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Boundedness of Vector Linéard Equation with Multiple Variable Delays

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Abstract: In this article, we consider a system of ordinary differential equations (ODEs) of second order with two variable time delays. We obtain new conditions for uniform ultimate bounded (UUB) solutions of the considered system. The technique of the proof is based on the Lyapunov–Krasovskii functional (LKF) method using a new LKF. The main result of this article extends and improves a recent result for ODEs of second order with a constant delay to a more general system of ODEs of second order with two variable time delays. In this particular case, we also give a numerical example to verify the application of the main result of this article.

Keywords: ODEs of second order; uniform ultimate; boundedness; LKF; variable time delay; second order

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1. Introduction

From the relevant literature, we know that ordinary differential equations (ODEs) of second order can denote numerous real world applications. For example, the Hill equation

$$\ddot{x} + a(t)x = 0,$$

has great importance in the study of the stability and instability of geodesics on Riemannian manifolds, and physicists can use this fact to investigate dynamics in the Hamiltonian systems (see Yang, Xiao-Song 1999 [1]).

Both the following linear and nonlinear ODEs of second order

$$(k(t)x')' + L(t)x = 0$$

and

$$x'' + g(x, x')x' + h(x) = e(t)$$

are frequently encountered as mathematical models of most dynamic processes in electromechanical systems (see the book of Ahmad and Rama Mohana Rao [2]).

Next, an ODE of the form

$$mx'' + \ell x' + kx = F(t),$$

where m, ℓ and k are positive constants, may represent the motion of a particle of mass m held by a linear spring, with the spring constant k, subject to an exterior disturbance $F(t)$ (see the book of Ahmad and Rama Mohana Rao [2]).

The nonlinear ODE of second order

$$x'' + f(x)x' + h(x) = 0$$

is called the Liénard equation. This equation can be used to model oscillating circuits (see Ahmad and Rama Mohana Rao [2]).

The electrophysiological behavior can be modeled as the ODE

$$\frac{dV}{dt} = \frac{I_{ion} + I_{stim}}{C_m},$$

where V is voltage, t is time, I_{ion} is the sum of all transmembrane ionic currents, I_{stim} is the externally applied stimulus current and C_m is cell capacitance per unit surface area (see Kim et al. [3]).

The desk dynamic relations, having the viscosity ratio E and mass M, together with the installed motors having the equivalent dissipation ratio D, are represented by the two-dimensional system of ODEs:

$$M\frac{dv}{dt} - m_1 r_1 \frac{dw_1}{dt}\sin\theta_1 = -Dv - Ey + m_1 r_1 w_1^2 \cos\theta_1, \tag{1}$$

$$\frac{dy}{dt} = v, \frac{dw_1}{dt} = j_1^{-1}(h_1 - q_1), \frac{d\theta_1}{dt} = w_1, \tag{2}$$

where v and w_1 are the desk vibration displacement speed, m_1 and r_1 are eccentricity mass and eccentricity creating the imbalance and j_1, h_1, q_1 are the rotating masses moment of inertia, electromagnetic torque and the moment of resistance on the shaft, respectively (see Baykov and Gordeev [4]).

On the other hand, in many real world applications, a system under consideration may be governed by a principle of causality, i.e., the future state of the system is independent of the past states and is determined solely by the present. For example, this fact can be represented by the following Liénard equation with the constant delay τ and its modified forms

$$x'' + f(x)x' + h(x(t-\tau)) = 0,$$

(see the book of Hale and Verduyn Lunel [5]).

It can also be seen from the literature that, in applied sciences, some practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information sciences, etc., are associated with these types of differential equations, i.e., certain ODEs of the second order with and without delay (see also the books of Èl'sgol'ts and Norkin [6], Krasovskii [7], Rihan [8], Gil' [9], Smith [10], Kuang [11] and Yoshizawa [12]). Indeed, numerous kinds of ODEs of second order with and without delays can be used to model real world problems. However, for the sake of brevity, we will not give more details about those models. Hence, we would like to say that, because of the effective roles of ODEs of second order in the real world problems and their applications, the qualitative behaviors of solutions of n-dimensional ODEs of second order with delays deserve to be studied.

