree of Doctor of Physical and Mathematical Sciences; he was the youngest Doctor of Sciences in the history of the USSR and the youngest Corresponding member of the Academy of Sciences of the Soviet Union (at 24 years old) (https://en.wikipedia.org/wiki/Sergey_Mergelyan (accessed on 20 December 2023)).

In 1951, S. Mergelyan proved his famous theorem on approximation by polynomials (Mergelyan, S. N. Certain questions of the constructive theory of functions. *Trudy Mat. Inst. Steklov* **1951**, 37, Acad. Sci. USSR, Moscow, 3–91). His theorem on the approximation of functions by polynomials has become classical among the theorems of Weierstrass and Runge.

The new terms "Mergelyan's Theorem" and "Mergelyan Sets" have found their place in textbooks and monographs on approximation theory.

S. Mergelyan's theorem answers the question about the possibility of polynomial approximation of the function of one complex variable: *Every function continuous on a compact set $K \subset \mathbb{C}$ and holomorphic in its interior can be represented in K by a uniformly converging sequence of polynomials if and only if the complement $\mathbb{C} \setminus K$ is connected* (Mergelyan, S. N. Uniform approximations of functions of a complex variable. *Uspekhi Mat. Nauk* **1952**, 7:2(48), 31–122).

S. Mergelyan's theorem completes a large cycle of research on polynomial approximations, which began in 1885, and consists of classical results by Weierstrass, Runge, Walsh, M. Lavrentiev, M. Keldysh, and others. In these papers, a function that is continuous on a compact set and holomorphic in its interior is approximated by a function that is holomorphic on the entire compact set (that is, in a neighborhood of this set). Polynomial approximation is then obtained using the Runge theorem (1885) that every function that is holomorphic on a compact set whose complement is connected can be represented in this set by a uniformly converging sequence of complex polynomials.



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S. Mergelyan's further results were devoted to the study of the approximation of continuous functions that satisfy the smoothness properties for an arbitrary set (1962) and the solution of Bernstein's approximation problem (1963).

I express my gratitude to the Leading Researcher of the Steklov Mathematical Institute of the Russian Academy of Sciences and Professor at Lomonosov Moscow State University, A.G. Sergeev, for supporting the idea of publishing Mergelyan's Dissertation.

I express my gratitude to Professors Heinrich Begehr (FU Berlin) and Paul Gauthier (Université de Montréal) for their efforts in reading the translation of this Dissertation, for their valuable comments in clarifying and correctly using the terminology of the theory of approximation, and for their help in editing the manuscript.

I express my sincere gratitude to two staff members at the Library of Natural Sciences of the Russian Academy of Sciences, Tatiana and Irina, for providing the original manuscript of Mergelyan's Dissertation, which is stored in the scientific collection of the library of the Steklov Mathematical Institute of the Russian Academy of Sciences.

I also express my gratitude to my graduate students M. Dorodnitsyn and A. Kovalev for typing this manuscript, and to my colleague V.N. Bobylev for their help in reading the manuscript and restoring the list of references.

S. N. Mergelyan “Best Approximations in the Complex Domain”

Introduction

Consider a finite closed domain $\overline{D}$ whose complement represents a connected set. Suppose that the function $f(z)$ is regular at interior points of $\overline{D}$ and continuous on $\overline{D}$. The infimum of the values

$$\max_{z \in \overline{D}} |f(z) - \mathcal{P}_n(z)|$$

with respect to all possible polynomials $\{\mathcal{P}_n(z)\}$ of degree $\leq n$ we denote by $\rho(n)$.

As is known [1], $\rho(n) \rightarrow 0$ as $n \rightarrow \infty$, and the rate of decrease of the number $\rho(n)$ is closely related to the properties of $f(z)$ on $\overline{D} - D = \Gamma$, as well as to the properties of the domain D .

In the case when the boundary D is an analytic curve, it is established [2] that if $f(z)$ has a continuous k^{th} derivative in $\overline{D}$, satisfying there the Lipschitz condition of order α , then there is a constant C for which

$$\rho(n) < \frac{C}{n^{k+\alpha}},$$

and, conversely, if $0 < \alpha < 1$, k is an integer and $\rho(n) < C n^{-k-\alpha}$, then the k^{th} derivative of $f(z)$ satisfies the Lipschitz condition of order α in $\overline{D}$.

Thus, in the case of an analytic domain, the dependence of $\rho(n)$ on $f(z)$ is similar to the dependence of the rate of best approximation on the properties of a function in the real domain.

From the results related to the investigation of the rate of best approximation in the case of non-analytic domains, we note the following two ideas.

Let $C(\rho)$ denote the distance of the image of the circle $|w| = \rho > 1$ under the conformal mapping $|w| > 1$ onto the complement of $\overline{D}$ to the boundary of D . If for some $\alpha > 0$

$$C(\rho) > \text{const} (\rho - 1)^\alpha,$$

then from inequality

$$\rho(n) < \frac{C}{n^{(k+\beta)\alpha}}, \quad (k \text{ is an integer, } 0 < \beta < 1),$$

it follows that the k^{th} derivative of $f(z)$ satisfies the Lipschitz condition of order β in $\overline{D}$ [2].

If $\theta(s)$ denotes the angle made by the tangent to the boundary of D at the point $z(s)$ with the axis OX , and $z(s)$ represents a point on Γ distant from some fixed point $z(0)$ at arc distance s , and $\theta(s)$ satisfies a Lipschitz condition of positive order, then from the fact that $f(z)$ is regular in D and satisfies the inequality

$$|f(z') - f(z'')| < K|z' - z''|^\delta, \quad z', z'' \in \overline{D}$$

it follows, as A. I. Markouchevitch showed [3], that for any positive ε

$$\rho(n) < \frac{C_1(\varepsilon)}{n^{\delta-\varepsilon}}. \quad (0.1)$$

The present paper is devoted to studying the rate of best approximation in the general case when D represents an arbitrary domain of the Carathéodory class. Some issues are also considered that are somehow related to the theory of best approximation.

In Section 1 we establish upper estimates for the quantities $\rho(n)$ for domains with various features, for example for domains with a corner point, convex domains, etc.

In Section 2 given the rate of approximation and the domain D , the necessary properties of the function are investigated, and the theorems are local in nature, since the same rate of approximation imposes different restrictions on the function at different boundary

points, depending on the behavior of the domain D near these points. Thus, it is possible to verify the accuracy of the estimates Section 1.

Direct and inverse theorems on the rate of best approximation in domains with a smooth boundary are highlighted in Section 3.

Here we show that the above-mentioned analogy between $\rho(n)$ and the rate of best approximation in the real domain $E_n(f)$, which holds for analytic domains, already disappears in the case of some of the domains with a smooth boundary.

Inequality 0.1 extends to arbitrary domains with a smooth boundary. Estimates for $\rho(n)$ are also given depending on the degree of smoothness of the boundary D .

It is known [4] that if two domains D_1 and D_2 have only one common boundary point, and f_1 and f_2 are regular in D_1, D_2 , respectively, and are continuous at the closures and take equal value at the common boundary point, then, as soon as in each of the domains D_1, D_2 the corresponding function can be uniformly approximated by polynomials, then there exists a sequence of polynomials uniformly converging in $\overline{D}_1$ to $f_1(z)$ and, at the same time, in D_2 to $f_2(z)$. In Section 4 the results concerning the study of the rate of simultaneous approximation in two touching domains are presented. (The Introduction specifies Section 4, but this Section 4 is missing from the manuscript. It is worth noting that the material stated as Section 4 is discussed in Section 3 (*editor's comment*)) In this case, the rate of approximation depends on a third factor—the relative position of the domains D_1 and D_2 . It is proved that if a certain relationship is satisfied between the rate of simultaneous approximation and the order of contact of the boundaries D_1 and D_2 , then the convergence of the sequence of polynomials $\{\mathcal{P}_n(z)\}$ to zero in $\overline{D}_1$ also automatically implies convergence to zero in $\overline{D}_2$.

In Section 5 some quasi-analytic classes of functions are introduced and criteria for belonging to them are given in terms of the best approximation. Related here is the question of the distribution of zeros of the analytic function $f(z)$ located on the boundary of the domain of regularity of $f(z)$ under the assumption that $f(z)$ is continuous in a closed domain.

In Section 6 the best approximation on various discontinuous sets is considered, and in some cases a dependency is established between the rate of approximation, the behavior of the function and the properties of the sets on which the approximation occurs.

I take this opportunity to express my deep gratitude to Academician M.V. Keldysh, whose advice and instructions provided me with great assistance in carrying out this work.

1. Direct Theorems for Domains with Different Types of Singularities

Let us present the formulation of one of Warschawski's results, which we will use in the future.

Let D be a Jordan domain bounded by a curve Γ passing through $z = 0$. Suppose that in the neighborhood $|z| \leq a$ of the point $z = 0$ the boundary Γ of the domain D consists of two arcs Γ_+ and Γ_- , the equations of which in polar coordinates are

$$\varphi = \Phi_+(\rho), \quad \varphi = \Phi_-(\rho) \quad (\Phi_+ < \Phi_-)$$

respectively. Let there also be limits

$$\lim_{\rho \rightarrow 0} \rho \frac{d\Phi_+}{d\rho}; \quad \lim_{\rho \rightarrow 0} \rho \frac{d\Phi_-}{d\rho}.$$

Let $w = w(z)$ denote the function that conformally maps the domain D onto the circle $|w - 1| < 1$ so that $z = 0$ goes to $w = 0$; $\theta(\rho) = \Phi_-(\rho) - \Phi_+(\rho)$.

Theorem 1.1 (S. E. Warschawski [5]). *If*

$$\int_0^a \left[\left(\frac{d\Phi_+(\rho)}{d\rho} \right)^2 + \left(\frac{d\Phi_-(\rho)}{d\rho} \right)^2 \right] \frac{\rho d\rho}{\theta(\rho)} < \infty,$$

then in the neighborhood of the point $z = 0$ we have

$$|w(z)| = C \exp \left\{ -\pi \int_{|z|}^a \frac{d\rho}{\rho \theta(\rho)} + O(1) \right\}, \quad (1.1)$$

where C does not depend on z .

This theorem makes it possible in a number of cases to investigate the behavior of conformal mapping functions in a closed domain, as well as to estimate the distance of the level lines of the Green's function to a boundary point, depending on the behavior of the boundary near this point.

Let D be a bounded domain with a simply connected complement; henceforth, by $\mathcal{Z}_R$ ($R > 1$) we mean the level line of the Green's function $G(z)$ of the complement to $\overline{D} - \mathcal{Z}_R$; $G(z) = \ln R$.

D_R is a bounded domain bounded by $\mathcal{Z}_R$. If $f(z)$ is regular in D_R and does not exceed unity there in absolute value, then for any integer $n \geq 1$ there is a polynomial $\mathcal{P}_n(z)$ of degree n , for which

$$\max_{z \in \overline{D}} |f(z) - \mathcal{P}_n(z)| < \frac{C_0}{(R-1)^3 R^n}, \quad (1.2)$$

where C_0 is an absolute constant (the degree of difference $R - 1$ can be reduced, however the question of determining it as accurately as possible does not interest us now).

The proof is easy to derive by composing a Fejér interpolation polynomial with uniformly distributed nodes, estimating the remainder term represented by the Cauchy integral, and considering that, firstly, if the diameter D is less than one, then

$$\text{length } \mathcal{Z}_R < \frac{C'_0}{R-1},$$

and secondly, the distance of any point $\mathcal{Z}_R$ to Γ exceeds $\frac{1}{C'_0}(R-1)^2$, where C'_0 is an absolute constant.

Let D now contain $z = 0$ and the equation of its boundary in polar coordinate be

$$\rho = \rho(\varphi),$$

where $\rho(\varphi) = \rho(\varphi + 2\pi)$ is a single-valued continuous function.

Theorem 1.2. *If all derivative numbers of the function $\ln \rho(\varphi)$ are uniformly bounded from above by the number k and the function $f(z)$ is regular in D , has a continuous m^{th} derivative in $\overline{D}$, the modulus of continuity of which is $w(\delta)$, then*

$$\rho_n(D; f) < \text{const} \left(\frac{\ln n}{n} \right)^{\frac{2m}{\pi} \arctg \frac{1}{k}} w \left(\left(\frac{\ln n}{n} \right)^{\frac{2}{\pi} \arctg \frac{1}{k}} \right). \quad (1.3)$$

Proof. Let ξ be an arbitrary boundary point of D , and let $w(z)$ map the complement of $\overline{D}$ to the circle $|w| < 1$ ($w(\infty) = 0$) so that ξ goes to $w = -1$.

It is easy to see that from the condition $|D \ln \rho(\varphi)| < k$ that it follows, in the notation of Warschawski's theorem

$$\theta(r) \geq 2 \arctg \frac{1}{k},$$

so putting $2 \operatorname{arctg} \frac{1}{k} = \varphi$ we have

$$|w(z) + 1| = C \exp \left\{ - \int_{|z-\xi|}^a \frac{\pi dr}{r\theta(r)} + O(1) \right\} \geq C_1 |z - \xi|^{\frac{\pi}{\varphi}},$$

that is, if $w(z)$ belongs to the circumference $|\frac{1}{w}| = R > 1$, then the distance z to ξ does not exceed $C_2(R - 1)^{\frac{\varphi}{\pi}}$.

