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Multiplicity of Normalized Solutions for the Fractional Schrödinger Equation with Potentials

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Abstract: We are concerned with the existence and multiplicity of normalized solutions to the fractional Schrödinger equation $\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = \lambda u + h(\varepsilon x)f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a, \end{cases}$, where $(-\Delta)^s$ is the fractional Laplacian, $s \in (0,1)$, $a, \varepsilon > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $h : \mathbb{R}^N \to [0, +\infty)$ are bounded and continuous, and f is L^2-subcritical. Under some assumptions on the potential V, we show the existence of normalized solutions depends on the global maximum points of h when ε is small enough.

Keywords: fractional Laplacian; normalized solution; mass critical exponent

MSC: 35A15; 35B33; 35Q55

1. Introduction

1.1. Background and Motivation

In this paper, we investigate the multiplicity of normalized solutions for the fractional Schrödinger equation as follows:

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - g(|\psi|^2)\psi \ \text{ in } \mathbb{R}^{\mathrm{N}}, \tag{1}$$

where $0 < s < 1$, i denotes the imaginary unit and $\psi(x,t)$ is a complex wave. A solution of (1) is called a standing wave solution if it has the form $\psi(x,t) = e^{-i\lambda t}u(x)$ for some $\lambda \in \mathbb{R}$. $(-\Delta)^s$ stands for the fractional Laplacian, and if u is small enough, it can be computed by the following singular integral:

$$(-\Delta)^s u = C(N,s)\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y.$$

Here, the symbol P.V. is the Cauchy principal value and $C(N,s)$ is a suitable positive normalizing constant.

The operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes [1], it originates from describing various phenomena in the field of applied science, such as fractional quantum mechanics, the barrier problem, Markov processes, and the phase transition phenomenon, see [2–5]. In recent decades, the study of the fractional Schrödinger equation has attracted wide attention, see, e.g., [6–9] and the references therein.

In [10], Alves considered the following class of elliptic problems with a L^2-subcritical nonlinear term:

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$$\begin{cases} -\Delta u = \lambda u + h(\varepsilon x) f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a. \end{cases} \tag{2}$$

By using the variational approaches, the author shows that problem (2) admits multiple normalized solutions if ε is small enough. Particularly, the numbers of the normalized solutions are at least the numbers of the global maximum points of h. Moreover, for the following class of problem:

$$\begin{cases} -\Delta u + V(\varepsilon x) u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \end{cases}$$

a similar result is also obtained for some negative and continuous potential V.

Motivated by [10], our interest is mainly focused on the fractional case with both potentials and weights. Actually, our purpose of this paper is devoted to the multiplicity of normalized solutions for the fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x) u = \lambda u + h(\varepsilon x) f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \end{cases} \tag{3}$$

where $s \in (0,1)$, $a, \varepsilon > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier.

In the local case, when $s = 1$, the fractional Laplace $(-\Delta)^s$ reduces to the local differential operator $-\Delta$. If $V(x) \equiv 0$, Jeanjean's [11] exploited the mountain pass geometry to deal with the existence of normalized solutions in purely L^2-supercritical, we refer [12–15] for more results in this type of problems. In [16], they considered the related problem for $q = 2 + \frac{4}{N}$. The multiplicity of normalized solutions for the Schrödinger equation or systems has also been extensively investigated, see [17–19].

For the non-potential case, a large body of literature is devoted to the following problem:

$$\begin{cases} -\Delta u = \lambda u + g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2. \end{cases} \tag{4}$$

In particular, for the case $g(u) = |u|^{p-1}u$, by assuming the H^1-precompactness of any minimizing sequences, Cazenave and Lions [20] showed the attainability of the L^2-constraint minimization problem and the orbital stability of global minimizers, it is assumed that $E_\alpha < 0$ for all $\alpha > 0$, and then the strict subadditivity condition as follows holds.

$$E_{\alpha+\beta} < E_\alpha + E_\beta. \tag{5}$$

However, when dealing with the general function g, it is difficult to show if (5) holds. Shibata [19] proved the subadditivity condition (5) using a scaling argument.

In addition, if $V(x) \not\equiv 0$, Ikoma and Miyamoto [21] studied the existence and nonexistence of a minimizer of the L^2-constraint minimization problem as follows:

$$e(a) = \inf\{E(u) | u \in H^1(\mathbb{R}^N), |u|_2^2 = a\},$$

where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 \mathrm{d}x + V(x)|u|^2) \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

V and f satisfy some suitable assumptions. They performed a careful analysis to exclude dichotomy and proved the precompactness of the modified minimizing sequence. When dealing with general nonlinear terms in mass subcritical cases, one can apply the subadditive inequality to prove the compactness of the minimizing sequence.

Zhong and Zou in [22] studied the existence of a ground state normalized solution to Schrödinger equations with potential under different assumptions, and presented a new approach to establish the strict subadditive inequality. Alves and Thin [23] studied the existence of multiple normalized solutions to the following class of elliptic problems:

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \end{cases} \tag{6}$$

where $\varepsilon > 0$, $V : \mathbb{R}^N \to [0, \infty)$ is a continuous function, and f is a differentiable function with L^2-subcritical growth. For the normalized solutions of the nonlinear Schrödinger equation with potential, we also see [24–26] and the references therein.

In the case $0 < s < 1$, few results are available. In the paper [27], the author proved some existence and asymptotic results for the fractional nonlinear Schrödinger equation. For the particular case of a combined nonlinearity of power type, namely, $f(t) = \mu|u|^{q-2}u + |u|^{p-2}u$, $h(x) = 1$ and $V(x) \equiv 0$, i.e $2 < q < p < 2_s^*$, Luo and Zhang [28] proved some existence and nonexistence results about the normalized solutions for L^2-subcritical, L^2-critical, and L^2-supercritical. Dinh [29] studied the existence and nonexistence of normalized solutions for the fractional Schrödinger equations as follows:

$$(-\Delta)^s u + V(x)u = |u|^{p-2}u, \ \text{ in } \mathbb{R}^N. \tag{7}$$

By using the concentration–compactness principle, he showed a complete classification for the existence and nonexistence of normalized solutions for the problem (7). For more results about the fractional Schrödinger equations, we can refer to [30,31] and the references therein.

1.2. Main Results

In what follows, we assume $f \in C^1(\mathbb{R}^N, \mathbb{R})$ is odd, continuous, and satisfies the following assumptions on f:

(f_1) $\lim\limits_{t\to 0} \frac{|f(t)|}{|t|^{q-1}} = c > 0$, where $2 < q < \bar{p} = 2 + \frac{4s}{N}$;

(f_2) $\lim\limits_{t\to \infty} \frac{|f(t)|}{|t|^{p-1}} = 0$, where $2 < p < \bar{p} = 2 + \frac{4s}{N}$;

(f_3) There exist $\alpha, \beta \in \mathbb{R}$ satisfying $2 < \alpha \le \beta < \bar{p}$ such that

$$0 < \alpha F(t) \le t f(t) \le F(t)\beta \ \text{ for any } t > 0.$$

Moreover, h and V satisfy the following assumptions.

(A_1) $h \in C(\mathbb{R}^N, \mathbb{R}^+)$, $0 < h_\infty = \lim\limits_{|x|\to+\infty} h(x) < \max\limits_{x\in\mathbb{R}^N} h(x) = h(a_i)$ for $1 \le i \le k$ with $a_1 = 0$ and $a_j \ne a_i$ if $i \ne j$;

(A_2) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(a_i) = \inf\limits_{x\in\mathbb{R}^N} V(x) < \lim\limits_{|x|\to+\infty} V(x) = 0$ for $1 \le i \le k$.