Moreover, since these equations are only of the second order, we would naturally be inclined to compute their solutions explicitly or numerically. However, as we know from practice, there are many such equations, e.g., linear equations with constant coefficients, for which this can be effectively carried out. This case is very difficult for the ODEs or systems of ODEs with delay. Hence, in the relevant literature, various methods or techniques have been improved to determine the qualitative behavior of ODEs of second order with and without delay(s) without solving them and without prior information of solutions. These methods and techniques are called the Lyapunov's direct method, the fixed point method, the LKF method, the Lyapunov–Razumikhin technique and so on. According to the stability theory of functional differential equations, when we use the Lyapunov–Krasovskii method as a basic tool in the proof, it is needed to construct or define a Lyapunov–Krasovskii functional such that it is positive definite and its derivative along the considered system has to be negative or negative semi-definite.

As for the motivation of this article, recently, Adeyanju [13] and Adeyanju and Tunç [14] considered the following n-dimensional ODEs of second order without delay and with the constant delay τ, respectively:

$$X'' + F(X, X')X' + H(X) = P(t, X, X')$$

and

$$X'' + F(X, X')X' + H(X(t-\tau)) = P(t, X, X').$$

Adeyanju [13] discussed boundedness, uniformly boundedness and some other behaviors of solutions of the first ODE via the Lyapunov second method. Next, Adeyanju and Tunç [14] constructed suitable new sufficient conditions that allow for the uniform ultimate boundedness of solutions of the second ODE with constant delay via the LKF method as a basic tool. On the other hand, we would like to outline some related works as follows: the stability and boundedness of solutions of vector differential equations by Lyapunov's second method (see [13,15–20]), qualitative behavior of ordinary and functional differential equations (see [2,21]), stability, boundedness and periodicity of solutions of delay differential equations of second order (see [22–24]) and existence, uniqueness, stability, etc., of ordinary and functional differential equations (see [12]); for some other results, see [1,14,25,26].

Motivated by the works of Adeyanju [13], Adeyanju and Tunç [14] and those mentioned above, we consider the following system of ODEs of second order with two multiple variable time delays:

$$X'' + a(t)F(t, X, X')X' + b(t)G(X) + \sum_{i=1}^{2} H_i(X(t - r_i(t))) = P(t, X, X'). \tag{3}$$

Equation (3) can be converted to the following differential system:

$$\begin{aligned} X' =& Y, \\ Y' =& -a(t)F(t, X, Y)Y - b(t)G(X) - \sum_{i=1}^{2} H_i(X) \\ & - \sum_{i=1}^{2} H_i(X) + \sum_{i=1}^{2} \int_{t-r_i(t)}^{t} J_{H_i}(X(s))Y(s)ds + P(t, X, Y), \end{aligned} \tag{4}$$

where $X,\ Y \in R^n$, $t \in R^+$, $R^+ = [0, \infty)$, $a \in C(R^+, (0, \infty))$, $b \in C^1(R^+, (0, \infty))$, $r_i \in C^1(R^+, R^+)$, $H_i \in C^1(R^n, R^n)$, $H_i(0) = 0$, $P \in C(R^+ \times R^{2n}, R^n)$, $G \in C^1(R^n, R^n)$, $G(0) = 0$ and $F \in C(R^+ \times R^{2n}, R^{2n})$ is a symmetric, positive definite matrix function depending on the arguments displayed explicitly. Moreover, the Jacobian matrices corresponding to $H_i(X)$, $G(X)$ are defined by

$$J_{H_1}(X) = \left(\frac{\partial h_{1i}}{\partial x_j}\right), J_{H_2}(X) = \left(\frac{\partial h_{2i}}{\partial x_j}\right), J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), (i, j = 1, 2, ..., n),$$

and $(h_{11}, h_{12}, ..., h_{1n}) \in H_1$, $(h_{21}, h_{22}, ..., h_{2n}) \in H_2$, respectively.

As for the contributions of this article, we would like to summarize some of them briefly in the following sentences. It can be observed that the ODEs of second order without and with delay of Adeyanju [13] and Adeyanju and Tunç [14], respectively, are particular cases of our Equation (3). The ODEs of Adeyanju [13] and Adeyanju and Tunç [14] are without delay and with a constant delay, respectively. However, our ODE (3) has two variable time delays. Next, Gözen [27] considered a system of ODEs of second order without delay. However, the system of ODEs of second order of this article and that in [27] are different mathematical models. In addition, in [27], the author discussed some qualitative properties of the considered system by using a Lyapunov function. Here, we proved the main result of this article via the LKF method. Finally, the differential system of Gözen [27], the considered problem and that in this paper are different from each other. Hence, we have contributions ranging from the cases of those without delay and a constant delay to the case of two variable time delays. This case is also a novelty. Furthermore, in the papers of Ademola et al. [22], Adeyanju and Adams [15], Omeike et al. ([16,17]), Tunç [23,26], Tunç and Tunç [19,24] and Tunç et al. [28], the boundedness and some other qualitative behaviors of solutions of certain scalar and vector ODEs of second order with or without delay have been investigated, and several interesting results have been obtained in these papers. It can be checked that the ODEs of second order of these papers are particular cases of the ODE (3) of second order, or they are included by the ODE (3) of second order for some particular choices of them. Hence, this paper has new contributions to the results of the mentioned papers. Finally, this is a theoretical work on the UUB solutions of functional differential equations.