Let us denote by $D^{(q)}$ the domain to which D goes under the transformation $w = qz$ ($q > 1$). From $|D \ln \rho(\varphi)| < k$ it follows that the existence of a constant $C_3 > 0$ for which the distance of the boundary $D^{(q)}$ to the boundary D exceeds $C_3(q - 1)$.

Thus, for some $C_4 > 0$ the domain D_R bounded by the level line Z_R is contained in the domain $D^{1+C_4(R-1)^{\frac{\varphi}{\pi}}}$. It follows that the function

$$f^{(m)} \left(\frac{z}{1 + C_4(R - 1)^{\frac{\varphi}{\pi}}} \right)$$

is analytic in D_R , uniformly bounded there in R ; therefore, according to the remark made above, there is a polynomial $P(z)$ of degree n such that

$$\max_{z \in \overline{D}} \left| f^{(m)} \left(\frac{z}{1 + C_4(R - 1)^{\frac{\varphi}{\pi}}} \right) - \mathcal{P}(z) \right| < \frac{C_5 M}{(R - 1)^3 R^n},$$

where

$$M = \max_{z \in \overline{D_R}} \left| f^{(m)} \left(\frac{z}{1 + C_4(R - 1)^{\frac{\varphi}{\pi}}} \right) \right|.$$

But

$$\max_{z \in \overline{D}} \left| f^{(m)}(z) - f^{(m)} \left(\frac{z}{1 + C_4(R - 1)^{\frac{\varphi}{\pi}}} \right) \right| < w \left(C_6(R - 1)^{\frac{\varphi}{\pi}} \right).$$

Let us put $R = 1 + \frac{4 \ln n}{n}$; we have

$$\max_{z \in \overline{D}} |f^{(m)}(z) - \mathcal{P}(z)| < \frac{C_7 M}{n(\ln n)^3} + w \left(C_6 \left(\frac{\ln n}{n} \right)^{\frac{\varphi}{\pi}} \right).$$

From the fact that $w(\delta)$ is the modulus of continuity of some function other than a constant, it follows that, firstly, $w(\delta) > C_8 \delta$, and secondly, $w(2^k \delta) < 2^k w(\delta)$, so we have

$$\max_{z \in \overline{D}} |f^{(m)}(z) - \mathcal{P}(z)| < C_9 w \left(\left(\frac{\ln n}{n} \right)^{\frac{\varphi}{\pi}} \right).$$

Next, we apply a well-known technique. The function

$$\int_0^z \left(f^{(m)}(z) - \mathcal{P}(z) \right) dz$$

satisfies a first-order Lipschitz condition with a Lipschitz constant equal to

$$C_9 w \left(\left(\frac{\ln n}{n} \right)^{\frac{\varphi}{\pi}} \right),$$

so, according to what has been proved, we can find a polynomial $Q(z)$ of degree $n + 1$ for which

$$\max_{z \in \overline{D}} |f^{(m-1)}(z) - Q(z)| < C_{10} \left(\frac{\ln n}{n} \right)^{\frac{\gamma}{\pi}} w \left(\left(\frac{\ln n}{n} \right)^{\frac{\gamma}{\pi}} \right).$$

Proceeding similarly with $f^{(m-2)}(z), \dots$, and finally with $f(z)$ we come to the proof of the theorem.

Thus, Theorem 1.2 gives an estimate of the rate of approximation for domains with corner points. $\square$

Let D be bounded by a finite number of smooth curves that make angles with each other, the internal openings of which do not exceed $2\pi\alpha$, and $f^{(m)}(z)$ be regular in D and satisfy the Lipschitz condition of order γ in $\overline{D}$.

Assuming the domain D is star-shaped with respect to one of its points, it is easy to show by the reasoning used in the proof of Theorem 1.2 that

$$\rho(D; f) < \frac{C(\varepsilon)}{n^{2(1-\alpha)(m+\gamma)-\varepsilon}}, \quad n = 1, 2, 3, \dots, \quad (1.4)$$

for any $\varepsilon > 0$.

We can free ourselves from the artificial restriction of star-shapedness by using an additional reasoning based on the “averaging” method of academician M.V. Keldysh [6], which we set out in §3 when proving Theorem 1.3.

If no additional restrictions are imposed on the smoothness of the boundary D , then, as will be seen, in §3, the numbers $C(\varepsilon)$ can increase arbitrarily quickly: for any positive function $f(z)$ there exists $N(\varepsilon)$ such that $f^{(m)}(z)$ satisfies a Lipschitz condition of order γ in $\overline{D}$; however,

$$\rho(D; f) > \frac{N(\varepsilon)}{n^{2(1-\alpha)(m+\gamma)-\varepsilon}}, \quad n_1(\varepsilon) < n < n_2(\varepsilon).$$

Now, let the domain D have an incoming point, and $\rho = \rho(\varphi)$ still means the equation of the boundary in polar coordinates; assume that $\rho'(\varphi)$ exists everywhere outside $\varphi = 0$, and also that at the point $(0; \rho(0))$ two arcs of the boundary of D touch the axis OX , and $\rho'(\varphi)$ decreases monotonically as $\varphi \rightarrow +0$, $\varphi \rightarrow 2\pi - 0$; by $d(\alpha)$ we denote the distance of the point $(0; \rho(0))$ to the level line $\mathcal{L}_{1+\alpha}$.

Theorem 1.3. *If $f(z)$ is regular in D , its m^{th} derivative has $w(\delta)$ modulus of continuity in $\overline{D}$, then*

$$\rho_n(D; f) < \text{const} \left(d \left(\frac{\ln n}{n} \right) \right)^m w \left(d \left(\frac{\ln n}{n} \right) \right). \quad (1.5)$$

Proof. The proof of this theorem is similar to the above proof of Theorem 1.2.

If, in particular, at the point $(0; \rho(0))$ we have the algebraic order of tangency of two arcs of the boundary

$$\rho(\varphi) \equiv \rho(0) + C\varphi^p, \quad 0 < \varphi < \varepsilon,$$

$$\rho(\varphi) \equiv \rho(0) + C(2\pi - \varphi)^p, \quad 2\pi - \varepsilon < \varphi < 2\pi,$$

then, using Warschawski's theorem, we have

$$d(\alpha) < \frac{\text{const}}{|\ln \alpha|^{\frac{1}{p-1}}}.$$

Hence, in this case

$$\rho_n(D; f) < \frac{\text{const}}{(\ln n)^{\frac{m}{p-1}}} w \left(\left(\frac{1}{\ln n} \right)^{\frac{m}{p-1}} \right)$$

In Section 2 it will be shown that this estimate is exact, in the sense that for some functions satisfying the conditions of Theorem 1.3 and two constants A_1 and A_2

$$A_1 \frac{1}{(\ln n)^{\frac{m}{p-1}}} w \left(\left(\frac{1}{\ln n} \right)^{\frac{1}{p-1}} \right) < \rho_n(f; D) < A_2 \frac{1}{(\ln n)^{\frac{m}{p-1}}} w \left(\left(\frac{1}{\ln n} \right)^{\frac{1}{p-1}} \right).$$

From the noted special cases we can conclude that in order to obtain a certain rate of approach in $\overline{D}$ there is no need to require that the function behaves well enough at all boundary points; the necessary and sufficient properties of the function for a given rate of approximation and domain depend at each point only on the behavior of the boundary of the domain near this point. $\square$

This circumstance constitutes a distinguishing feature of the best approximations in the complex domain from the best approximations in the real domain.

2. Inverse Theorems

Let D denote a domain bounded by the Jordan curve Γ . By $d(\xi; \alpha)$ we denote the distance of the image of the circle $|w| = 1 + \alpha$ ($\alpha > 0$) under a conformal mapping of the exterior of the unit circle to the complement of $\overline{D}$ to a boundary point ξ .

Let B_ξ denote an arbitrary subdomain of D having only the property that the ratio of the distance of any point of B_ξ to ξ to the distance of the same point to Γ is bounded from above uniformly with respect to all points of B_ξ . The class of functions that are regular in some domain G and whose k^{th} derivative satisfies a Lipschitz condition of order $\alpha' \leq 1$ in $\overline{G}$ is denoted by $Z(G; k + \alpha')$ ($\alpha' > 0$).

It should be noted that by the modulus of continuity $w(\delta)$ of the function $f(z)$ in $\overline{G}$ we mean the supremum of the quantities

$$|f(z') - f(z'')|$$

by all possible pairs z', z'' belonging to $\overline{G}$ and such that z' can be connected to z'' by a rectifiable curve lying entirely in $\overline{G}$ and by length not exceeding δ ; for some domains this definition obviously does not coincide with the definition of the modulus of continuity that is given in the real domain; accordingly, a different meaning, generally speaking, is attached to the satisfaction of the Lipschitz condition in a closed domain $\overline{G}$.

It is easy to see that the quantity $w(\delta)$ is closely related to the properties of the function $f(z)$, while

$$w_1(\delta) = \max_{|z' - z''| \leq \delta, z', z'' \in \overline{G}} |f(z') - f(z'')|$$

represents an artificial formation in relation to $f(z)$.

Indeed, for any function $\mu(\delta)$ decreasing monotonically to zero, one can construct a domain such that the fact that $w_1(\xi) < \mu(\xi)$ implies infinite differentiability of $f(z)$ at individual points on the boundary of the domain. As a similar example, we can take a domain with an incoming point $z = 0$ and a sufficiently large order of contact of two boundary arcs at $z = 0$.

The following proposition also applies to this question, the proof of which we will not dwell on.

Proposition 2.1. *If two domains D_1 and D_2 have one common boundary point $z = 0$, and the function $f(z)$ in D_i coincides with the function $f_i(z)$ that is regular in D_i and continuous in $\overline{D}_i$ ($i = 1, 2$), and $f_1(0) = f_2(0)$, then for any function $v(\delta)$ decreasing towards zero one can specify such a large order of contact of the boundaries of D_1 and D_2 at $z = 0$ that from the inequality*

$$w_1(\delta) < v(\delta) \tag{2.1}$$

it follows that if one of the functions $f_1(z)$, $f_2(z)$ is identically equal to zero in the corresponding domain, then the same can be stated regarding the other function, i.e., pairs of functions satisfying (2.1) constitute a quasi-analytic class.

This Proposition can be deduced from a theorem to be proved later.

Theorem 2.1. *If*

$$\liminf_{n \rightarrow \infty} \frac{\ln \rho(n)}{\ln d(\xi; \frac{1}{n})} = A, \quad (2.2)$$

then for any $\varepsilon > 0$, $f(z) \in \mathcal{Z}(B_\xi; A - \varepsilon)$; if $A = \infty$, $f(z)$ is infinitely differentiable in $\overline{B}_\xi$.

Proof. For any integer $k > 0$ we define the integer n_k from the condition

$$d(\xi; \frac{1}{n_k}) \geq \frac{1}{2^k} > d(\xi; \frac{1}{n_k + 1}). \quad (2.3)$$

The numbers $\{n_k\}$ obviously constitute an increasing sequence. Let z denote an arbitrary point of $\overline{B}_\xi$, $r > 0$, p be some integer, and $\mathcal{P}_m(z)$ be a polynomial of degree m that least deviates from $f(z)$ in $\overline{D}$.

We have, obviously,

$$\mathcal{P}_{n_k}^{(p)}(z) = \mathcal{P}_{n_{k-1}}^{(p)}(z) = \frac{p!}{2\pi i} \int_{|t-z|=r} \frac{\mathcal{P}_{n_k}(t) - \mathcal{P}_{n_{k-1}}(t)}{(t-z)^{p+1}} dt \quad (*)$$

But from the definition of B_ξ it follows that there exists $C > 0$ such that the disc $|t-z| < C d(\xi; \varepsilon)$ is contained entirely in the domain $D_{1+\varepsilon}$, bounded by the outer line of the level $\mathcal{Z}_{1+\varepsilon}$ ($\varepsilon > 0$).

It is known that if $\max_{z \in \overline{D}} |Q_n(z)| = M$, then $\max_{z \in D_\rho} |Q_n(z)| \leq M\rho^n$, $\rho > 1$. Keeping in mind that

$$\max_{z \in \overline{D}} |\mathcal{P}_{n_k}(z) - \mathcal{P}_{n_{k-1}}(z)| \leq \rho(n_k) + \rho(n_{k-1}) \leq 2\rho(n_{k-1})$$

and setting $r = C d(\xi; \frac{1}{n_k})$, we apply this to the estimate of the difference under the integral in (*)

$$\begin{aligned} |\mathcal{P}_{n_k}^{(p)}(z) - \mathcal{P}_{n_{k-1}}^{(p)}(z)| &\leq \frac{p!}{2\pi} \int_{|t-z|=C d(\xi; \frac{1}{n_k})} \frac{2\rho(n_{k-1})(1+\frac{1}{n_k})^{n_k}}{|t-z|^{p+1}} |dt| \leq \\ &\frac{2e p!}{C^p} \frac{\rho(n_{k-1})}{(d(\xi; \frac{1}{n_k}))^p} \leq \frac{2^{p+1} e p!}{C^p} \frac{\rho(n_{k-1})}{(d(\xi; \frac{1}{n_{k-1}}))^p}, \end{aligned} \quad (2.4)$$

(the last inequality follows from (2.3)).