The problem (3) is variational, and the associated energy functional is given by the following:

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x)F(u)\mathrm{d}x, \ u \in H^s(\mathbb{R}^N), \tag{8}$$

with

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

It is easy to know that $I_\varepsilon \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$I'_\varepsilon(u)\varphi = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)u\varphi \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x)f(u)\varphi \mathrm{d}x, \ \ \forall \varphi \in H^s(\mathbb{R}^N).$$

The solutions to (3) can be characterized as critical points of the function $I_\varepsilon(u)$ constrained on the sphere as follows:

$$S_a = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a \right\}. \tag{9}$$

Now, we are ready to state the main result of this paper.

Theorem 1. *Suppose $(A_1),(A_2)$, $(f_1)-(f_3)$ hold, then there exists $\varepsilon_1 > 0$ such that problem (3) admits at least k couples $(u_j, \lambda_j) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $\varepsilon \in (0, \varepsilon_1)$ with $\int_{\mathbb{R}^N} |u_j|^2 \mathrm{d}x = a$, $\lambda < 0$ and $I_\varepsilon(u_j) < 0$ for $j = 1, 2, \cdots, k$.*

The paper is organized as follows: In Section 2, we study the autonomous problem and give some useful results, which will be used later. Section 3 is devoted to the non-autonomous problem. In Section 4, the proof of Theorem 1, is given.

2. The Autonomous Problem

In this section, we focus on the existence of a normalized solution for the autonomous problem.

$$\begin{cases} (-\Delta)^s u + \eta u = \lambda u + \mu f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \end{cases} \tag{10}$$

where $s \in (0,1)$, $a, \mu > 0$, $\eta \leq 0$, and $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier. With the assumptions $(f_1)-(f_3)$, it is standard to show that the solutions to (10) can be characterized as critical points of the function as follows:

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{\eta}{2} \int_{\mathbb{R}^N} u^2 \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u) \mathrm{d}x, \tag{11}$$

restricted to the sphere S_a given in (9). Meanwhile, set

$$J_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

and

$$\Upsilon_a = \inf_{S_a} J_0(u).$$

Theorem 2. *Suppose that f satisfies the conditions $(f_1)-(f_3)$. Then, problem (10) has a couple (u, λ) solution, where u is positive, radial and $\lambda < \eta$.*

The proof of Theorem 2 is standard. For the sake of convenience, we give the details. Before the proof, some lemmas are given below.

Lemma 1. *Assume u is a solution to (10), then $u \in S_a \cap P$, where*

$$P := \left\{ u \in H^s(\mathbb{R}^N) | \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{N\mu}{s} \int_{\mathbb{R}^N} F(u) \mathrm{d}x - \frac{N\mu}{2s} \int_{\mathbb{R}^N} f(u) u \mathrm{d}x = 0 \right\}.$$

Proof. Let u be a solution (10), then we obtain the following:

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + (\eta - \lambda) \int_{\mathbb{R}^N} u^2 \mathrm{d}x - \mu \int_{\mathbb{R}^N} f(u) u \mathrm{d}x = 0. \tag{12}$$

In addition, one can show that u satisfies the Pohozaev identity as follows:

$$(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + N(\eta - \lambda) \int_{\mathbb{R}^N} u^2 \mathrm{d}x - 2N\mu \int_{\mathbb{R}^N} F(u) = 0,$$

which combined with (12) gives the following:

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{N\mu}{s} \int_{\mathbb{R}^N} F(u) \mathrm{d}x - \frac{N\mu}{2s} \int_{\mathbb{R}^N} f(u) u \mathrm{d}x = 0.$$

□

Lemma 2. *Assume $(f_1) - (f_2)$, then we have the following:*
(i) J is bounded from below on S_a;
(ii) Any minimizing sequence for J is bounded in $H^s(\mathbb{R}^N)$.

Proof. (i) According to the assumptions $(f_1) - (f_2)$, there exist $C_1, C_2 > 0$ such as the following:

$$|F(t)| \le C_1 |t|^q + C_2 |t|^p, \;\; \forall t \in \mathbb{R}, \tag{13}$$

where $q, p \in (2, 2 + \frac{4s}{N})$. By the fractional Gagliardo–Nirenberg–Sobolev inequality [32],

$$\int_{\mathbb{R}^N} |u|^\alpha \le C(s, N, \alpha) \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \right)^{\frac{N(\alpha-2)}{4s}} \left(\int_{\mathbb{R}^N} |u|^2 \right)^{\frac{\alpha}{2} - \frac{N(\alpha-2)}{4s}}, \tag{14}$$

for some positive constant $C(s, N, \alpha) > 0$. Then, (13) and (14) give the following:

$$\begin{aligned} J(u) \ge\; & \frac{1}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \eta u^2) \mathrm{d}x - \frac{\mu C_1 C(s, N, q)}{q} a^{\frac{q}{2} - \frac{N(q-2)}{4s}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^{\frac{N(q-2)}{4s}} \\ & - \frac{\mu C_2 C(s, N, p)}{p} a^{\frac{p}{2} - \frac{N(p-2)}{4s}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^{\frac{N(p-2)}{4s}}. \end{aligned} \tag{15}$$

Since $q, p \in (2, 2 + \frac{4s}{N})$, we infer that $0 < \frac{N(q-2)}{4s}, \frac{N(p-2)}{4s} < 1$. Therefore $J(u)$ is bounded from below on S_a.

(ii) Since $u \in S_a$, the conclusion immediately follows from (15). □

The lemma above guarantees the following:

$$E_a = \inf_{u \in S_a} J(u),$$

is well-defined. Now, we study the properties of the function J defined in (10) restrict to S_a and prove Theorem 2.

Lemma 3. *For any $a > 0$ and $\eta \le 0$, there holds $E_a < 0$. In particular, we have $E_a < \frac{\eta a}{2}$.*

Proof. From the condition (f_1), we know that $\lim_{t \to 0} \frac{qF(t)}{t^q} = c > 0$, then there exists $\zeta > 0$ as follows:

$$\frac{qF(t)}{t^q} \ge \frac{c}{2}, \; \forall t \in [0, \zeta]. \tag{16}$$

In fact, taking $u \in S_a \cap L^\infty(\mathbb{R}^N)$ as a fixed non-negative function, we define the following:

$$(\tau * u)(x) = e^{\frac{N}{2}\tau} u(e^\tau x), \;\; \text{for all } x \in \mathbb{R}^N \text{ and all } \tau \in \mathbb{R},$$

then $\tau * u \in S_a$ and

$$\int_{\mathbb{R}^N} F((\tau * u)(x)) \mathrm{d}x = e^{-N\tau} \int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}} u(x)) \mathrm{d}x.$$

Moreover, for $\tau < 0$ and $|\tau|$ large enough, we have the following:

$$0 \le e^{\frac{N}{2}\tau} u(x) \le \zeta, \ \ \forall x \in \mathbb{R}^N,$$

which combines with (16) to give the following:

$$\int_{\mathbb{R}^N} F((\tau * u)(x))\mathrm{d}x \ge \frac{c}{2q} e^{\frac{(q-2)N\tau}{2}} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x.$$

Hence, as follows:

$$\begin{aligned} J(\tau * u) = &\ \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u)|^2 \mathrm{d}x + \frac{\eta a}{2} - \mu \int_{\mathbb{R}^N} F(\tau * u)\mathrm{d}x \\ \le &\ \frac{1}{2} e^{2s\tau} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{\eta a}{2} - \frac{\mu c}{2q} e^{\frac{(q-2)N\tau}{2}} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x. \end{aligned} \tag{17}$$