Let $[a]$ denote, as usual, the integer part of a and $\{a\} = a - [a]$. Let us put $p = [A - \varepsilon]$; since $A > 0$ and $\varepsilon > 0$ is small enough, then $\{A - \varepsilon\} > 0$. From the conditions of the theorem it follows that

$$|\mathcal{P}_{n_k}^{(p)}(z) - \mathcal{P}_{n_{k-1}}^{(p)}(z)| < \frac{2^{p+1} e p!}{C^p} \left(d\left(\xi; \frac{1}{n_{k-1}}\right) \right)^{\{A-\varepsilon\}} \leq p! \frac{C_1^p}{2^{k\{A-\varepsilon\}}}. \quad (2.5)$$

Consequently, the series

$$\mathcal{P}_{n_0}^{(p)}(z) + \sum_{k=1}^{\infty} [\mathcal{P}_{n_k}^{(p)}(z) - \mathcal{P}_{n_{k-1}}^{(p)}(z)]$$

representing $f^{(p)}(z)$ in D uniformly converges in $\overline{B}_\xi$, i.e., $f^{(p)}(z)$ is continuous in $\overline{B}_\xi$.

Let us estimate its modulus of continuity in the closed domain $\overline{B_\xi}$. Let z', z'' be points of B_ξ . We have

$$|f^{(p)}(z') - f^{(p)}(z'')| \leq \sum_{k=1}^n |\mathcal{P}_{n_k}^{(p)}(z') - \mathcal{P}_{n_{k-1}}^{(p)}(z') - \mathcal{P}_{n_k}^{(p)}(z'') + \mathcal{P}_{n_{k-1}}^{(p)}(z'')| + \\ + |\mathcal{P}_{n_0}^{(p)}(z') - \mathcal{P}_{n_0}^{(p)}(z'')| + \sum_{k=n+1}^{\infty} |\mathcal{P}_{n_k}^{(p)}(z') - \mathcal{P}_{n_{k-1}}^{(p)}(z') - \mathcal{P}_{n_k}^{(p)}(z'') + \mathcal{P}_{n_{k-1}}^{(p)}(z'')|. \quad (2.6)$$

But the common term of the sum just written does not exceed

$$\max_{z \in \overline{B_\xi}} |\mathcal{P}_{n_k}^{(p)}(z) - \mathcal{P}_{n_{k-1}}^{(p)}(z)| \int_{z'}^{z''} ds,$$

where the integration path lies entirely in $\overline{B_\xi}$.

The first multiplier, similar to the above, can be easily estimated using the Cauchy integral, and with respect to the second we assume that

$$\int_{z'}^{z''} ds \leq \delta.$$

As a result, we obtain

$$|\mathcal{P}_{n_k}^{(p)}(z') - \mathcal{P}_{n_{k-1}}^{(p)}(z') - \mathcal{P}_{n_k}^{(p)}(z'') + \mathcal{P}_{n_{k-1}}^{(p)}(z'')| \leq C_1^p p! 2^{k\{A-\varepsilon\}}.$$

But

$$|\mathcal{P}_{n_0}^{(p)}(z') - \mathcal{P}_{n_0}^{(p)}(z'')| < C_2^p \delta.$$

We estimate the general term of the last sum on the right side of (2.6) based on (2.5). Thus, we have

$$|f^{(p)}(z') - f^{(p)}(z'')| < C_3^p p! \left(\delta \sum_{k=1}^n 2^{k\{A-\varepsilon\}} + \delta + \sum_{k=n+1}^{\infty} \frac{1}{2^{k\{A-\varepsilon\}}} \right).$$

Now assuming $n = \left\lfloor \frac{|\ln \delta|}{\ln 2} \right\rfloor$ and taking any pair of points z', z'' from $\overline{B_\xi}$ with the condition $\int_{z'}^{z''} ds \leq \delta$, we have

$$w_p(\delta) < \text{const} \cdot \delta^{\{A-\varepsilon\}},$$

where $w_p(\delta)$ is the modulus of continuity of $f^{(p)}(z)$ in $\overline{B_\xi}$.

Consequently, the inclusion $f(z) \in \mathcal{Z}(\overline{B_\xi}; A - \varepsilon)$ is proven for any $\varepsilon > 0$. $\square$

Corollary 2.1. *From the above reasoning it can be seen that for $A = \infty$ $f(z)$ is infinitely differentiable in $\overline{B_\xi}$.*

Thus, an arbitrarily slow approximation rate $\rho(n)$ ensures infinite differentiability of the approximated function at some boundary points, if only the domain is located appropriately near these points.

Applying Warschawski's result on conformal mapping stated above, in many cases it is possible, by estimating $d(\xi; \alpha)$, to formulate the previous theorem directly in terms of the boundary of the domain.

Let the domain D contain, for some $\alpha > 0$, the segment $(-\alpha, 0)$ and its boundary near $z = 0$ be determined by the equation

$$|y| = \varphi(x), \quad 0 < x < \alpha,$$

$$C_1 x^m < \varphi(x) < C_2 x^m, \quad m > 1,$$

that is, we have a domain with an incoming point of algebraic tangent order.

In this case, according to Theorem 1.1,

$$d(0; \varepsilon) > \frac{C}{|\ln \varepsilon|^{\frac{1}{m-1}}}; \quad (2.7)$$

therefore, it can be proved that

Corollary 2.2. *If for some $C > 0$*

$$\rho(n) < \frac{C}{(\ln n)^p}, \quad p > 0,$$

then $f(z) \in \mathcal{Z}(B_0; p(m-1) - \varepsilon)$ for any $\varepsilon > 0$; if the proof is carried out carefully, then it can be shown that even $f(z) \in \mathcal{Z}(B_0; p(m-1))$, and this result is quite accurate, in the sense that, as it follows from Theorem 2.1, it is invertible: if $f(z) \in \mathcal{Z}(D; p(m-1))$, then $\rho(n) < \frac{C}{(\ln n)^p}$.

Now, let $A = \infty$. We denote

$$M_n = \max_{z \in \overline{B}_\xi} |f^{(n)}(z)|.$$

The rate of increase of the numbers M_n depends on the rate of decrease of the numbers $\rho(n)$ and will be closer to the rate of increase of $C^n n!$, the closer $\rho(n)$ to any geometric progression. Namely, we will show that for any $\mu, 0 < \mu < 1$

$$M_n < C^n n! \sum_{k=1}^{\infty} \frac{(\rho(n_{k-1}))^\mu}{\left(d\left(\xi; \frac{1-\mu}{n_k} \ln \frac{1}{\rho(n_k)}\right)\right)^n}. \quad (2.8)$$

In the case of $\rho(n) < q^n$ ($q < 1$) this shows that $M_n < C_0^n n!$ (C_0 does not depend on n), i.e., $f(z)$ is analytic in $\overline{D}$.

Indeed, for any $\alpha > 0$, using the Cauchy integral, one can obtain the estimate

$$|\mathcal{P}_{n_k}^m(z) - \mathcal{P}_{n_{k-1}}^m(z)| < C^m m! \frac{\rho(n_{k-1})(1+\alpha)^{n_k}}{(d(\xi; \alpha))^m}; \quad (2.9)$$

putting $\alpha = \frac{1-\mu}{n_k} \ln \frac{1}{\rho(n_k)}$ and taking into account (2.9) we obtain, summing over k , the required inequality. In particular, let

$$d(\xi; \alpha) \simeq \alpha^\beta \quad (0 < \beta < 1) \quad \text{and} \quad \rho(n) < e^{-n^\alpha} \quad (0 < \alpha < 1).$$

In this case, it is easy to calculate the right-hand side of (2.8):

$$M_n < C^n n! n^{n \frac{\beta(1-\alpha)}{\alpha}} \quad (2.10)$$

For α close to one, the estimate (2.10) does not differ much from the exact one. Indeed, consider the function

$$\mathcal{F}(z) = \int_0^1 e^{-\varphi^{-1}(d^{-1}(x))} \sqrt{z-x} dx, \quad (2.11)$$

where $\varphi(n) = \frac{1}{n} \ln \frac{1}{\rho(n)}$, and φ^{-1} and ρ^{-1} are functions inverse to the corresponding ones, and z belongs to the domain D , symmetrically located relative to the OX axis so that the distance of its boundary point $z = 0$ to the level line $\mathcal{Z}_{1+\varepsilon}$ is equal to $d(0; \varepsilon) = d(\varepsilon)$ and $\overline{D}$ does not intersect with $\arg z = 0$ anywhere other than $z = 0$.

Since the function

$$\mathcal{F}_\alpha(z) = \int_{d(0;\alpha)}^1 e^{-\varphi^{-1}(d^{-1}(x))} \sqrt{z-x} dx$$

is regular in the domain $D_{1+\alpha}$ bounded by $\mathcal{L}_{1+\alpha}$, then, as is known, there exists a polynomial $\mathcal{P}_n(z)$ so that

$$\max_{z \in \overline{D}} |\mathcal{F}_\alpha(z) - \mathcal{P}_n(z)| < C \frac{1}{\alpha^2(1+\alpha)^n} \quad (C = \text{const}).$$

But

$$\max_{z \in \overline{D}} |\mathcal{F}(z) - \mathcal{F}_\alpha(z)| < \int_0^{d(0;\alpha)} e^{-\varphi^{-1}(d^{-1}(x))} \sqrt{z-x} dx < e^{-\varphi^{-1}(\alpha)}.$$

Setting $\alpha = \varphi([n\varphi(n)])$ we obtain

$$\max_{z \in \overline{D}} |\mathcal{F}(z) - \mathcal{P}_n(z)| < C\rho(n).$$

Let us estimate $|\mathcal{F}^{(n)}(0)|$ from below. We have

$$|\mathcal{F}^{(n)}(0)| = \frac{1 \cdot 3 \dots 2n-1}{2^n} \int_0^1 \left(\frac{1}{x}\right)^{\frac{2n+1}{2}} e^{-\varphi^{-1}(d^{-1}(0;x))} dx.$$

Estimating the integral and assuming the existence of $d''(0;\alpha)$, $\varphi''(x)$, we obtain

$$|\mathcal{F}^{(n)}(0)| \geq C^n n! \int_1^\infty \frac{d'(0; \varphi(x)) \varphi'(x) e^{-x} dx}{(d(0; \frac{1}{x} \ln \frac{1}{\rho(x)}))^n}. \quad (2.12)$$

In our case $d(0;\alpha) \simeq \alpha^\beta$ and $\rho(x) \simeq e^{-x^\alpha}$; substituting in (2.12) we have

$$M_n \geq |\mathcal{F}^{(n)}(0)| > C^n n! n^{n\beta(1-\alpha)},$$

that is, for small $1-\alpha$ the estimate (2.10) does not differ much from the exact one.

So, for an arbitrarily slow rate of approximation, the corresponding structure of the domain can guarantee sufficiently “good” properties of the function at some boundary points. Therefore, the question arises: is it possible, by means of an appropriate construction of the domain, to ensure that the rate of approximation, different from the progression, would guarantee the analyticity of the function at some points on the boundary?

Regarding this question, the following can be stated:

Theorem 2.2. For any function $\varphi(n) > 0$ satisfying, for any $k > 1$, the condition

$$\lim_{n \rightarrow \infty} k^n \varphi(n) = \infty \quad (2.13)$$

and any bounded domain D of the Carathéodory class, there exists a function $f(z)$ regular in D , continuous in $\overline{D}$, the rate of approximation to which satisfies the inequality

$$\rho(n) < \varphi(n), \quad (2.14)$$

however for which the boundary of D is a cut.

Proof. Let us denote, for brevity,

$$\rho_n = 1 + \theta \frac{\ln n}{n} + \frac{1}{n} \ln \frac{1}{\varphi(n)} \quad (3 < \theta < 4). \quad (2.15)$$

From (2.13) it follows that $\rho_n \rightarrow 1$ on the outer line of the level $\mathcal{Z}_{\rho_n}$; we mark a point α_n so that for the set $\{\alpha_i\}$, each boundary point of D would be a limit point. This is obviously possible, since $\rho_n \rightarrow 1$ implies that $\mathcal{Z}_{\rho_n} \rightarrow \Gamma$.

Let δ_n denote the distance of the point α_n to $\mathcal{Z}_{\rho_{2n}}$. Let us create the function

$$f(z) = \sum_{k=1}^{\infty} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)}, \quad 0 < \varepsilon_i < 1.$$

We have, obviously,

$$\max_{z \in \bar{D}} \left| f(z) - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)} \right| \leq \frac{1}{2^{\frac{n}{2}}} < \frac{\varphi(n)}{2}, \quad n > n_0.$$

In addition, since the function $\sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)}$ is regular in $\bar{D}_{\rho_n}$ and uniformly bounded in n , then there exists a polynomial $\mathcal{P}_n(z)$ for which

$$\max_{z \in \bar{D}} \left| \mathcal{P}_n(n) - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)} \right| \leq \frac{C}{(\rho_n - 1)^3 \rho_n^n} < \frac{\varphi(n)}{2}, \quad n > n_1.$$

Hence,

$$\rho_n(f; D) \leq \max_{z \in \bar{D}} |f(z) - \mathcal{P}_n(z)| < \varphi(n).$$

Let us show that the boundary of D is a cut-off for $f(z)$, if only ε_k decreases sufficiently quickly.