Since $q \in (2, 2 + \frac{4s}{N})$, increasing $|\tau|$ if necessary, we deduce the following:

$$\frac{1}{2} e^{2s\tau} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \frac{\mu c}{2q} e^{\frac{(q-2)N\tau}{2}} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x = K_\tau < 0.$$

Hence, we obtain the following:

$$J(\tau * u) \le K_\tau + \frac{\eta a}{2} < 0,$$

and then $E_a < 0$. In particular, we have $E_a < \frac{\eta a}{2}$. The proof is complete. $\square$

In the following, we adopt some idea introduced in [22] to obtain the subadditive inequality:

Lemma 4. *For $\mu > 0, \eta \le 0$ and let $a, b > 0$, then*
(i) *$a \mapsto E_a$ is nonincreasing;*
(ii) *$a \mapsto E_a$ is continuous;*
(iii) *$E_{a+b} \le E_a + E_b$. If E_a or E_b can be attained, then $E_{a+b} < E_a + E_b$.*

Proof. (i) For any $\varepsilon > 0$ small, there exist $u \in S_a \cap C_0^\infty(\mathbb{R}^N)$ and $v \in S_{b-a} \cap C_0^\infty(\mathbb{R}^N)$ such that

$$J(u) \le E_a + \varepsilon, \quad J_0(v) \le \Upsilon_{b-a} + \varepsilon.$$

Since u and v have compact support, by using a parallel translation, we can take R large enough, satisfying the following:

$$\tilde{v}(x) = v(x - R), \quad supp\ u \cap supp\ \tilde{v} = \emptyset.$$

Then $u + \tilde{v} \in S_b$ and

$$\begin{aligned} E_b \le J(u + \tilde{v}) = &\ \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|(u+\tilde{v})(x) - (u+\tilde{v})(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \frac{\eta}{2}|u+\tilde{v}|_2^2 - \mu \int_{\mathbb{R}^N} F(u+\tilde{v})\mathrm{d}x \\ = &\ J(u) + J(\tilde{v}) + \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\tilde{v}(x)-\tilde{v}(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Suppose that

$$supp\ u \subset B_R(0) \ \text{ and } \ supp\ \tilde{v} \subset B_{3R}(0) \backslash B_{2R}(0),$$

we obtain the following:

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\tilde{v}(x)-\tilde{v}(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^{2N}} \frac{u(x)\tilde{v}(x)-2u(x)\tilde{v}(y)+u(y)\tilde{v}(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \iint_{\mathbb{R}^{2N}} \frac{-2u(x)\tilde{v}(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Noting that $|x-y| \geq R$ is large enough, we have the following:

$$E_b \leq J(u+\tilde{v}) \leq J(u)+J(\tilde{v})+\varepsilon \leq J(u)+J_0(v)+\varepsilon \leq E_a+\Upsilon_{b-a}+3\varepsilon \leq E_a+3\varepsilon. \quad (18)$$

Here, we used the fact $\Upsilon_{b-a} < 0$. Then by (18) and the arbitrariness of ε, we obtain that $E_b \leq E_a$ for any $b > a > 0$.

(ii) We prove the following two claims:

Claim 1: $\lim\limits_{h\to 0^+} E_{a-h} \leq E_a$.

For $\varepsilon > 0$, by the definition of E_a, there exists $u \in S_a$ such that

$$E_a \leq J(u) \leq E_a+\varepsilon. \quad (19)$$

Setting

$$t = t(h) = \left(\frac{a-h}{a}\right)^{\frac{1}{N}},$$

and $u_t(x) = u(\frac{x}{t})$, we obtain the following:

$$\lim_{h\to 0^+} t = 1 \ \text{ and } \ |u_t|_2^2 = t^N a = a-h. \quad (20)$$

Then, by using (i), we have $J(u_t) \geq E_{a-h}$. In addition,

$$\begin{aligned} J(u_t) &= \frac{t^{N-2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{\eta t^N}{2}\int_{\mathbb{R}^N} u^2\mathrm{d}x - \mu t^N \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ &= \frac{t^{N-2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + t^N\left(J(u) - \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right) \\ &= t^N J(u) + \frac{t^{N-2s}(1-t^{2s})}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x, \end{aligned}$$

by (19) and (20), we obtain the following:

$$\lim_{h\to 0^+} E_{a-h} \leq E_a+\varepsilon.$$

Since ε is arbitrary, the claim holds.

Claim 2: $\lim\limits_{h\to 0^+} E_{a+h} \geq E_a$.

Actually, we consider the case $h = \frac{1}{n}, n \in \mathrm{N}$. Take $u_n \in S_{a+\frac{1}{n}}$ such that $J(u_n) \leq E_{a+\frac{1}{n}} + \frac{1}{n}$. Set

$$v_n(x) := \sqrt{\frac{na}{na+1}}u_n(x).$$

By Lemma 2, we know $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Moreover, we have the following:

$$|v_n|_2^2 = \frac{na}{na+1}|u_n|_2^2 = \frac{na}{na+1}\left(a+\frac{1}{n}\right) = a.$$

Hence, we obtain $u_n \in S_a$. On the other hand,

$$||v_n - u_n||_{H^s(\mathbb{R}^N)} = \left(1 - \sqrt{\frac{na}{na+1}}\right)||u_n||_{H^s(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty.$$

Then

$$E_a \le \liminf_{n\to+\infty} J(v_n) = \liminf_{n\to+\infty}[J(u_n) + o_n(1)] = \lim_{h\to0^+} E_{a+h}.$$

Thus, we obtain the following:

$$\lim_{h\to0^+} E_{a+h} \ge E_a.$$

Moreover, $E_{a-h} \ge E_a \ge E_{a+h}$ holds due to (i). Hence, we obtain the following:

$$\lim_{h\to0^+} E_{a-h} \ge E_a \ge \lim_{h\to0^+} E_{a+h}.$$

We complete the proof of (ii).
(iii) Firstly, we prove that

$$E_{\theta a} \le \theta E_a \text{ for } \theta > 1 \text{ closing to } 1.$$

For any $\varepsilon > 0$, we take $u \in S_a \cap P$ such that

$$J(u) \le E_a + \varepsilon.$$

Setting $\tilde{u}(x) = u(\nu^{-\frac{1}{N}}x)$ for $\nu \ge 1$, by the assumption, we have $|\tilde{u}|_2^2 = \nu a$ and the following:

$$\begin{aligned} J(\tilde{u}) = & \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 \mathrm{d}x + \frac{\eta}{2}\int_{\mathbb{R}^N} \tilde{u}^2 \mathrm{d}x - \mu\int_{\mathbb{R}^N} F(\tilde{u})\mathrm{d}x \\ = & \frac{1}{2}\nu^{\frac{N-2s}{N}}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{\eta\nu}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x - \mu\nu\int_{\mathbb{R}^N} F(u)\mathrm{d}x. \end{aligned}$$

Then, we obtain the following:

$$\frac{d}{d\nu}J(\tilde{u}) = \frac{N-2s}{2N}\nu^{-\frac{2s}{N}}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{\eta}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x - \mu\int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Since $u \in P$, we know the following:

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{N\mu}{s}\int_{\mathbb{R}^N} F(u) - \frac{N\mu}{2s}\int_{\mathbb{R}^N} f(u)u\mathrm{d}x = 0.$$

Thus

$$\begin{aligned} \frac{d}{d\nu}J(\tilde{u}) - J(u) = & \left(\frac{N-2s}{2N}\nu^{-\frac{2s}{N}} - \frac{1}{2}\right)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x \\ = & \left(\frac{N-2s}{2N}\nu^{-\frac{2s}{N}} - \frac{1}{2}\right)\frac{N\mu}{s}\int_{\mathbb{R}^N}[\frac{1}{2}f(u)u - F(u)]\mathrm{d}x \\ = & \left(\frac{N-2s}{2s}\mu\nu^{-\frac{2s}{N}} - \frac{N\mu}{2s}\right)\int_{\mathbb{R}^N}[\frac{1}{2}f(u)u - F(u)]\mathrm{d}x. \end{aligned}$$

Obviously, if $\xi > 0$ is small, it follows that

$$\frac{N-2s}{2s}\mu\nu^{-\frac{2s}{N}} - \frac{N\mu}{2s} = \frac{N\mu(\nu^{-\frac{2s}{N}} - 1)}{2s} - \mu\nu^{-\frac{2s}{N}} < 0, \text{ for } \nu \in [1, 1+\xi]. \tag{21}$$

Then by (21) and the condition (f_3), we obtain the following:

$$\frac{d}{dv}J(\tilde{u})-J(u)\leq(\frac{N-2s}{2s}\mu v^{-\frac{2s}{N}}-\frac{N\mu}{2s})(\frac{\alpha-2}{2})\int_{\mathbb{R}^N}F(u)\mathrm{d}x<0.$$

Namely, the following:

$$\frac{d}{dv}J(\tilde{u})-J(u)<0,\ \text{for}\ \ \forall v\in[1,1+\xi].$$

Therefore, for any $\theta\in(1,1+\xi)$, we have the following:

$$J(\tilde{u})-J(u)=\int_1^\theta\frac{d}{dv}J(\tilde{u})dv<\int_1^\theta J(u)dv=J(u)(\theta-1).$$

Then, it is easy to see that

$$E_{\theta a}\leq J(\tilde{u})\leq\theta J(u)\leq\theta(E_a+\varepsilon).$$

Since the arbitrariness of ε, we obtain the following:

$$E_{\theta a}\leq\theta E_a,\ \theta\in(1,1+\xi).$$

If E_a is attained, we can take u as a minimizer in the above step, and obtain the strictly inequality as follows:

$$E_{\theta a}\leq J(\tilde{u})<\theta J(u)=\theta E_a,\ \ \theta\in(1,1+\xi).$$

Furthermore, following the proof of (i), since E_a is nonincreasing, if $E_a<0$, for any $b\in(a,+\infty)$, we can obtain some uniform $\xi>0$ satisfying

$$E_{\theta c}\leq\theta E_c,\ \forall\theta\in[1,1+\xi),\forall c\in[a,b].$$

Now, for any $a>0$ with $E_a<0$ and $\theta>1$, we take $\xi>0$ such that

$$E_{(1+k)c}\leq(1+k)E_c,\forall k\in[0,\xi),\forall c\in[a,\theta b].$$

Then, we may choose $k_0\in(0,\xi)$ and $n\in\mathbb{N}$ such that

$$(1+k_0)^n<\theta<(1+k_0)^{n+1},$$

and so

$$\begin{aligned}E_{\theta a}=E_{(1+k_0)\frac{\theta}{1+k_0}a}\leq&\ (1+k_0)E_{\frac{\theta}{1+k_0}a}\leq(1+k_0)^2E_{\frac{\theta}{(1+k_0)^2}a}\\ \leq&\ (1+k_0)^nE_{\frac{\theta}{(1+k_0)^n}a}\leq(1+k_0)^n\frac{\theta}{(1+k_0)^n}E_a=\theta E_a.\end{aligned}$$

Then, if E_a is attained, we obtain that $E_{\theta a}<\theta E_a$ for any $\theta>1$. For $0<b\leq a$, we obtain the following:

$$E_{a+b}=E_{\frac{a+b}{a}a}\leq\frac{a+b}{a}E_a=E_a+\frac{b}{a}E_a=E_a+\frac{b}{a}E_{\frac{a}{b}b}\leq E_a+E_b.$$

If E_a or E_b is attained, we obtain the following:

$$E_a=E_{\frac{a}{b}b}<\frac{a}{b}E_b,\tag{22}$$

and thus $E_{a+b}<E_a+E_b$. The proof is complete. □

The next compactness lemma on S_a is useful in the study of the autonomous problem as well as the non-autonomous problem.

Lemma 5. *Let $\{u_n\} \subset S_a$ be a minimizing sequence with respect to E_a. Then, for some subsequence, one of the following alternatives holds:*

(i) *$\{u_n\}$ is strongly convergent;*

(ii) *There exists $\{y_n\} \subset S_a$ with $|y_n| \to \infty$ such that the sequence $v_n(x) = u_n(x + y_n)$ is strongly convergent to a function $v \in S_a$ with $J(v) = E_a$.*

Proof. By Lemma 2, we know J is coercive on S_a, the sequence $\{u_n\}$ is bounded, so $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ for some subsequence. Now we consider the following three possibilities:

(1) If $u \not\equiv 0$ and $|u|_2^2 = b \neq a$, we must have $b \in (0, a)$. Set $v_n = u_n - u$, by the Brézis-Lieb Lemma [33], we obtain the following:

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + o_n(1). \tag{23}$$

Since F is a C^1 function and has a subcritical growth in the Sobolev sense, we can see the following:

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x + \int_{\mathbb{R}^N} F(u)\mathrm{d}x + o_n(1). \tag{24}$$

Furthermore, setting $d_n = |v_n|_2^2$, and by using

$$|u_n|_2^2 = |v_n|_2^2 + |u_n|_2^2 + o_n(1),$$

we obtain that $d_n \in (0, a)$ for n large enough and $|v_n|_2^2 \to d$ with $a = b + d$. Hence, the following:

$$\begin{aligned}
E_a + o_n(1) = \;& J(u_n) \\
= \;& \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \frac{\eta}{2}|v_n|_2^2 - \mu \int_{\mathbb{R}^N} F(v_n)\mathrm{d}x \\
& + \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \frac{\eta}{2}|u|_2^2 - \mu \int_{\mathbb{R}^N} F(u)\mathrm{d}x + o_n(1) \\
= \;& J(v_n) + J(u) + o_n(1) \\
\geq \;& E_{d_n} + E_b + o_n(1).
\end{aligned}$$

Letting $n \to +\infty$, by Lemma 4, we find the following:

$$E_a \geq \; E_d + E_b > E_a,$$

which is absurd. This possibility can not exist.

(2) If $|u_n|_2^2 = |u|_2^2 = a$, it is well-known that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Then, by (13) and (14), we have the following:

$$\begin{aligned}
\int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x \leq \;& C_1 \int_{\mathbb{R}^N} |u_n - u|^q \mathrm{d}x + C_2 \int_{\mathbb{R}^N} |u_n - u|^p \mathrm{d}x \\
\leq \;& C\Big(\int_{\mathbb{R}^N} |u_n - u|^2 \mathrm{d}x\Big)^{\frac{q}{2} - \frac{N(q-2)}{4s}} + C\Big(\int_{\mathbb{R}^N} |u_n - u|^2 \mathrm{d}x\Big)^{\frac{p}{2} - \frac{N(p-2)}{4s}}.
\end{aligned}$$

Hence, we obtain $\int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x \to 0$. From (24), we obtain the following:

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

These limits together with $E_a = \lim_{n\to+\infty} J(u_n)$ provide the following:

$$\begin{aligned} E_a = & \lim_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2 + \eta u_n^2)\mathrm{d}x - \mu\int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ \geq & \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + \eta u^2)\mathrm{d}x - \mu\int_{\mathbb{R}^N} F(u)\mathrm{d}x = J(u) \\ \geq & E_a. \end{aligned}$$

Since $u \in S_a$, we infer that $E_a = J(u)$, then $||u_n||^2 \to ||u||^2$, where $||\ ||$ denotes the usual norm in $H^s(\mathbb{R}^N)$. Thus, $u_n \to u$ in $H^s(\mathbb{R}^N)$, which implies that (i) occurs.