Let $f(z)$ be analytically extendable through the continuum $K \in \Gamma$ into the domain G . We enclose each of the points α_v in a circle K_v with the center at α_v and radius r_v so small that none of the pairs of discs $|z - \alpha_i| < r_i$ and $|z - \alpha_j| < r_j$ will have common points. In the domain G we fix a subdomain G^* lying outside all circles K_v , $v = 1, 2, \dots$

Let $\gamma(z)$ denote the sum of angles at which the continuum K is visible from point z in domain G ; by $\varphi_n(z)$ we denote the angle at which the disc $|z - \alpha_n| \leq r_n$ is visible from z in G . Also, let

$$\gamma_n = \min_{z \in G^*} [\gamma(z) - \sum_{k=1}^n \varphi_k(z)].$$

It is easy to see that G^* can be chosen so that

$$\min_{z \in G^*} \gamma(z) > 0.$$

We will assume, in addition, that the r_v decreases so quickly to zero that the numbers $\gamma_n > 0$, $n = 1, 2, 3, \dots$, and the function

$$r_n(z) = f(z) - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)}$$

already defined according to the assumption in $D + G + K$ for $\varepsilon_k < r_k$ is bounded in the domain G_n obtained from G by removing $\sum_{v=1}^n K_v$ by a number M independent of n . On the boundary continuum K of the domain G_n we have

$$|r_n(z)| < \left| \sum_{\left[\frac{n}{2}\right]+1}^{\infty} \frac{\delta_k \varepsilon_k}{2^k(z - \alpha_k)} \right| < \sum_{\left[\frac{n}{2}\right]+1}^{\infty} \frac{\varepsilon_k}{2^k}.$$

Now let us choose ε_k so that

$$1. \varepsilon_k < r_k, \quad 2. \left[\sum_{[\frac{n}{2}]+1}^{\infty} \frac{\varepsilon_i}{2^i} \right]^{\gamma_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

According to the well-known Lindelöf theorem, if in some domain B there is an analytic function $\mathcal{F}(z)$ such that on some arc γ of the boundary B visible from the point $z_0 \in B$ at an angle $\frac{2\pi}{q}$

$$|\mathcal{F}(z)| \leq m,$$

and on the remaining part of the boundary B

$$|\mathcal{F}(z)| \leq N,$$

then

$$|\mathcal{F}(z_0)| < N^{1-\frac{1}{q}} m^{\frac{1}{q}}.$$

Applying this theorem to our case and considering as B the domain G_n , and $\mathcal{F}(z)$ the function $r_n(z)$, we obtain

$$\max_{z \in G^*} |r_n(z)| \leq M \text{const} \left[\sum_{[\frac{n}{2}]+1}^{\infty} \frac{\varepsilon_i}{2^i} \right]^{\gamma_n} \rightarrow 0,$$

that is, the series

$$\sum_{k=1}^{\infty} \frac{\delta_k \varepsilon_k}{2^k (z - \alpha_k)}$$

in the domain G^* represents the analytic continuation of $f(z)$.

But it is easy to see that if we consider the sum of a series outside $\overline{D}$, then for it the boundary of D is a cut, that is, we have obtained a contradiction, since, on the other hand, this sum represents in G^* a function that can be analytically extended into the domain D . $\square$

Now consider the case when

$$A = \liminf_{n \rightarrow \infty} \frac{\ln \rho_n(f; D)}{\ln d(\xi; \frac{1}{n})} = 0.$$

In this case, $f(z)$ may already be non-differentiable and not satisfy any Lipschitz condition of positive order; however, its modulus of continuity $\omega(\delta)$ in $\overline{B}_\xi$ satisfies the following constraint.

Theorem 2.3. For some $C > 0$ independent of δ

$$\omega(\delta) < C \min_{n \geq 1} \left\{ \rho_n(f; D) + \frac{\delta}{d(\xi; \frac{1}{n})} \right\}.$$

It will be shown later that, in the general case, this estimate cannot be improved.

Proof. Indeed, let $\mathcal{P}_n(z)$ be a polynomial that least deviates from $f(z)$ in $\overline{D}$ of degree n . We have

$$|f(z') - f(z'')| \leq |f(z') - \mathcal{P}_n(z')| + |f(z'') - \mathcal{P}_n(z'')| + |\mathcal{P}_n(z') - \mathcal{P}_n(z'')| \leq 2\rho_n(f; D) + \max_{z \in \overline{B}_\xi} |\mathcal{P}_n(z)| \int_{z'}^{z''} ds.$$

But $\max_{z \in \overline{B}_\xi} |\mathcal{P}_n(z)|$ is easy to estimate using the Cauchy integral so that

$$\omega(\delta) < C \min_{n \geq 1} \left\{ \rho_n(f; D) + \frac{\delta}{d(\xi; \frac{1}{n})} \right\}.$$

In particular, if $d(\xi; \alpha) > \alpha^\beta$ and $\rho(f; D) < \frac{1}{(\ln)^\gamma}$ ($\beta > 0, \gamma > 0$), we have

$$\omega(\delta) < \frac{\text{const}}{|\ln \delta|^\gamma}.$$

□

Let us note one corollary of Theorem 2.1.

Corollary 2.3. *Let the domain D be convex or bounded by a polygonal line with a finite number of segments. If $f(z)$ can be approached in $\overline{D}$ with the rate*

$$\rho_n(f; D) < \frac{C}{n^p},$$

then $f(z) \in \mathcal{Z}(\overline{D}; \frac{p}{2})$.

Indeed, the inequality $d(\xi; \alpha) > \alpha^2$ follows for convex regions and polygons from Lindelöf's principle, since there always exists a segment located in $\overline{D}$ and having one of its ends at the boundary point ξ ; it is enough now to compare the lines of the external level of D and the complement to the mentioned segment.

Note that these theorems apply not only to Jordan domains, but to any domains of the Carathéodory class.

3. Estimation of the Rate of Approximation for Domains with a Smooth Boundary

Theorem 3.1. *If the domain D is bounded by a smooth curve and $f(z)$ is analytic in D and continuous in $\overline{D}$, and the k^{th} derivative of the function $f(z)$ satisfies a Lipschitz condition of order α , $0 < \alpha \leq 1$, in $\overline{D}$, then for any $\varepsilon > 0$,*

$$\rho_n(f; D) < \frac{\text{const}}{n^{k+\alpha-\varepsilon}}, \quad (3.1)$$

where const does not depend on n .

Proof. It is sufficient to provide the proof for the case $k = 0$. The general case is derived from here by a well-known trick.

Let $\varepsilon > 0$ be an arbitrarily small fixed number; let us assume that the projection of D onto the axis OX contains the interval $[0; 1]$ and the straight lines $x = a, x = b$ ($0 < a < b < 1$) are drawn so that they do not touch the boundary of D and, therefore, intersect it at a finite number of points.

The part of D located to the right of $x = a$ will be denoted by D_a , and the part of D located to the left of $x = b$ by D_b . The open sets D_a and D_b obviously consist of a finite number of simply connected regions located at a positive distance from each other:

$$D_a = \sum_{i=1}^p D_a^{(i)}, \quad D_b = \sum_{i=1}^q D_b^{(i)}.$$

Let $z = z(s)$ denote the parametric equation of the boundary of the domain D , and the parameter s represent the length of the arc $\Gamma = \overline{D} - D$ from some fixed point $z(0)$ to $z(s)$. We choose the number $\rho < 1$ so close to unity that the angle made by the normal to the boundary Γ at a point $z(s)$ with the internal level line $\mathcal{Z}_\rho$ ($\rho < 1$) at the point closest to $z(s)$

would differ from $\frac{\pi}{2}$ by less than ω for any value of s . The ring-shaped region enclosed between Γ and $\mathcal{Z}_\rho$ we denote by l .

The part of l located by the inner normal to Γ at $z(0)$ and the inner normal to Γ at $z(s)$ are drawn until their intersection with $\mathcal{Z}_\rho$, denoted by $\sigma(s)$.

Let us choose δ so that the set

$$\Delta(s; \delta) = \sigma(s + \delta) - \bar{\sigma}(s)$$

represents a star-shaped domain with respect to some point $z(s; \delta)$ for all values s ($\delta = \delta(s)$). (For $\delta = \delta(s)$, it is sufficient to take the length of the segment of the internal normal to Γ at $z(s)$, enclosed between $z(s)$ and the points closest to $z(s)$).

$\Delta(s; \delta)$ is a curvilinear quadrilateral with angles close to right angles and depends on ρ . We denote by J_s the line passing through $z(s)$ and orthogonal at $z(s)$ to Γ , and by J'_s the line passing through $z(s + \frac{\delta}{2})$ and parallel to J_s . It is obvious that the numbers ω , ρ and δ can, in addition, be chosen so that the part of l located between J_s and J'_s would belong to the intersection of the domains $\Delta(s; \delta)$ and $\sigma(s + \frac{3}{4}\delta)$ for all s .

Consider one of the components D_a —for example, $D_a^{(1)}$. We denote the part of $\sigma(s)$ located in $D_a^{(1)}$ by $\sigma_1(s)$. Suppose that $z(0)$ lies outside $\bar{D}_a$; let s_0 be the largest of the s for which $\sigma(s)$ has no common points with $D_a^{(1)}$, and let s_1 be the smallest of those s for which the intersection of l and $D_a^{(1)}$ coincides with $\sigma_1(s)$.

Let $[s_0; s_1]$ denote the largest interval that has the property that $\sigma_1\sigma(s_1) - \bar{\sigma}(s_0)$ belongs to $D_a^{(1)}$.

The part $\sigma_1(s)$ located on the same side of the line J_s as J'_s will be denoted by $\sigma_{11}(s)$. The set of points located from $\sigma_{11}(s)$ at a distance less than λ will be denoted by $\sigma_{11\lambda}(s)$; $\sigma_{1\lambda}(s)$ is the set of points separated from $\sigma_1(s)$ by a distance less than λ .

Suppose that there exists a polynomial $\pi_1(z)$ of degree n with the following properties:

1. $|\pi_1(z) - f(z)| < \frac{C_1}{n^{1-\varepsilon}}, \quad z \in \bar{\sigma}_1(s + \frac{3}{4}\delta)$
2. $|\pi_1(z)| < M, \quad z \in \bar{\sigma}_{1\lambda_n}(s + \frac{3}{4}\delta), \quad s_0 < s < s_1,$

where M does not depend on n and s , s is fixed, $\lambda_n = \frac{1}{n^{1-\varepsilon}}$.

The domain $\Delta(s; \delta)$ is star-shaped with respect to $z(s; \delta)$, so the function

$$f_1(z) = f\left(z(s; \delta) + \frac{z - z(s; \delta)}{1 + \lambda}\right)$$

is analytic in the domain $\Delta_\lambda(s; \delta)$ obtained from $\Delta(s; \delta)$ by stretching with respect to the point $z(s; \delta)$ $1 + \lambda$ times; there is a constant $C_2 > 0$ for which the image of the circle $|w| = 1 + 2\lambda$ when mapping $|w| > 1$ onto the complement of $\bar{\Delta}(s; \delta)$ is located entirely in $\Delta_{C_2\lambda^{1-\varepsilon}}(s; \delta)$ (we are only interested in small values $\lambda > 0$).

This follows from the fact that the distance of the points of the level line $\mathcal{Z}_R$ ($R > 1$) of the domain D , bounded by a finite number of smooth curves making angles with each other, the internal openings of which do not exceed π , up to the boundary D does not exceed $C_3(R - 1)^{1-\varepsilon}$ uniformly with respect to the points $\mathcal{Z}_R$.

According to the remark (property 1), there is a polynomial $\pi_2(z)$ of degree n such that

$$\max_{z \in \mathcal{Z}_{1+\lambda}} |\pi_2(z) - f_1(z)| < \frac{C_4}{\lambda^3(1 + \lambda)^n}$$

(by $\mathcal{Z}_{1+\lambda}$ we mean the level line of the complement to $\Delta(s; \delta)$).

At the same time, in $\Delta(s; \delta)$, the inequality will be satisfied,

$$|\pi_2(z) - f(z)| < \frac{C_4}{\lambda^3(1 + \lambda)^n} + C_5\lambda^{\alpha(1-\varepsilon)},$$

since for $z \in \Delta(s; \delta)$

$$|f(z) - f_1(z)| \leq K(C_2\lambda^{1-\varepsilon})^\alpha \max |z - z(s; \delta)| < C_5\lambda^{\alpha(1-\varepsilon)}.$$

Let us cover the domain $\Delta_{C_2\lambda^{1-\varepsilon}}(s; \delta)$ with the domain $G(K)$, and $\sigma_{11\lambda}(s)$ by the domain $B(K)$ so that $G(K)$ and $B(K)$ would be at a positive distance from each other exceeding $K > 0$, as well as the distance of any point $G(K)$ to $\Delta_{C_2\lambda^{1-\varepsilon}}(s; \delta)$; likewise, the distance of any point $B(K)$ to $\sigma_{11\lambda}(s)$ would exceed K (it is obviously possible to find such $K > 0$, independent of λ).

As is well known, there exists a polynomial $\pi_3(z)$ of degree n such that

$$\begin{aligned} |\pi_3(z) - \pi_1(z)| &< C_6q^n, \quad z \in \bar{\sigma}_{11\lambda}(s), \\ |\pi_3(z) - \pi_2(z)| &< C_6q^n, \quad z \in \bar{\Delta}_{C_2\lambda^{1-\varepsilon}}(s; \delta), \end{aligned}$$

where $q < 1$ depends on K . $\square$

Now let G be a bounded domain with a simply connected complement, whose projection onto the OJ axis is greater than one, and $f(z)$ be regular in G and continuous in $\bar{G}$. $G_{1+\lambda}$ is the domain bounded by the outer line of level $\mathcal{L}_{1+\lambda}$ of the domain G ; $\pi_4(z)$ and $\pi_5(z)$ are two polynomials of degree n with the following properties :

- (1) $|\pi_4(z)| < M$ in the part $\bar{G}_{1+\lambda}$, which is located above the straight line $y = c$; similarly $|\pi_5(z)| < M$ in the part $G_{1+\lambda}$, located below the straight line $y = d$; we assume that the segment $(c; d)$ belongs to the projection of G onto the axis OJ , and $d - c = 1$.
- (2) In the part G located in the half-plane $y \leq d$,

$$|\pi_5(z) - f(z)| < \frac{C_7}{n^{\alpha-\varepsilon}};$$

in the part G located in the half-plane $y \geq c$,

$$|\pi_4(z) - f(z)| < \frac{C_7}{n^{\alpha-\varepsilon}}.$$

Let us additionally assume that the distance of any point $\mathcal{L}_{1+\lambda}$ to $\bar{G} - G$ is less than $C_8\lambda_{1-\varepsilon}$.

Lemma 3.1. *There exists a polynomial π_6 of degree n for which*

1. *in the domain $G_{1+\frac{\lambda}{2}}$*

$$|\pi_6(z)| < 2M,$$

2. *and in the domain G*

$$|\pi_6(z) - f(z)| < \frac{C_9}{n^{\alpha-3\varepsilon}}.$$

Proof. The proof is based on the averaging method of Academician M.V. Keldysh [6].

Let us denote

$$\varphi(z; t) = \begin{cases} \pi_4(z), & t \geq \frac{c+d}{2}, \\ \pi_5(z), & t < \frac{c+d}{2}. \end{cases}$$

Let $\zeta = \xi + i\eta$. The formula

$$\varphi(z) - \frac{1}{2\pi i} \int_{\mathcal{L}_{1+\lambda}} \frac{\varphi(t)dt}{t-z} = \frac{1}{4\pi} \iint_{G_{1+\lambda}} \frac{\varphi(\zeta; \eta + \frac{1}{2}) - \varphi(\zeta; \eta - \frac{1}{2})}{\zeta - z} d\zeta d\eta \quad (3.2)$$

was obtained in [6].

Let us split the integral on the right side (3.2) into two parts: over the domain G and over the remaining domain $G_{1+\lambda} - \bar{G}$. By adding and subtracting the fractions under the integral on the right side (3.2) $f(z)$ in the numerator, we make sure that the first term does not exceed

$$\frac{C_{10}}{n^{\alpha-\varepsilon}} \iint_G \frac{d\zeta d\eta}{|\zeta - z|} \leq \frac{C_{11}}{n^{\alpha-\varepsilon}}.$$

We estimate the second term based on property 1, which is satisfied by the polynomials $\pi_4(z)$ and $\pi_5(z)$:

$$\left| \frac{1}{4\pi} \iint_{G_{1+\lambda} - \bar{G}} \frac{\varphi(\zeta; \eta + \frac{1}{2}) - \varphi(\zeta; \eta - \frac{1}{2})}{\zeta - z} d\zeta d\eta \right| \leq 2M \frac{1}{4\pi} \iint_{G_{1+\lambda} - \bar{G}} \frac{d\zeta d\eta}{|\zeta - z|} \leq C_{12} \lambda^{1-\varepsilon} \ln \frac{1}{\lambda}.$$

But obviously

$$|\varphi(z) - f(z)| \leq \int_{\varphi - \frac{1}{2}}^{\varphi + \frac{1}{2}} |\varphi(z; t) - f(z)| dt < \frac{C_7}{n^{\alpha-\varepsilon}}$$

so that in the domain G

$$\left| \frac{1}{2\pi i} \int_{\mathcal{Z}_{1+\lambda}} \frac{\varphi(t)}{t-z} dt - f(z) \right| < \frac{C_7}{n^{\alpha-\varepsilon}} + \frac{C_{12} \ln n}{n^{(1-\varepsilon)^2}} < \frac{C_{13}}{n^{\alpha-3\varepsilon}}.$$

The function $\frac{1}{2\pi i} \int_{\mathcal{Z}_{1+\lambda}} \frac{\varphi(t)}{t-z} dt$ is analytic in $G_{1+\lambda_n}$ and, as follows from (3.2), is bounded there by a constant independent of n , so we can find a polynomial $\pi_7(z)$ of degree n so that

$$\max_{z \in \bar{G}_{1+\frac{\lambda_n}{2}}} \left| \pi_7(z) - \frac{1}{2\pi i} \int_{\mathcal{Z}_{1+\lambda}} \frac{\varphi(t)}{t-z} dt \right| < \frac{C_{14}}{(\frac{1+\lambda_n}{1+\frac{\lambda_n}{2}} - 1)^3 (\frac{1+\lambda_n}{1+\frac{\lambda_n}{2}})^n} < \frac{C_{15}}{n}, \quad n \geq n_0.$$

Consequently, the polynomial $\pi_7(z)$ satisfies all the conditions of the lemma. $\square$

Note that the assumption $d - c = 1$ is not significant and was made for simplicity. In the case of $d - c \neq 1$, only the constant C_9 , which is included in the estimate as a factor, will change. It is also unimportant that the parallels of the OJ axis are taken as two straight lines; the general case can be reduced to this case by rotation.

After this remark, we can apply the lemma to the case when as G we have the domain $\sigma_1(s + \delta)$, as straight lines $y = c$, $y = d$ – straight lines J_s , J'_s and polynomials $\pi_4(z)$ and $\pi_5(z)$ – polynomials $\pi_3(z)$ and $\pi_1(z)$, respectively.

Thus, assuming the existence of a polynomial $\pi_1(z)$ that approximates the function $f(z)$ in $\sigma_1(s + \frac{3}{4}\delta)$ with the rate $C n^{-\alpha+\varepsilon}$ and is bounded in a certain neighborhood $\sigma_1(s + \frac{3}{4}\delta)$, we come to the existence of a polynomial $\pi_7(z)$ approximating $f(z)$ in $\sigma_1(s + \delta)$ with rate $C n^{-\alpha+C_{16}\varepsilon}$ and bounded in the corresponding neighborhood $\sigma_1(s + \delta)$.

But it is easy to see that $\sigma_1(s_0)$ is star-shaped with respect to one of its points and for it the corresponding polynomial $\pi_1(z)$ exists; the same can be stated with respect to $\sigma_1(s_0 + \frac{3}{4}\delta(s_0)) - \bar{\sigma}_1(s_0)$; next we divide the interval (s_0, s_1) into parts of the corresponding length:

$$s_0, s^{(1)} = s_0 + \frac{\delta(s_0)}{4}, s^{(2)} = s_1 + \frac{\delta(s_1)}{4}, \dots, s^{(m)} = s_{m-1} + \frac{\delta(s_{m-1})}{4} > s_1,$$

and, successively applying the averaging process described above to the domains $\sigma_1(s_0)$, $\sigma_1(s^{(1)})$, ..., $\sigma_1(s_1)$, we finally obtain a polynomial $\pi_8(z)$ of degree n satisfying the inequalities

$$\begin{aligned} |f(z) - \pi_8(z)| &< \frac{C_{17}}{n^{\alpha-C_{18}\varepsilon}} \quad \text{in the domain } \sigma_1(s_1), \\ |\pi_8(z)| &\leq C_{20}M \quad \text{in the domain } \sigma_{1C_{19}\lambda_n^{1-\varepsilon}}(s_1). \end{aligned} \quad (3.3)$$

Carrying out a similar reasoning for each of the components D_a and keeping in mind that they are all located at a positive distance from each other, independent of a , we can assume that the inequalities (3.3) are satisfied in part l_a of domain l located to the right of $x = a$; we also find the polynomial $\pi_9(z)$ that approximates $f(z)$ in part l_b of domain l located to the left of $x = b$, with a rate $n^{-\alpha+C_{21}\varepsilon}$ and bounded in the appropriate neighborhood l_b . Finally, we apply the averaging process to the domains l_a and l_b and the polynomials $\pi_8(z)$ and $\pi_9(z)$.

It is easy to see that, slightly changing the proofs of formula (3.2), we can obtain the formula for our case:

$$\varphi(z) - \frac{1}{2\pi i} \int_{\mathcal{Z}_{1+\lambda_n}} \frac{\varphi(t)}{t-z} dt - \frac{1}{2\pi i} \int_{\mathcal{Z}_\rho} \frac{\varphi(t)}{t-z} dt = \frac{1}{4\pi} \iint_{d_n} \frac{\varphi(\zeta; \eta + \frac{1}{2}) - \varphi(\zeta; \eta - \frac{1}{2})}{\zeta - z} d\zeta d\eta \quad (3.4)$$

(the notation is similar to the notation above; the distance between b and a is set equal to one for simplicity).

Let $z \in \mathcal{Z}_{1+\lambda}$; since $|\varphi(t) - f(t)| \leq \frac{C_{22}}{n^{\alpha-C_{23}\varepsilon}}$ for $t \in \mathcal{Z}_\rho$ ($\rho < 1$), then, due to the analyticity of $f(z)$ in the domain D , we have

$$\left| \frac{1}{2\pi i} \int_{\mathcal{Z}_\rho} \frac{\varphi(t)}{t-z} dt \right| < \frac{C_{24}}{n^{\alpha-C_{25}\varepsilon}}.$$

We estimate the double integral on the right-hand side of (3.4) in the same way as was shown in the proof of Lemma 3.1.

The polynomial $\pi_{10}(z)$ of degree n is found so that

$$\max_{z \in \overline{D}} \left| \frac{1}{2\pi i} \int_{\mathcal{Z}_{1+\lambda_n}} \frac{\varphi(t)}{t-z} dt - \pi_{10}(z) \right| < \frac{1}{n}.$$

As a result we have

$$\left| \pi_{10}(z) - f(z) \right| < \frac{C_{26}}{n^{\alpha-C_{27}\varepsilon}},$$

where C_{27} does not depend on n .

Choosing $\varepsilon < \frac{\varepsilon_1}{C_{27}}$ sufficiently small, we arrive at the proof of Theorem 3.1.

If $\omega(\delta)$ denotes the modulus of continuity of $f(z)$ in D , then the following proposition can be proven in a completely analogous manner.

Theorem 3.2. For any $\varepsilon > 0$ there is a constant C such that

$$\rho_n(f; D) < C \omega\left(\frac{1}{n^{1-\varepsilon}}\right). \quad (3.5)$$

Let us now consider the connection between the rate of best approximation and the properties of functions under some additional restrictions on the smoothness of the boundary.

Let $\gamma(\delta)$ denote the modulus of continuity of the function $z'(s)$ ($z(s)$ is the parametric equation of the boundary; s is the length of the arc Γ).

Theorem 3.3. If $f(z) \in \mathcal{Z}(\overline{D}; p)$ and

$$\int_0^a \frac{\gamma(x)}{x} dx < \infty, \quad (3.6)$$

then

$$\rho_n(f; D) < \text{const} \left(\frac{\ln n}{n} \right)^p; \quad (3.7)$$

if

$$\int_\varepsilon^a \frac{\gamma(x)}{x} dx > |\ln \ln \varepsilon| |\ln \ln \ln \varepsilon|, \quad (3.8)$$

then, generally speaking, there is a function $f(z) \in \mathcal{Z}(D; p)$ such that

$$\limsup_{n \rightarrow \infty} \frac{\rho_n(f; D) n^p}{(\ln n)^p} = \infty. \quad (3.9)$$

Proof. Let $z = z(s)$ be some point on Γ . Placing the origin of the polar coordinates at z , we notice that the angle $z(s - \Delta s), z(s), z(s + \Delta s)$ differs from π by less than $2\gamma(\Delta s)$; but the quantity $\theta(\Delta s)$ for a small Δs can be replaced by the vertex angle $z(s)$ in the triangle $z(s - \Delta s), z(s), z(s + \Delta s)$; hence,

$$|\pi - \theta(\Delta s)| < 3\gamma(\Delta s), \quad |\Delta s| < \varepsilon_0.$$

From here we conclude that

$$\left| \pi \int_\rho^a \frac{dr}{r\theta(r)} - \ln \frac{1}{\rho} \right| \leq \frac{4}{\pi} \int_0^a \frac{\gamma(x)}{x} dx = \text{const},$$

i.e.,

$$\left| e^{-\pi \int_\rho^a \frac{dr}{r\theta(r)}} - \rho \right| < (C_0 - 1)\rho$$

so that

$$d(z; \alpha) < C_0 \alpha. \quad (3.10)$$

Let us assume that D is star-shaped with respect to the point z_0 . Using the technique we used to prove Theorem 1.2 and keeping in mind (3.10), we come to the proof in the case when $f(z)$ satisfies a Lipschitz condition of order δ . The general case is deduced from here in the same way as in the proof of Theorem 1.2.