(3) If $u \equiv 0$; that is, $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. We claim that there exists $\beta > 0$ such that

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2\mathrm{d}x \geq \beta, \ \text{ for some } R > 0. \tag{25}$$

Indeed, otherwise by Lemma 2.2 of [34], we have $u_n \to 0$ in $L^l(\mathbb{R}^N)$ for all $l \in (2, \frac{2N}{N-2s})$. Thus

$$\begin{aligned} E_a + o_n(1) = J(u_n) = & \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x + \frac{\eta}{2}\int_{\mathbb{R}^N} u_n^2\mathrm{d}x - \mu\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \\ = & \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x + \frac{\eta}{2}\int_{\mathbb{R}^N} u_n^2\mathrm{d}x + o_n(1), \end{aligned}$$

which contradicts the Lemma 3.

Hence, from this case, (25) holds and $|y_n| \to +\infty$, then we consider $\tilde{u}_n(x) = u(x + y_n)$, obviously $\{\tilde{u}_n\} \subset S_a$ and it is also a minimizing sequence with respect to J_a. Moreover, there exists $\tilde{u} \in H^s(\mathbb{R}^N)\backslash\{0\}$ such that $\tilde{u}_n(x) \rightharpoonup \tilde{u}$ in $H^s(\mathbb{R}^N)$. Following as in the first two possibilities of the proof, we infer that $\tilde{u}_n(x) \to \tilde{u}$ in $H^s(\mathbb{R}^N)$, which implies that (ii) occurs. This proves the lemma. □

In what follows, we begin to prove Theorem 2.

Proof of Theorem 2. By Lemmas 2 and 3, there exists a bounded minimizing sequence $\{u_n\} \subset S_a$ satisfying $J(u_n) \to E_a$. Then applying Lemma 5, there exists $u \in S_a$ such that $J(u) = E_a$. By the Lagrange multiplier, there exists $\lambda \in \mathbb{R}$ such that

$$J'(u) = \lambda\Phi'(u) \ \text{ in } \ (H^s(\mathbb{R}^N))', \tag{26}$$

where $\Phi(u) : H^s(\mathbb{R}^N) \to \mathbb{R}$ is given by the following:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2\mathrm{d}x, \quad u \in H^s(\mathbb{R}^N).$$

Therefore, from (26), we have the following:

$$(-\Delta)^s u + \eta u = \lambda u + \mu f(u) \ \text{ in } \ \mathbb{R}^N. \tag{27}$$

By Lemma 1, we can obtain the following:

$$\begin{aligned}(\lambda-\eta)\int_{\mathbb{R}^N}u^2\mathrm{d}x &= \int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x-\mu\int_{\mathbb{R}^N}f(u)u\mathrm{d}x\\ &=-\frac{N\mu}{s}\int_{\mathbb{R}^N}F(u)\mathrm{d}x+\frac{N\mu}{2s}\int_{\mathbb{R}^N}f(u)u\mathrm{d}x-\mu\int_{\mathbb{R}^N}f(u)u\mathrm{d}x\\ &=-\frac{\mu}{s}\left[\int_{\mathbb{R}^N}(NF(u)-\frac{N-2s}{2}f(u)u)\mathrm{d}x\right].\end{aligned}$$

Furthermore, according to the condition (f_3) and the claim 3, we must have $\lambda<\eta$.

Next, we will prove that u can be chosen to be positive. Obviously, we have $J(u)=J(|u|)$. Moreover, since $u\in S_a$ shows that $|u|\in S_a$, we infer that

$$E_a=J(u)=J(|u|)\geq E_a,$$

which implies that $J(|u|)=E_a$, and so, we can replace u by $|u|$. Furthermore, if u^* denotes the symmetrization radial decreasing rearrangement of u (see Section 1 [35]), we observe the following:

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\leq\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y,\\ &\int_{\mathbb{R}^N}|u|^2\mathrm{d}x=\int_{\mathbb{R}^N}|u^*|^2\mathrm{d}x\text{ and }\int_{\mathbb{R}^N}F(u)\mathrm{d}x=\int_{\mathbb{R}^N}F(u^*)\mathrm{d}x,\end{aligned}\tag{28}$$

then $u^*\in S_a$ and $J(u^*)=E_a$, it follows that we can replace u by u^*. Similarly, as in [36], one can show that $u(x)>0$ for any $x\in\mathbb{R}$. This completes the proof. □

3. The Non-Autonomous Problem

In this section, we first give some properties of the functional $I_\varepsilon(u)$ given by (8) restricted to the sphere S_a, and then prove Theorem 1. Define the following energy functionals:

$$I_\infty(u)=\ \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x-h_\infty\int_{\mathbb{R}^N}F(u)\mathrm{d}x,$$

and for $i=1,2,\cdots,k$,

$$I_{a_i}(u)=\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x+\frac{V(a_i)}{2}\int_{\mathbb{R}^N}u^2\mathrm{d}x-h(a_i)\int_{\mathbb{R}^N}F(u)\mathrm{d}x.$$

Moreover, denoted by $E_{\varepsilon,a}$, $E_{a_i,a}$, and $E_{\infty,a}$ the following real numbers:

$$E_{\varepsilon,a}=\inf_{u\in S_a}I_\varepsilon(u),\ \ E_{a_i,a}=\inf_{u\in S_a}I_{a_i}(u),\ \ E_{\infty,a}=\inf_{u\in S_a}I_\infty(u).$$

The next two lemmas establish some crucial relations involving the levels $E_{\varepsilon,a}$, $E_{\infty,a}$, and $E_{a_i,a}$. For any $\alpha,\beta\in\mathbb{R}$, set

$$J_{\alpha\beta}(u)=\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x+\frac{\beta}{2}\int_{\mathbb{R}^N}u^2\mathrm{d}x-\alpha\int_{\mathbb{R}^N}F(u)\mathrm{d}x,$$

where

$$E_{\alpha\beta,a}=\inf_{u\in S_a}J_{\alpha\beta}(u).$$

Lemma 6. *Fix $a>0$, let $0<h_1<h_2$ and $V_2<V_1\leq 0$. Then $E_{h_2V_2,a}<E_{h_1V_1,a}<0$.*

Proof. The proof is standard and we omit the details. □

Lemma 7. $\limsup\limits_{\varepsilon\to0^+}E_{\varepsilon,a}\leq E_{a_i,a}<E_{\infty,a}<0, i=1,2,\cdots,k.$

Proof. By the proof of the Theorem 2, choose $u_0 \in S_a$ such that $I_{a_i}(u_0) = E_{a_i,a}$. For $1 \le i \le k$, we define

$$u = u_0\left(x - \frac{a_i}{\varepsilon}\right),\ x \in \mathbb{R}^N.$$

Then $u \in S_a$ for all $\varepsilon > 0$, we have the following:

$$E_{\varepsilon,a} \le I_\varepsilon(u) = \frac{1}{2}|(-\Delta)^{\frac{s}{2}}u_0|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + a_i)u_0^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x + a_i)F(u_0)\mathrm{d}x.$$

Letting $\varepsilon \to 0^+$, by the Lebesgue dominated convergence theorem, we deduce the following:

$$\limsup_{\varepsilon \to 0^+} E_{\varepsilon,a} \le \lim_{\varepsilon \to 0^+} I_\varepsilon(u) = I_{a_i}(u_0) = E_{a_i,a}. \tag{29}$$

Noting that $E_{\infty,a}$ can be achieved, due to $0 < h_\infty < h(a_i)$ and $V(a_i) < 0$, we have the following:

$$E_{a_i,a} < E_{\infty,a} < 0.$$

It completes the proof. □

Hence, by Lemma 7, there exists $\varepsilon_1 > 0$ satisfying $E_{\varepsilon,a} < E_{\infty,a}$ for all $\varepsilon \in (0, \varepsilon_1)$. In the following, we always assume that $\varepsilon \in (0, \varepsilon_1)$. The next three lemmas will be used to prove the $(PS)_c$ condition for I_ε restricts to S_a at some levels.