Note that the star-shaped assumption can be eliminated by applying the averaging process described above, breaking up the domain D into overlapping star-shaped parts.

Let us now show that in the case of (3.8) the theorem is, generally speaking, incorrect, i.e., that for (3.8) there is, generally speaking, a function $f(z) \in \mathcal{Z}(D; p)$ for which the inequality (3.7) does not hold for any constant C .

Consider a domain D for which the modulus of continuity $z'(s)$ does not exceed $\gamma(\delta)$ and in some neighborhood of the point $z = 0$

$$\theta(r) = \pi + \gamma(r),$$

(such domains obviously exist). Using Warschawski's theorem, it is easy to obtain the inequality

$$d(0; x) > \text{const} \frac{x}{|\ln x| |\ln \ln x|}.$$

Now suppose that for some function $f(z)$ regular in D and continuous in $\overline{D}$ there exists a polynomial $\mathcal{P}_n(z)$ of degree n such that

$$\max_{z \in \overline{D}} |f(z) - \mathcal{P}_n(z)| < C \left(\frac{\ln n}{n} \right)^\alpha.$$

Let us estimate the modulus of continuity $f(z)$ in D :

$$|f(z') - f(z'')| \leq \sum_{k=1}^n \left| \{ \mathcal{P}_{2^k}(z') - \mathcal{P}_{2^{k-1}}(z') \} - \{ \mathcal{P}_{2^k}(z'') - \mathcal{P}_{2^{k-1}}(z'') \} \right| + \\ + |\mathcal{P}_1(z') - \mathcal{P}_1(z'')| + \sum_{k=n+1}^{\infty} \left| \mathcal{P}_{2^k}(z') - \mathcal{P}_{2^{k-1}}(z'') - \mathcal{P}_{2^k}(z'') + \mathcal{P}_{2^{k-1}}(z'') \right|.$$

From (2.7) we find

$$|\mathcal{P}'_{2^k}(z) - \mathcal{P}'_{2^{k-1}}(z)| < C \left(\frac{k}{2^k} \right)^{\alpha} \frac{2^k}{k \ln k} = C \frac{2^{k(1-\alpha)}}{k^{1-\alpha} \ln k}.$$

Taking this into account, we obtain

$$|f(z') - f(z'')| < C\delta \sum_{k=2}^n \frac{2^{k(1-\alpha)}}{k^{1-\alpha} \ln k} + C\delta + C \sum_{k=n+1}^{\infty} \frac{k^{\alpha}}{2^{k\alpha}}.$$

Evaluating the sums and setting n equal to the integer part of the solution to the equation

$$2^x = \frac{x}{\delta} \ln \ln \ln \frac{1}{\delta},$$

we find

$$\omega(\delta) = \sup |f(z') - f(z'')| < C \frac{\delta^{\alpha}}{\ln \ln \ln \delta}. \quad (3.11)$$

Let us now assume that $f(z) \in \mathcal{Z}(D; \alpha)$, but does not satisfy the additional condition (3.11) (for example, $f(z) = z^{\alpha}$); then, according to what has been proven, the following relation must be satisfied:

$$\lim_{n \rightarrow \infty} \sup \rho_n(f; D) \left(\frac{n}{\ln n} \right)^{\alpha} = \infty.$$

If, for example, $\gamma(r) = \frac{1}{|\ln r|^{\lambda}}$, then for $\lambda > 1$ the estimate (3.7) is correct; if $\lambda = 1$ or $\lambda < 1$, then it ceases to be valid in the general case.

Let

$$\gamma(r) = \frac{C}{|\ln r| |\ln \ln r| \dots |\ln_s r|^{\lambda}}.$$

□

Theorem 3.4. If $\lambda = 1$, then, generally speaking, for some $f(z) \in \mathcal{Z}(D; \alpha)$

$$\rho_n(f; D) > \text{const} \left(\frac{\ln_s n}{n} \right)^{\alpha};$$

if $\lambda < 1$, then for some $f(z) \in \mathcal{Z}(D; \alpha)$ the following inequality holds:

$$\rho_n(f; D) > \text{const} \left\{ \frac{\exp \frac{1}{1-\lambda} (\ln_s n)^{1-\lambda}}{n} \right\}^{\alpha}.$$

We do not present the proof, since it is similar to the proof of Theorem 3.3.

Let (3.6) hold and

$$\rho_n(f; D) < \frac{\text{const}}{n^{k+\alpha}}, \quad (3.12)$$

where k is an integer, $0 < \alpha \leq 1$.

Theorem 3.5. *If $\alpha < 1$, then $f(z) \in \mathcal{L}(D; k + \alpha)$; if $\alpha = 1$, then this conclusion, as is known, is not true for analytical domains. In the case of $\alpha = 1$, in order for $f^{(k)}(z)$ to satisfy a first-order Lipschitz condition, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \rho_n(f; D) n^k < \infty. \quad (3.13)$$

Proof. Using Warschawski's theorem it is easy to show that, given condition (3.6), there is a constant μ for which the inequality

$$d(\zeta; x) > \mu x$$

holds for all boundary points of ζ .

From here we conclude that

$$|\mathcal{P}'_{2^k}(z) - \mathcal{P}'_{2^{k-1}}(z)| < 2\rho_{2^{k-1}}(f; D)2^k. \quad (3.14)$$

Next, we proceed in exactly the same way as when proving the corresponding result in the real domain (S.N. Bernstein's theorem).

We now prove the second part of the theorem when $\alpha = 1$.

Condition (3.13) is necessary: assuming for simplicity $k = 0$, consider the function

$$f(z) = \sum_1^{\infty} z^n (\rho_n - \rho_{n+1})$$

in the unit circle, where $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n \geq \dots \rightarrow 0$ is an arbitrary sequence of monotonically decreasing numbers for which

$$\sum \rho_n = \infty.$$

The partial sums of its Taylor series approach it in $|z| \leq 1$ with a rate of ρ_n ; however, $f(z)$ does not satisfy a first-order Lipschitz condition, since for any arbitrarily large N , one can find n and $\delta < \frac{1}{n}$ so that

$$f(1) - f(1 - \delta) \geq \delta \sum_{k=1}^n \rho_k (1 - \delta)^k > \delta \frac{1}{e} \sum_{k=1}^n \rho_k > \delta N.$$

We prove the sufficiency of condition (3.13). Taking into account (3.14) it is easy to obtain the inequality

$$|\mathcal{P}_{2^n}^{(k)}(z') - \mathcal{P}_{2^{n-1}}^{(k)}(z') - \mathcal{P}_{2^n}^{(k)}(z'') + \mathcal{P}_{2^{n-1}}^{(k)}(z'')| \leq |z' - z''| 2 \cdot 2^{n(k+1)} \rho_{2^{n-1}}(f; D).$$

Based on (2.6) we have

$$|f^{(k)}(z') - f^{(k)}(z'')| < C_1 \delta \sum_{n=1}^N 2^{n(k+1)} \rho_{2^{n-1}}(f; D) + C_1 \delta + \sum_{i=N+1}^{\infty} 2^{ik} \rho_{2^i}(f; D).$$

Let us choose N depending on δ so that

$$\sum_{i=N+1}^{\infty} 2^{ik} \rho_{2^i}(f; D) < \delta.$$

□

Let us note the following proposition, the proof of which we will not dwell on.

Lemma 3.2. Let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \cdots \geq \alpha_n \geq \cdots \rightarrow 0$ be an arbitrary sequence of numbers monotonically decreasing towards zero, and $n_1 < n_2 < \cdots < n_k < \dots$ be an increasing sequence of integers.

In order for, from the convergence of the series

$$\sum_{i=1}^{\infty} \alpha_i,$$

the convergence of the series

$$\sum_{k=1}^{\infty} n_k \alpha_{n_k}$$

would always follow, it is necessary and sufficient that there exists a positive number ε such that for all k , starting from a sufficiently large one,

$$\frac{n_{k+1}}{n_k} > 1 + \varepsilon.$$

In particular, it follows that

$$\sum_{n=1}^{\infty} 2^{n(k+1)} \rho_{2^n}(f; D) < \infty,$$

that is, $f(z)$ satisfies a first-order Lipschitz condition.

As for the general case of domains with a smooth boundary that do not satisfy condition (3.6), then $f(z) \in \mathcal{Z}(D; p)$ giving an estimate $\rho_n(f; D)$ that relates to the entire class of domains with smooth boundaries and is better than

$$\rho_n(f; D) < \frac{C(\varepsilon)}{n^{p-\varepsilon}},$$

is impossible, since the following can be proven:

Proposition 3.1. Let $\psi(\delta)$ be a monotone function decreasing to zero for $\delta \rightarrow +0$ and for any $\varepsilon > 0$ satisfying the condition

$$\lim_{\delta \rightarrow 0} \frac{\delta^\varepsilon}{\psi(\delta)} = 0.$$

There is a domain D_1 with a smooth boundary such that the rate of approximation $n^{-k-\alpha}$ ($0 < \alpha \leq 1$) guarantees for some boundary points ζ the inequality

$$|f^{(k)}(z') - f^{(k)}(z'')| < \psi(|z' - z''|) |z' - z''|^\alpha \cdot \text{const}$$

(z' and z'' belong to $\overline{B_\zeta}$) and there is another domain D_2 , also with a smooth boundary, and a function $f(z)$ such that $\rho_n(f; D_2) < \frac{1}{n^{k+\alpha}}$; however, for some boundary points ζ

$$\sup_{z', z'' \in \overline{B_\zeta}} |f^{(k)}(z') - f^{(k)}(z'')| > \text{const} \frac{|z' - z''|^\alpha}{\psi(|z' - z''|)}.$$

The proof of this is similar to the proof given above for the second part of Theorem 3.3. From Theorem 2.3 it follows that if in a domain with a smooth boundary the inequality

$$\rho_n(f; D) < M(\ln n)^{-p},$$

holds, then

$$\omega(\delta) < \frac{C_1 M}{|\ln \delta|^p},$$

where C_1 does not depend on M and δ ; and vice versa, according to Theorem 3.2, from the inequality

$$\omega(\delta) < \frac{N}{|\ln \delta|^p}$$

it follows that

$$\rho_n(f; D) < \frac{C_2 N}{(\ln n)^p},$$

where C_2 does not depend on N and n .

From a comparison of these facts, it is easy to conclude that there exist domains with a smooth boundary and functions $f(z)$ that are regular in D , the modulus of continuity of which in $\overline{D}$ is denoted by $\omega(\delta)$, for which

$$A_1 \omega\left(\frac{1}{n}\right) < \rho_n(f; D) < A_2 \omega\left(\frac{1}{n}\right),$$

as well as

$$A_3 \min_{n \geq 1} \left\{ \rho_n(f; D) + \frac{\delta}{d(\zeta; \frac{1}{n})} \right\} < \omega(\delta) < A_4 \min_{n \geq 1} \left\{ \rho_n(f; D) + \frac{\delta}{d(\zeta; \frac{1}{n})} \right\},$$

where A_1, A_2, A_3 and A_4 do not depend on n and δ .

4. This Section is Missing from the Manuscript

As we noted above, Section 4 is indicated in the Introduction, but Section 4 is missing from the manuscript. The material intended for Section 4 is set out in Section 3.

5. Some Quasi-Analytic Classes of Functions

Academician S.N. Bernstein in 1923 showed [7] that the class of functions defined on $[0; 1]$, for which

$$E_n(f) < q^n, \quad n = n_1, n_2, n_3, \dots, n_k, \dots,$$

where $n_1 < n_2 < n_3 < \dots < n_k < \dots$ is a particular sequence of integers, is a quasi-analytic class, in the sense that if any two of its functions coincide on any part of the interval $[0; 1]$, then they are identical.

Below we give some other quasi-analytic classes of functions defined using best approximations.

Let $\varphi(n)$ be a positive function of the integer argument n . The class of functions for which

$$E_n(f) < \varphi(n), \quad n = n_1, n_2, \dots,$$

where $n_1 < n_2 < n_3 < \dots < n_k < \dots$ is some sequence of integers, is denoted by $\mathcal{Q}_{\varphi(n)}$.

Let $\psi(\delta)$ be a monotone function satisfying the condition

$$\lim_{\delta \rightarrow 0} \frac{\psi(\delta)}{\delta^n} = 0, \quad n = 1, 2, 3, \dots,$$

where δ_n means the root of the equation $\psi(\delta) = \delta^n C^n$ for some fixed $C > 0$.

Theorem 5.1. *If*

$$\varphi(n) = O(\delta_n^n),$$

then the class $\mathcal{Q}_{\varphi(n)}$ is quasi-analytic in the sense that if with respect to any two of its functions $f_1(x)$ and $f_2(x)$ it is known that

$$|f_1(x) - f_2(x)| < \psi(|x - x_0|),$$

where $x_0 \in [0; 1]$, then $f_1(x) \equiv f_2(x)$.

If, in particular, a function $f(x)$ of class $Q_{\varphi(n)}$ decreases around some point as $\psi(\delta)$, then $f(x) \equiv 0$ is required.

Proof. Let $f_1(x) - f_2(x) = f(x)$ and

$$|f(x) - \mathcal{P}_n(x)| < K\delta_n^n < K\psi(\delta_n), \quad n = n_1, n_2, \dots;$$

since on the interval $(x_0 - \varepsilon; x_0 + \varepsilon)$ $|f(x)| < \psi(\varepsilon)$, then

$$\max_{|x-x_0|<\varepsilon} |\mathcal{P}_n(x)| < K\psi(\delta_n) + \psi(\varepsilon).$$

Let $x_1 \in [0; 1]$ and $\mathcal{L}_{C_1}$ ($C_1 > 1$) be the level line of the complement of $[x_0 - \varepsilon; x_0 + \varepsilon]$, passing through x_1 ($x_1 \in [x_0 - \varepsilon; x_0 + \varepsilon]$). We choose a point x_1 so close to x_0 that the following inequality holds:

$$C C_1 \varepsilon < \theta < 1$$

(a simple calculation shows that C_1 depends on x_1 in such a way that this can always be performed). Then, as is known,

$$|\mathcal{P}_n(x_1)| < \frac{\theta^n}{(C\varepsilon)^n} (K\psi(\delta_n) + \psi(\varepsilon)), \quad n = n_1, n_2, \dots$$

Let $\varepsilon = \delta_n$; hence, it follows that

$$\lim_{k \rightarrow \infty} \mathcal{P}_{n_k}(x_1) = 0.$$

The same can be said regarding the points x , $|x - x_0| < |x_1 - x_0|$, so $f(x) = 0$ on the segment $[x_0; x_1]$ and

$$|\mathcal{P}_n(x)| < \psi(\delta_n)K, \quad x \in [x_0; x_1], \quad n = n_1, n_2, \dots$$

If x is any point in $[0; 1]$, and $\mathcal{L}_q$ is the level line of the complement of $[x_0; x_1]$, passing through x , then

$$|\mathcal{P}_n(x)| < q^n \psi(\delta_n)K = KC^n \delta_n^n q^n, \quad n = n_1, n_2, \dots,$$

and since for sufficiently large n the inequality $C\delta_n q < \theta < 1$ holds, then $\mathcal{P}_{n_k}(x) \rightarrow 0$, that is, $f(x) \equiv 0$.

If, in particular, $\psi(\delta) = \exp(-\delta^{-\lambda})$, then

$$\psi(\delta_n) = \exp\left(-\frac{n \ln n}{\lambda}\right).$$

□

Let $\mathcal{F}$ be a closed bounded set that does not break up the plane, and M be an infinite set of points belonging to $\mathcal{F}$ ($\mathcal{F}$ is assumed to be infinite).

We denote by $\mathcal{U}_{\varphi(n)}$ the class of functions continuous on $\mathcal{F}$ for which

$$\rho_n(\mathcal{F}; f) < \varphi(n), \quad n = n_1, n_2, \dots$$

Theorem 5.2. For any infinite set M there is a positive function $\psi_M(n)$ for which the class $\mathcal{U}_{\psi_M(n)}$ is quasi-analytic in the sense that from the coincidence of any two of its functions $f_1(z)$ and $f_2(z)$ on the set M their identity on $\mathcal{F}$ follows.

Proof. Let $\mathcal{P}_n(z)$ be a polynomial of best approximation of the function $f(z) = f_1(z) - f_2(z)$ of degree n on the set $\mathcal{F}$. On M we choose some countable part of $\{x_i\}$, $i = 0, 1, \dots$ and

represent the polynomial $\mathcal{P}_n(x)$ in the form of an interpolation polynomial with nodes at points $x_0, x_1, \dots, x_n$:

$$\mathcal{P}_n(z) = \sum_{i=0}^n \mathcal{P}_n(x_i) \frac{\omega_n(z)}{(z - x_i)\omega'_n(x_i)}, \quad (\omega_n(z) = \prod_0^n (z - x_i)).$$

Let $\varphi_M(n)$ denote

$$\max_{z \in \mathcal{F}} \frac{|\omega_n(z)|}{|z - x_i| |\omega'_n(x_i)|}, \quad i = 0, 1, \dots, n.$$

The numbers $\varphi_M(n)$ depend, obviously, only on M .

It is enough now to set $\psi_M(n)$ equal to $o(\frac{1}{n\varphi_M(n)})$, since from $f(x_k) = 0$ follows

$$|\mathcal{P}_n(x_k)| < \psi_M(n) \quad \text{and} \quad |\mathcal{P}_n(x)| \leq o(\psi_M(n)\varphi_M(n)n) = o(1), \quad n = n_1, n_2, \dots$$

If, in particular, $\mathcal{F}$ is the segment $[0; 1]$, and the set M consists of points of the form $\{\frac{1}{n^p}\}$, $p > 0$, $n = 1, 2, 3, \dots$, then it is easy to calculate that

$$\psi_M(n) = e^{-pn \ln n}.$$

If M is more sparse, and consists of points $\{q^n\}$, $0 < q < 1$, $n = 1, 2, 3, \dots$, then

$$\psi_M(n) = q^{\frac{n(n+1)}{2}}.$$

Thus, the class of functions for which

$$E_n(f) < e^{-Cn \ln n}, \quad n = n_1, n_2, \dots$$

has two properties characteristic of the class of analytic functions.

Let $\mathcal{F}$ be the circle $|z| \leq 1$, and M be the set of zeros $f(z)$ located on the circumference $|z| = 1$, and let $f(z)$ be regular in $|z| < 1$, continuous in $|z| \leq 1$. Applying the theorem to this case, we conclude that the class of functions $f(z)$ for which any infinite set is a set of uniqueness includes functions with an arbitrarily bad (in the sense of slowness of decrease) modulus of continuity. $\square$

Let $\omega(\delta)$ be the modulus of continuity of $f(z)$ in $|z| \leq 1$. Suppose that the zeros of $f(z)$ located on $|z| = 1$ are condensed to the point $z = 1$. To characterize the rate of condensation to the limit point, we introduce the function $\lambda(r)$; $\lambda_1(\varphi)$ is the distance $e^{i\varphi}$ to M ; if $e^{ir} \notin M$, then

$$\lambda(r) = \max_{\varphi \leq r} \lambda_1(\varphi).$$

If $e^{ir} \in M$, then

$$\lambda(r) = \lim_{\varphi \rightarrow r} \lambda(\varphi).$$

Theorem 5.3. If

$$\lambda(r) < \omega^{-1}(e^{-\gamma(r)}) \quad \text{and} \quad \int_0^a \gamma(r) dr = +\infty,$$

then $f(z) \equiv 0$.

Proof. Indeed, we have

$$|f(e^{i\varphi})| = |f(e^{i\varphi}) - f(e^{i\varphi_n})| \leq \min_{n \geq 1} \omega(\varphi - \varphi_n) \leq \omega[\lambda(\varphi)] < e^{-\gamma(\varphi)}.$$

But it is known that if the circumference $|z| = 1$ is divided into n arcs $\Phi_1, \Phi_2, \dots, \Phi_n$ with lengths $l_1, l_2, \dots, l_n$, respectively, and

$$\max_{z \in \Phi_k} |f(z)| = M_k,$$

then for some constant C and any point z located in the disc $|z| \leq \theta < 1$,

$$|f(z)| < M_1^{Cl_1} M_2^{Cl_2} \dots M_n^{Cl_n}, \quad C = C(\theta).$$

In our case, on the arc $(e^{i\varphi_n}; e^{i\varphi_{n+1}})$ $|f(z)| < e^{-\gamma(\varphi_n)}$; hence,

$$\max_{|z| \leq \theta < 1} |f(z)| < e^{C[\gamma(\varphi_1)\Delta\varphi_1 + \gamma(\varphi_2)\Delta\varphi_2 + \dots + \gamma(\varphi_n)\Delta\varphi_n]}.$$

But if $\int_0^a \gamma(r) dr = +\infty$, then the left-hand side can be made as small as desired, i.e., $f(z) \equiv 0$.

If, in particular, $f(z)$ satisfies a Lipschitz condition of some positive order in $|z| \leq 1$ and vanishes at points $\{e^{\frac{2\pi i}{\ln n}}\}$, $n = 2, 3, \dots$, then $f(z) = 0$. $\square$

Theorem 5.4. *If $f(z)$ can be approached in $f(z)$ with the rate*

$$\rho_n(f) < e^{-\frac{n}{\ln n}}, \quad (5.1)$$

then the cardinality of the set of zeros $f(z)$ is at most $\aleph_0$.

Proof. Suppose that this is not so, that is, the set M is irreducible. We denote by $M^{(n)}$ the derived set of the set $M^{(n-1)}$ ($M^{(0)} = M$).

From each M^n we choose one point z_n , which can be done since M is irreducible. Let α be one of the limit points of the set $\{z_i\}$.

It is easy to see that the set M^n has the property that $f^{(n)}(z)$ vanishes on $M^{(n)}$. From (5.1) it follows that $f(z)$ is infinitely differentiable in $|z| \leq 1$, and

$$\sum_1^\infty \frac{1}{\sqrt[n]{M_n}} = \infty.$$

But $M \supseteq M^{(1)} \supseteq M^{(2)} \supseteq \dots$. Consequently, at point α all derivatives of $f(z)$ vanish; hence, according to the mentioned Carleman theorem, it follows that $f(z) = 0$. $\square$

Remark 5.1. *From the proof it is clear that more can be stated, namely, if $f(z)$ is regular in $|z| < 1$, and*

$$\rho_n(f; |z| \leq 1) < e^{-\frac{n}{\ln n}},$$

then $f(z)$ has only a finite number of zeros in $|z| \leq 1$.

6. Best Approximation on Closed Sets

Let E be a closed set of points located in the plane of the complex variable z , and let $f(z)$ be a function defined and continuous on E . We denote by $\rho(n)$ the infimum of the numbers

$$\max_{z \in E} |f(z) - \mathcal{P}_n(z)|$$

by any polynomials of degree $\leq n$.

Suppose that E is bounded, is not dense anywhere, and does not break up the plane; according to Lavrentiev's theorem [8], $\rho(n) \rightarrow 0$ always for $n \rightarrow \infty$.

Theorem 6.1. Let $M(r)$ be an arbitrary function growing to $+\infty$ as $r \rightarrow \infty$ faster than any power of r , i.e.,

$$\lim_{r \rightarrow \infty} \frac{M(r)}{r^n} = \infty, \quad n - \text{any},$$

and E be an unbounded set.

There is a function $\varphi(r) > 0$ such that from the inequality

$$\rho(n) < \varphi(n), \quad n = 1, 2, 3, \dots,$$

it follows that the function $f(z)$ can be continued from the set E to the entire plane so that the function $\mathcal{F}(z)$ obtained as a result of the continuation will be an entire function satisfying the condition

$$\max_{|z|=r} |\mathcal{F}(z)| < M(r).$$

Proof. Without loss of generality, it can be assumed that the origin is a limit point of the set E . The polynomial of best approximation of the function $f(z)$ on E of degree n will be denoted by $\mathcal{P}_n(z)$.

Let the points $z_0, z_1, z_2, \dots, z_n, \dots$ belonging to E converge to $z = 0$ and

$$\omega_n(z) = (z - z_0)(z - z_1) \cdots (z - z_n).$$

We denote by $\psi(n)$

$$\max_{1 \leq k \leq n} \max_{|z| \leq 1} \frac{|\omega_n(z)|}{|z - z_n| |w'_n(z_k)|} \cdot \frac{n}{n+1}.$$

Representing the difference $\mathcal{P}_n(z) - \mathcal{P}_{n+1}(z)$ in the form of an interpolation polynomial with nodes at $z_0, z_1, \dots, z_n$, from here we find

$$\max_{|z| \leq 1} |\mathcal{P}_n(z) - \mathcal{P}_{n+1}(z)| < n\psi(n)\varphi(n).$$

We have

$$\max_{|z|=R>1} |f(z)| \leq \max_{|z|=R>1} |\mathcal{P}_0(z)| + \sum_{n=1}^{\infty} \max_{|z|=R>1} |\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)| \leq \sum_{n=1}^{\infty} n\psi(n)\varphi(n)R^n.$$

Let $A(x)$ denote the integer part $\frac{\ln M(x)}{2 \ln x}$, and $A^{-1}(x)$ be the function inverse to $A(x)$. Assuming

$$\varphi(n) < \frac{1}{n\psi(n)} e^{-n[A^{-1}(n)+1]},$$

we obtain

$$\sum_1^{\infty} n\varphi(n)\psi(n)R^n = \sum_{n=1}^{A(R)} n\varphi(n)\psi(n)R^n + \sum_{A(R)+1}^{\infty} n\varphi(n)\psi(n)R^n \leq \frac{M(R)}{2} + \frac{M(R)}{2} = M(R),$$

that is, the theorem is proven. $\square$

Theorem 6.2. Whatever the functions $\varphi(n)$ and $\omega(\delta)$, $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, there exists a set $\mathcal{P} \subseteq [0, 1]$ and $f(x)$ so that

$$\rho_n(f; \mathcal{P}) < \varphi(n), \quad n = 1, 2, 3, \dots;$$

however,

$$\omega_{a,b}(\delta) > \omega(\sigma), \quad \delta < \delta(a; b),$$

where $\omega_{a,b}(\delta)$ is the modulus of continuity of $f(x)$ on the portion $\mathcal{P}$ contained in the segment $[a, b]$ ($\omega_{a,b}(\delta) = 1$, if $\mathcal{P}[a, b] = 0$).