Lemma 8. *Assume $\{u_n\} \subset S_a$ such that $I_\varepsilon(u_n) \to c$ as $n \to +\infty$ with $c < E_{\infty,a} < 0$, then*

$$\delta := \liminf_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n(x)|^2 \mathrm{d}x > 0.$$

Proof. We argue by contradiction and assume that $\delta = 0$, then up to a subsequence, we have $u_n \to 0$ in $L^l(\mathbb{R}^N)$ for all $l \in (2, \frac{2N}{N-2s})$, by the Lebesgue dominated convergence theorem and $(f_1) - (f_2)$, we infer the following:

$$\int_{\mathbb{R}^N} h(\varepsilon x)F(u_n)\mathrm{d}x \to 0 \ \text{ as } \ n \to +\infty. \tag{30}$$

Since $V(x) \to 0$ as $|x| \to \infty$, one can show the following:

$$\int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x = o_n(1),$$

which combined with (30) gives the following:

$$0 > c = I_\varepsilon(u_n) + o(1) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2 \mathrm{d}x + o(1) \ge 0,$$

which is a contradiction. □

Lemma 9. *Under the assumption of Lemma 8, assume $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then $u \not\equiv 0$.*

Proof. By Lemma 8, we have that

$$\liminf_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n(x)|^2 \mathrm{d}x > 0.$$

So if $u \equiv 0$, there exists $\{y_n\}$ satisfying $|y_n| \to \infty$, let $\tilde{u}_n = u_n(x + y_n)$, obviously $\{\tilde{u}_n\} \subset S_a$, we have the following:

$$\begin{aligned}
c + o_n(1) =& I_\varepsilon(u_n) \\
=& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x \\
=& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_n) \tilde{u}_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x + \varepsilon y_n) F(\tilde{u}_n) \mathrm{d}x \\
=& I_\infty(\tilde{u}_n) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x + \varepsilon y_n) - V_\infty) \tilde{u}_n^2 \mathrm{d}x + \int_{\mathbb{R}^N} (h_\infty - h(\varepsilon x + \varepsilon y_n)) F(\tilde{u}_n) \mathrm{d}x \\
=& I_\infty(\tilde{u}_n) + o_n(1) \geq E_{\infty,a} + o_n(1),
\end{aligned}$$

which is absurd, because $c < E_{\infty,a} < 0$. This proves the lemma. □

Lemma 10. *Let $\{u_n\} \subset S_a$ be a $(PS)_c$ sequence of I_ε restricted to S_a with $c < E_{\infty,a} < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^s(\mathbb{R}^N)$. If $v_n = u_n - u_\varepsilon \not\to 0$ in $H^s(\mathbb{R}^N)$, there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_1)$ such that*

$$\liminf_{n \to +\infty} |u_n - u_\varepsilon|_2^2 \geq \beta.$$

Proof. Setting the functional $\Phi : H^s(\mathbb{R}^N) \to \mathbb{R}$ given by the following:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x.$$

It follows that $S_a = \Phi^{-1}(\{a/2\})$. Then, by Willem (Proposition 5.12 [33]), there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$||I_\varepsilon'(u_n) - \lambda_n \Phi'(u_n)||_{(H^s(\mathbb{R}^N))'} \to 0 \ \text{ as } n \to +\infty. \tag{31}$$

By the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$, we know $\{\lambda_n\}$ is a bounded sequence, thus there exists λ_ε such that $\lambda_n \to \lambda_\varepsilon$ as $n \to +\infty$. This together with (31) leads to the following:

$$I_\varepsilon'(u_\varepsilon) - \lambda_\varepsilon \Phi'(u_\varepsilon) = 0 \ \text{ in } \ (H^s(\mathbb{R}^N))',$$

and then

$$||I_\varepsilon'(v_n) - \lambda_n \Phi'(v_n)||_{(H^s(\mathbb{R}^N))'} \to 0 \ \text{ as } n \to +\infty. \tag{32}$$

By a straightforward calculation, we have the following:

$$\begin{aligned}
E_{\infty,a} >& \liminf_{n \to +\infty} I_\varepsilon(u_n) \\
=& \liminf_{n \to +\infty} \left(I_\varepsilon(u_n) - \frac{1}{2} I_\varepsilon'(u_n) u_n + \frac{1}{2}\lambda_n a + o_n(1) \right) \\
=& \liminf_{n \to +\infty} \left[\int_{\mathbb{R}^N} \frac{h(\varepsilon x)}{2} f(u_n) u_n \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x + \frac{1}{2}\lambda_n a + o_n(1) \right] \\
\geq& \frac{1}{2}\lambda_\varepsilon a,
\end{aligned}$$

implying the following:

$$\lambda_\varepsilon \leq \frac{2E_{\infty,a}}{a} < 0, \ \text{for all } \varepsilon \in (0, \varepsilon_1). \tag{33}$$

From (32), we obtain the following:

$$|(-\Delta)^{\frac{s}{2}} v_n|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x) |v_n|^2 \mathrm{d}x - \lambda_\varepsilon |v_n|_2^2 - \int_{\mathbb{R}^N} h(\varepsilon x) f(v_n) v_n \mathrm{d}x = o_n(1), \tag{34}$$

which combined with (33) gives the following:

$$|(-\Delta)^{\frac{s}{2}} v_n|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2 \mathrm{d}x - \frac{2E_{\infty,a}}{a} \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \le \int_{\mathbb{R}^N} h(\varepsilon x) f(v_n) v_n \mathrm{d}x + o_n(1).$$

Then

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x + C_3 \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \le C_2 \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x + o_n(1), \tag{35}$$

for some constant $C_3 > 0$ that does not depend on $\varepsilon \in (0, \varepsilon_1)$. If $u_n \not\to u_\varepsilon$ in $H^s(\mathbb{R}^N)$; that is $v_n \not\to 0$ in $H^s(\mathbb{R}^N)$, the last inequality ensures that there exists $C_0 > 0$ independent of ε such that

$$\liminf_{n \to +\infty} |v_n|_p^p \ge C_0. \tag{36}$$

Then, by the fractional Gagliardo-Nirenberg-sobolev inequality,

$$\int_{\mathbb{R}^N} |v_n|^\alpha \le C(s, N, \alpha) \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \Big)^{\frac{N(\alpha-2)}{4s}} \Big(\int_{\mathbb{R}^N} |v_n|^2 \Big)^{\frac{\alpha}{2} - \frac{N(\alpha-2)}{4s}},$$

for some positive constant $C(s, N, \alpha) > 0$. We have

$$\begin{aligned} \liminf_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^p \le\; & C(s, N, p) \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \Big)^{\frac{N(p-2)}{4s}} \Big(\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^2 \Big)^{\frac{p}{2} - \frac{N(p-2)}{4s}} \\ \le\; & C(s, N, p) K^{\frac{N(p-2)}{4s}} \Big(\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^2 \Big)^{\frac{p}{2} - \frac{N(p-2)}{4s}}, \end{aligned} \tag{37}$$

where $K > 0$ is a suitable constant independent of $\varepsilon \in (0, \varepsilon_1)$ satisfying the condition $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \le K$ for all $n \in \mathbb{N}$. Now, the lemma follows from (36) and (37). □

Next we will give the compactness lemma.