Proof. We denote by $\mu(n)$ a positive function satisfying the inequalities

$$\mu(n) < \nu(n), \quad 2\mu(2n) < \mu(n), \quad \mu(1) = 1, \quad n \geq n_0.$$

Let $x'_1 = 0$, $x'_2 = 1$ and $A'_1 = 0$, $A'_2 = 1$. Suppose that the points $\{x_j^{(k)}\}$ and the numbers $\{A_j^{(k)}\}$ are constructed for $j = 1, 2, \dots, 2^k$, $k = 1, 2, \dots, i$.

Let us define $\{x_j^{(i+1)}\}$ and $\{A_j^{(i+1)}\}$, $j = 1, 2, 3, \dots, 2^{i+1}$.

By $\mathcal{P}_i(x)$ we denote a polynomial of degree 2^i taking in $x_j^{(i)}$ the values $A_j^{(i)}$ ($j = 1, 2, 3, \dots, 2^i$), and by $\omega_i(\delta)$ we denote the modulus of continuity of $\mathcal{P}_i(x)$. We also assume that for $k = 1, 2, \dots, 2^{i-1}$

$$x_{2k}^{(i)} - x_{2k-1}^{(i)} = \delta_i > 0. \quad (6.1)$$

We determine the number δ_{i+1} from the conditions

$$1. \omega(\delta_{i+1}) < \mu(2^{i+3}), \quad 2. 0 < 2\delta_{i+1} < \delta_i, \quad 3. \omega_i(\sigma_{i+1}) < \mu(2^{i+2}). \quad (6.2)$$

As $\{x_j^{i+1}\}$, $j = 1, 2, \dots, 2^{i+1}$, we take a set of points satisfying the condition

$$x_{4k-3}^{(i+1)} = x_{2k-1}^{(i)}; \quad x_{4k}^{(i+1)} = x_{2k}^{(i)}; \quad x_{4k-2}^{(i+1)} = x_{2k-1}^{(i)} + \delta_{i+1}; \quad x_{4k-1}^{(i+1)} = x_{2k}^{(i)} - \delta_{i+1}, \quad k = 1, 2, \dots, 2^{i-1}.$$

It follows that if (6.1) was fulfilled for any $i = i_0$, then it will be fulfilled for $i = i_0 + 1$. But the numbers $\{x_j^{(1)}\}$ ($j = 1, 2$) are defined and for $i_0 = 0$ (6.1) holds, so $\{x_j^{(i+1)}\}$ are defined and (6.1) holds. We determine the numbers from the relations

$$\begin{aligned} A_{4k-3}^{(i+1)} &= A_{2k-1}^{(i)}; \quad A_{4k}^{(i+1)} = A_{2k}^{(i)}; \\ A_{4k-2}^{(i+1)} &= A_{2k-1}^{(i)} + \mu(2^{i+2}); \quad A_{4k-1}^{(i+1)} = A_{2k}^{(i)} - \mu(2^{i+2}), \quad k = 1, 2, \dots, 2^{i-1}. \end{aligned} \quad (6.3)$$

Since A'_1 and A'_2 are defined, then for every $i \geq 1$ $\{A_j^{(i)}\}$, ($j = 1, 2, \dots, 2^i$), and from (6.3) it follows that $A_1^{(i)}, A_2^{(i)}, A_3^{(i)}, \dots, A_{2^i}^{(i)}$ constitutes a monotonically increasing sequence

$$0 < A_1^{(i)} < A_2^{(i)} < \dots < A_{2^i}^{(i)} < 1.$$

Let us denote

$$\mathcal{F}_i = \sum_{k=1}^{2^{i-1}} [x_{2k-1}^{(i)}; x_{2k}^{(i)}], \quad \mathcal{P} = \prod_{i=1}^{\infty} \mathcal{F}_i.$$

$[a, b]$ is a segment with ends at a and b ($a < b$).

$\mathcal{P}$ obviously represents a perfect set, which is the closure of the set $\{x_j^{(i)}\}$.

At the point $x_j^{(i)}$ let us set the function $f(x)$ equal to $A_j^{(i)}$ for $j = 1, 2, \dots, 2^i$, $i = 1, 2, \dots$; at the limit points of the set $\{x_j^{(i)}\}$ x we define $f(x)$ as the supremum of the numbers $A_j^{(i)}$ for some $x_j^{(i)} < x$.

From the monotonicity of $A_1^{(i)}, A_2^{(i)}, \dots, A_{2^i}^{(i)}$, it follows that $f(x)$ is monotonically increasing and continuous on $\mathcal{P}$.

Let us prove that $f(x)$ satisfies the required conditions. Let $[a, b]$ be an arbitrary segment located at $[0, 1]$ and containing the points $\mathcal{P}$.

Let i be sufficiently large and let $[x_{2k-1}^{(i)}; x_{2k}^{(i)}] \subseteq [a, b]$. Since $f(x_{2k}^{(i)}) - f(x_{2k-1}^{(i)}) = \mu(2^{i+1})$ and $x_{2k}^{(i)} - x_{2k-1}^{(i)} = \delta_i$, then

$$\omega_{a,b}(\delta_i) \geq \mu(2^{i+1}).$$

Now let δ be an arbitrary, sufficiently small number. Let us choose i so that

$$\delta_{i+1} < \delta \leq \delta_i.$$

Obviously, based on (6.2) we have

$$\omega_{a,b}(\delta) \geq \omega_{a,b}(\delta_{i+1}) \geq \mu(2^{i+2}) > \omega(\delta_i) \geq \omega(\delta),$$

i.e.,

$$\omega_{a,b}(\delta) > \omega(\delta), \quad \delta \leq \delta(a, b).$$

Now, let x be an arbitrary point of $\mathcal{P}$, $i > 0$ be an arbitrary integer, k be determined from the condition $x \in [x_{2k-1}^{(i)}; x_{2k}^{(i)}]$. Due to the monotonicity of $f(x)$ we have

$$|f(x) - f(x_{2k-1}^{(i)})| \leq \psi(2^{i+2}), \quad |x - x_{2k-1}^{(i)}| \leq \delta_i.$$

But since

$$\mathcal{P}_{i-1}(x_k^{(i-1)}) = f(x_k^{(i-1)}), \quad k = 1, \dots, 2^i,$$

$$|f(x) - \mathcal{P}_{i-1}(x)| \leq |f(x) - f(x_{2k-1}^{(i+1)})| +$$

$$+ |\mathcal{P}_{i-1}(x) - \mathcal{P}_{i-1}(x_{2k-1}^{(i)})| + |\mathcal{P}_{i-1}(x_{2k-1}^{(i)}) - \mathcal{P}_{i-1}(x_k^{(i-1)})| \leq \psi(2^{i+1}), \quad (\psi(x) \equiv \mu(x)).$$

Hence,

$$\rho(2^i) < \psi(2^{i+1}).$$

Let $n > 0$ be an arbitrary integer and $2^i < n \leq 2^{i+1}$. We have

$$\rho(n) \leq \rho(2^i) < \psi(2^{i+1}) < \psi(n),$$

that is, the theorem is proven. $\square$

The above two qualitative results determine the formulation of the problem of studying the best approximation on closed sets depending on the properties of these sets and on the behavior of the approximated functions on them.

Without dwelling here on all sorts of problems in this direction, let us consider one of them.

According to a well-known classical result, if $\mathcal{P}$ is a segment or closed domain and the function $f(z)$ can be approached with a progression rate on $\mathcal{P}$, i.e.,

$$\rho_n(f; \mathcal{P}) < q^n, \quad 0 < q < 1,$$

then $f(z)$ is analytic at every point of $\mathcal{P}$.

Let us consider the question for which, more generally, set $\mathcal{P}$ the previous result continues to be valid.

Theorem 6.3. Suppose the set $\mathcal{P}$ is perfect, with the connected complement, and for any function $\varphi(z)$ continuous on $\mathcal{P}$, there exists a function $u(z)$ harmonic on the complement of $\mathcal{P}$, taking at the point $\mathcal{P}$ value $\varphi(z)$. If for some function $f(z)$ defined on $\mathcal{P}$

$$\rho(n) < q^n, \quad 0 < q < 1, \quad n = 1, 2, 3, \dots, \quad (6.4)$$

then $f(z)$ is analytic at every point of $\mathcal{P}$.

Remark 6.1. Theorem 6.3 can also be formulated in local form: if a given point z_0 is regular (in the sense of the Dirichlet problem), then (6.4) implies the analyticity of $f(z)$ at this point.

Remark 6.2. Thus, the possibility of extending the above-mentioned classical result to a more general type of set is influenced not by the measure (linear or flat), but by a more subtle characteristic of the set—its capacity.

Proof of Theorem 6.3. Let $\mathcal{P}_n(z)$ denote a polynomial of degree n whose maximum modulus on $\mathcal{P}$ is M . Taking into account the fact that $\ln |\mathcal{P}_n(z)| - \ln M$ is a subharmonic function, non-positive on $\mathcal{P}$ and behaving at infinity as $n \ln |z|$, we obtain

$$|\mathcal{P}_n(z)| \leq M e^{nG(z)},$$

where $G(z)$ is the Green's function of the complement to $\mathcal{P}$, which vanishes, according to the regularity criterion for the Bouligand point, on the set $\mathcal{P}$.

For any $\varepsilon > 0$ we denote by G_ε an open set containing $\mathcal{P}$ and such that

$$\max_{z \in \overline{G_\varepsilon}} |G(z)| \leq \varepsilon.$$

Thus,

$$\max_{z \in \overline{G_\varepsilon}} |\mathcal{P}_n(z)| < M(1 + \varepsilon)^n.$$

Taking into account (6.4), we have

$$\max_{z \in \overline{G_\varepsilon}} |\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)| < 2q^{n-1}(1 + \varepsilon)^n.$$

Choosing ε so that $q(1 + \varepsilon) < 1$, we notice that the series

$$\mathcal{P}_0(z) + \sum_{n=1}^{\infty} [\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)]$$

converges uniformly on $\overline{G_\varepsilon}$, i.e., $f(z)$ is analytic on the set $\mathcal{P}$. $\square$

In the case where $\mathcal{P}$ has only one limit point, the capacity of $\mathcal{P}$ is zero; hence, the previous theorem says nothing.

Theorem 6.4. For any positive function $\varphi(n)$ satisfying the condition

$$\lim_{n \rightarrow \infty} \frac{\ln \varphi(n)}{n} = -\infty, \quad (6.5)$$

there exists a countable set E on $[0; 1]$, having only one limit point and such that if the inequality

$$\rho(n) < \varphi(n)$$

holds for some function $f(x)$ defined on E , it follows that $f(x)$ takes its values from some entire function $\mathcal{F}(z)$.

Proof. Let us denote $l_n = 2^{\sqrt[n]{\varphi(n)}}$, $\omega_n(x) = \prod_{i=0}^n (x - \frac{i}{n} l_n)$,

$$M_n = \max_{|z| \leq 2, 0 \leq i \leq n} \frac{|\omega_n(z)|}{\left| z - \frac{i}{n} l_n \right| \left| \omega'_n\left(\frac{i}{n} l_n\right) \right|}.$$

We denote the set of points $\{\frac{\varphi(n)}{M_n n} k\}$ ($k = 0, 1, 2, \dots, [\frac{n M_n}{\varphi(n)} l_n]$) and $\{\frac{l_n}{n} k\}$ ($k = 0, 1, 2, \dots, n$) by E_n . Let us show that the set

$$E = \sum_{n=1}^{\infty} E_n$$

is what we are looking for.

Indeed, from (6.5) it follows that E has only one limit point $x = 0$. In addition, if by $\mathcal{P}_n(z)$ we denote the polynomial of best approximation $f(z)$ of degree n on E , then, representing the difference $\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)$ in the form of an interpolation polynomial with nodes at points

$$0, \frac{l_n}{n}, \frac{2l_n}{n}, \dots, l_n,$$

we find

$$\max_{|z| \leq 1} |\mathcal{P}'_n(z) - \mathcal{P}'_{n-1}(z)| \leq M_n n.$$

But

$$\left| \mathcal{P}_n\left(\frac{\varphi(n)}{M_n n} k\right) - \mathcal{P}_{n-1}\left(\frac{\varphi(n)}{M_n n} k\right) \right| < 2\varphi(n-1), \quad k = 0, 1, 2, \dots, \left[\frac{M_n n}{\varphi(n)} l_n \right].$$

The distance of any point on the segment $[0; l_n]$ to E does not exceed $\frac{\varphi(n)}{M_n n}$; therefore,

$$\max_{0 \leq x \leq l_n} |\mathcal{P}_n(x) - \mathcal{P}_{n-1}(x)| \leq 3\varphi(n-1).$$

Let z be any point in the plane, and let $\mathcal{Z}_q$ be the level line of the complement of $[0; l_n]$ passing through z ($z \notin [0; l_n]$).

It is easy to see that $q = q(n; z) < \frac{C(z)}{l_n}$, where $C(z)$ depends only on z .

It follows from this that

$$|\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)| < \frac{3C(z)}{l_n^n} \varphi(n) < \frac{C_1(z)}{2^n},$$

that is, the series

$$\mathcal{P}_0(z) + \sum_1^{\infty} [\mathcal{P}_n(z) - \mathcal{P}_{n-1}(z)]$$

converges in the entire plane, and its sum is an entire function coinciding on E with $f(x)$, i.e., the theorem is proven. $\square$

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