Lemma 11. *Let*

$$0 < \rho_0 < \min\Big\{ E_{\infty,a} - E_{a_i,a}, \frac{\beta}{a}(E_{\infty,a} - E_{a_i,a}) \Big\}.$$

Then, for each $\varepsilon \in (0, \varepsilon_1)$, the functional I_ε satisfies the $(PS)_c$ condition restricts to S_a if $c < E_{a_i,a} + \rho_0$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε restricts to S_a and $c < E_{a_i,a} + \rho_0$. It follows that $c < E_{\infty,a} < 0$, since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we let $u_n \rightharpoonup u_\varepsilon$ in $H^s(\mathbb{R}^N)$. By Lemma 9, $u_\varepsilon \not\equiv 0$. Denote $v_n = u_n - u_\varepsilon$, if $u_n \to u_\varepsilon$ in $H^s(\mathbb{R}^N)$, the proof is complete. If $u_n \not\to u_\varepsilon$ in $H^s(\mathbb{R}^N)$, by Lemma 10,

$$\liminf_{n \to +\infty} |v_n|_2^2 \ge \beta.$$

Set $b = |u_\varepsilon|_2^2$, $d_n = |v_n|_2^2$ and suppose that $|v_n|_2^2 \to d > 0$, then we obtain $d \ge \beta > 0$ and $a = b + d$. From $d_n \in (0, a)$ for n large enough, we obtain the following:

$$c + o_n(1) = I_\varepsilon(u_n) = I_\varepsilon(v_n) + I_\varepsilon(u_\varepsilon) + o_n(1). \tag{38}$$

Since $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, we can follow the lines in the proof of Lemma 9. Then

$$I_\varepsilon(v_n) \ge E_{\infty,d_n} + o_n(1), \tag{39}$$

by (38) and (39), we obtain the following:

$$\begin{aligned} c + o_n(1) = I_\varepsilon(u_n) \ge\; & E_{\infty,d_n} + I_\varepsilon(u_\varepsilon) + o_n(1) \\ \ge\; & E_{\infty,d_n} + E_{a_i,b} + o_n(1). \end{aligned}$$

Letting $n \to \infty$, by the in Equation (22), we have the following:

$$\begin{aligned} c \geq E_{\infty,d} + E_{a_i,b} \geq & \ \frac{d}{a} E_{\infty,a} + \frac{b}{a} E_{a_i,a} \\ = & \ E_{a_i,a} + \frac{d}{a}(E_{\infty,a} - E_{a_i,a}) \\ \geq & \ E_{a_i,a} + \frac{\beta}{a}(E_{\infty,a} - E_{a_i,a}), \end{aligned}$$

which is a contradiction, because $c < E_{a_i,a} + \frac{\beta}{a}(E_{\infty,a} - E_{a_i,a})$. Therefore, we can obtain $u_n \to u_\varepsilon$ in $H^s(\mathbb{R}^N)$. $\square$

In what follows, let us fix $\bar{\rho}, \bar{r} > 0$ satisfying the following:

(1) $\overline{B_{\bar{\rho}}(a_i)} \cap \overline{B_{\bar{\rho}}(a_j)}$ for $i \neq j$ and $i, j \in \{1, \dots k\}$;
(2) $\cup_{i=1}^{k} B_{\bar{\rho}}(a_i) \subset B_{\bar{r}}(0)$;
(3) $Q_{\frac{\bar{\rho}}{2}} = \cup_{i=1}^{l} \overline{B_{\frac{\bar{\rho}}{2}}(a_i)}$.

We set the function $G_\varepsilon : H^s(\mathbb{R}^N) \backslash \{0\} \to \mathbb{R}^N$ by

$$G_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$ denotes the characteristic function; that is, the following:

$$\chi(x) = \begin{cases} x, & \text{if } |x| \leq \bar{r}, \\ \bar{r}\frac{x}{|x|}, & \text{if } |x| > \bar{r}. \end{cases}$$

The next two lemmas will be useful to obtain important (PS) sequences for I_ε restricted to S_a.

Lemma 12. *For $\varepsilon \in (0, \varepsilon_1)$, there exists $\delta_1 > 0$ such that if $u \in S_a$ and $I_\varepsilon(u) \leq E_{a_i,a} + \delta_1$, then*

$$G_\varepsilon(u) \in Q_{\frac{\bar{\rho}}{2}}, \forall \varepsilon \in (0, \varepsilon_1).$$

Proof. If the lemma does not occur, there must be $\delta_n \to 0$, $\varepsilon_n \to 0$ and $\{u_n\} \subset S_a$ such that

$$I_{\varepsilon_n}(u_n) \leq E_{a_i,a} + \delta_n \text{ and } G_{\varepsilon_n}(u_n) \notin Q_{\frac{\bar{\rho}}{2}}, \forall \varepsilon \in (0, \varepsilon_1). \tag{40}$$

So we have

$$E_{a_i,a} \leq I_{a_i}(u_n) \leq I_{\varepsilon_n}(u_n) \leq E_{a_i,a} + \delta_n,$$

then

$$\{u_n\} \subset S_a \text{ and } I_{a_i}(u_n) \to E_{a_i,a}.$$

According to Lemma 5, we have one of the following two cases:

(i) $u_n \to u$ in $H^s(\mathbb{R}^N)$ for some $u \in S_a$;
(ii) There exists $\{y_n\} \subset S_a$ with $|y_n| \to \infty$ such that the sequence $v_n(x) = u_n(x + y_n)$ in $H^s(\mathbb{R}^N)$ to some $v \in S_a$.

For (i): By the Lebesgue dominated convergence theorem,

$$G_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \chi(0)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} = 0 \in Q_{\frac{\bar{\rho}}{2}}.$$

Then $G_{\varepsilon_n}(u_n) \in Q_{\frac{\bar{\rho}}{2}}$ for n large enough, which contradicts (40).

For (ii): We will study the following two cases: (I) $|\varepsilon_n y_n| \to +\infty$; (II) $\varepsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$.

If (I) holds, the limit $v_n \to v$ in $H^s(\mathbb{R}^N)$ provides the following:

$$I_{\varepsilon_n}(u_n) = \frac{1}{2}|(-\Delta)^{\frac{s}{2}} v_n|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n y_n)|v_n|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon_n x + \varepsilon_n y_n)F(v_n)\mathrm{d}x$$
$$\to I_\infty(v) \text{ as } n \to +\infty. \tag{41}$$

Since $I_\varepsilon(u_n) \le E_{a_i,a} + \delta_n$, we deduce the following:

$$E_{\infty,a} \le I_\infty(v) \le E_{a_i,a},$$

which contradicts $E_{a_i,a} < E_{\infty,a}$ in Lemma 7.

If (II) holds, by (41), we obtain the following:

$$I_{\varepsilon_n}(u_n) \to I_{h(y)V(y)}(v) \text{ as } n \to +\infty,$$

and then $E_{h(y)V(y),a} \le I_{h(y)V(y)}(v) \le E_{a_i,a}$. By Lemma 6, we must have $h(y) = h(a_i)$ and $V(y) = V(a_i)$. Namely, $y = a_i$ for some $i = 1, 2, \ldots, k$. Hence,

$$G_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x)|u_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x + \varepsilon_n y_n)|v_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x}$$
$$\to \frac{\int_{\mathbb{R}^N} \chi(y)|v|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x} = 0 \in Q_{\frac{\bar{\rho}}{2}},$$

which implies that $G_{\varepsilon_n}(u_n) \in Q_{\frac{\bar{\rho}}{2}}$ for n large enough, which contradicts (40). The proof is complete. □

From now on, we will use the following notations:

- $\theta_\varepsilon^i := \{u \in S_a : |G_\varepsilon(u) - a_i| \le \bar{\rho}\}$;
- $\partial\theta_\varepsilon^i := \{u \in S_a : |G_\varepsilon(u) - a_i| = \bar{\rho}\}$;
- $\beta_\varepsilon^i = \inf_{u \in \theta_\varepsilon^i} I_\varepsilon(u)$;
- $\bar{\beta}_\varepsilon^i = \inf_{u \in \partial\theta_\varepsilon^i} I_\varepsilon(u)$.

Lemma 13. *Let ρ_0 be defined in lemma 11. Then,*

$$\beta_\varepsilon^i < E_{a_i,a} + \rho_0 \text{ and } \beta_\varepsilon^i < \bar{\beta}_\varepsilon^i, \text{ for } \forall \varepsilon \in (0, \varepsilon_1).$$

Proof. Let $u \in S_a$ satisfy

$$I_{a_i}(u) = E_{a_i,a}.$$

For $1 \le i \le k$, we define the following:

$$\hat{u}_\varepsilon^i(x) := u\Big(x - \frac{a_i}{\varepsilon}\Big), \; x \in \mathbb{R}^N.$$

Then $\hat{u}_\varepsilon^i(x) \in S_a$ for all $\varepsilon > 0$ and $1 \le i \le k$. Direct calculations give the following:

$$I_\varepsilon(\hat{u}_\varepsilon^i) = \frac{1}{2}|(-\Delta)^{\frac{s}{2}} u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + a_i)|u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x + a_i)F(u)\mathrm{d}x,$$

and then

$$\lim_{\varepsilon \to 0} I_\varepsilon(\hat{u}_\varepsilon^i) = I_{a_i}(u) = E_{a_i,a}, \tag{42}$$

we know

$$G_\varepsilon(\hat{u}_\varepsilon^i) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x + a_i)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} \to a_i \text{ as } \varepsilon \to 0^+.$$

So $\hat{u}_\varepsilon^i(x) \in \theta_\varepsilon^i$ for ε small enough, which combined with (42) implies the following:

$$I_\varepsilon(\hat{u}_\varepsilon^i) < E_{a_i,a} + \frac{\delta_1}{2}, \ \forall \varepsilon \in (0, \varepsilon_1).$$

Decreasing δ_1 if necessary, we know the following:

$$\beta_\varepsilon^i < E_{a_i,a} + \rho_0, \ \forall \varepsilon \in (0, \varepsilon_1).$$

For any $u \in \partial\theta_\varepsilon^i$, which is $u \in S_a$ and $|G_\varepsilon(u) - a_i| = \bar{\rho}$, we obtain the following: $|G_\varepsilon(u)| \notin Q_{\frac{\bar{\rho}}{2}}$.

Then by Lemma 12,

$$I_\varepsilon(u) > E_{a_i,a} + \delta_1, \ \text{ for all } u \in \partial\theta_\varepsilon^i \text{ and } \varepsilon \in (0, \varepsilon_1),$$

which implies the following:

$$\bar{\beta}_\varepsilon^i = \inf_{u \in \partial\theta_\varepsilon^i} I_\varepsilon(u) \geq E_{a_i,a} + \delta_1.$$

Then, we have

$$\beta_\varepsilon^i < \bar{\beta}_\varepsilon^i, \ \text{ for all } \varepsilon \in (0.\varepsilon_1).$$

□

4. Proof of Theorem 1

Proof. By Lemma 13, for each $i \in \{1, 2, \ldots, k\}$, we can use the Ekeland's variational principle to find a sequence $\{u_n^i\} \subset S_a$ satisfying

$$I_\varepsilon(u_n^i) \to \beta_\varepsilon^i \ \text{ and } \ I_\varepsilon(w) \geq I_\varepsilon(u_n^i) - \frac{1}{n}||w - u_n^i||, \ \forall w \in \theta_\varepsilon^i.$$

Recalling Lemma 13, $\beta_\varepsilon^i < \bar{\beta}_\varepsilon^i$, and so $u_n^i \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i$ for n large enough.

Let $w \in T_{u_n^i} S_a$, there exists $\delta > 0$ such that the path $\gamma : (-\delta, \delta) \to S_a$ defined by the following:

$$\gamma(t) = a \frac{(u_n^i + tw)}{|u_n^i + tw|_2},$$

belongs to $C^1((-\delta, \delta), S_a)$ and satisfies

$$\gamma(t) \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i \ \ \forall t \in (-\delta, \delta), \ \gamma(0) = u_n^i \text{ and } \gamma'(0) = w.$$

Then for any $t \in (0, \delta)$,

$$\frac{I_\varepsilon(\gamma(t)) - I_\varepsilon(\gamma(0))}{t} = \frac{I_\varepsilon(\gamma(t)) - I_\varepsilon(u_n^i)}{t} \geq -\frac{1}{n}\left\|\frac{\gamma(t) - u_n^i}{t}\right\| = -\frac{1}{n}\left\|\frac{\gamma(t) - \gamma(0)}{t}\right\|.$$

Taking the limit of $t \to 0^+$, we obtain the following: $I_\varepsilon'(u_n^i)(w) \geq -\frac{1}{n}||w||$. Replacing w by $-w$, we obtain $|I_\varepsilon'(u_n^i)(w)| \leq \frac{1}{n}||w||$.

Then, we have the following:

$$\sup\{|I_\varepsilon'(u_n^i)(w)| : ||w|| \leq \delta_n\} \leq \frac{1}{n}.$$

Consequently,

$$I_\varepsilon(u_n^i) \to \beta_\varepsilon^i \ \text{ as } n \to +\infty \ \text{ and } \ ||I_\varepsilon|_{S_a}'(u_n^i)|| \to 0 \text{ as } n \to +\infty,$$

that is, $\{u_n^i\}$ is a $(PS)_{\beta_\varepsilon^i}$ for I_ε restricts to S_a. Since $\beta_\varepsilon^i < E_{a_i,a} + \rho_0$, it follows from Lemma 11 that there exists u^i such that $u_n^i \to u^i$ in $H^s(\mathbb{R}^N)$. Then, we obtain the following:

$$u^i \in \theta_\varepsilon^i, I_\varepsilon(u_n^i) = \beta_\varepsilon^i \text{ and } I_\varepsilon|'_{S_a}(u_n^i) = 0.$$

Moreover,

$$G_\varepsilon(u^i) \in \overline{B_{\bar{\rho}}(a_i)}, \quad G_\varepsilon(u^j) \in \overline{B_{\bar{\rho}}(a_j)},$$

and

$$\overline{B_{\bar{\rho}}(a_i)} \cap \overline{B_{\bar{\rho}}(a_j)} = \emptyset \text{ for } i \neq j,$$

which implies that $u^i \neq u^j$ for $i \neq j$ while $1 \leq i, j \leq k$, we can understand I_ε has at least k nontrivial critical points for any $\varepsilon \in (0, \varepsilon_1)$. Therefore, we obtain the theorem. □

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