Throughout this article, we will use the following abbreviations, respectively:

$$F(t,X,Y) = F(.), J_G(Y) = J_G(.), J_{H_i}(X) = J_{H_i}(.) \text{ and } P(t,X,Y) = P(.).$$

Furthermore, for any two vectors X, Y in R^n, the symbol $\langle X, Y\rangle$ denotes the usual scalar product in R^n, i.e., $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; therefore, $\|X\|^2 = \langle X, X\rangle$.

2. Basic Concepts

Consider a system of differential equations

$$\frac{dx}{dt} = F(t,x), \tag{5}$$

where $t \in R^+ = [0,\infty)$, $x \in R^n$ and $F \in C(R^+ \times R^n, R^n)$.

Definition 1 (Yoshizawa [12])**.** *The solutions of* (5) *are equi-ultimately bounded for bound B if there exists a $B > 0$ and if, corresponding to any $\alpha > 0$ and $t_0 \in I$, there exists a $T(t_0,\alpha) > 0$ such that $x_0 \in S_\alpha$ implies that $\|x(t;x_0,t_0)\| < B$ for all $t \geq t_0 - T(t_0,\alpha)$.*

Definition 2 (Yoshizawa [12])**.** *The solutions of* (5) *are uniform ultimately bounded for bound B if Definition 1 is independent of t_0.*

Definition 3 (Yoshizawa [12])**.** *The solutions of* (5) *are equi-bounded if, for any $\alpha > 0$ and $t_0 \equiv I$, there exists a $\beta(t_0,\alpha) > 0$ such that, if $x_0 \in S_\alpha$, $\|x(t;x_0,t_0)\| < B(t_0,\alpha)$ for all $t \geq t_0$.*

Definition 4 (Yoshizawa [12])**.** *The solutions of* (5) *are uniform bounded if the B in Definition 3 is independent of t_0.*

3. Boundedness

The main results of this article with regard to the UUB solutions of the system of the ODEs (3) of second order with two multiple variable time delays are given in the following Theorem 1.

Theorem 1. *In addition to the basic conditions with regard to the system* (4)*, we assume that there exist positive constants $a(t), b(t), D_0, D_1, \delta_f, \delta_h, \Delta_f, \Delta_h, \varepsilon, \alpha$ and ξ such that the following conditions hold for all $X, Y \in R^n$ and $t \in R^+$:*

(i) *$H_i(0) = 0, H_i(X) \neq 0, (X \neq 0)$, the Jacobian matrices $J_{H_1}(.), J_{H_2}(.)$ and $J_G(X)$ exist and they are symmetric and positive definite such that $\delta_{h_i} \leq \lambda_i(J_{H_i}(X)) \leq \Delta_{h_i}$, $\Delta h = \max\{\Delta_{h_i}\}, \delta_h = \min\{\delta_{h_i}\}$ $\delta_g \leq \lambda(J_G(X)) \leq \Delta_g$, where, in order, $\lambda_i(J_{H_1}(X))$ and $\lambda_i(J_{H_2}(X))$ and $\lambda(J_G(X))$ are the eigenvalues of $J_{H_1}(X)$ and $J_{H_2}(X)$, $J_G(X)$;*

(ii) *The eigenvalues* $\lambda_i(J_F(X,Y))$ *of* $F(.)$ *satisfy* $\delta_f = \alpha - \varepsilon \le \lambda_i(J_F(X,Y)) \le \alpha$;

(iii)

$$0 \le r_i(t) \le \gamma_i, \gamma_i \in R, \gamma_i > 0, \gamma = \max\{\gamma_i\}, {r'}_i(t) \le \xi_i, \xi_i \in R,$$
$$\xi_i > 0, \xi = \max\{\xi_i\},\ 0 < \xi < 1; 1 \le a(t), 1 \le b(t), b'(t) \le 0;$$

(iv)

$$\|P(.)\| \le D_0 + D_1\{\|X\| + \|Y\|\}.$$

Then, the solutions of (4) *are UUB provided that*

$$0 < \gamma < \min\left\{\frac{4\alpha\delta_h - \alpha\varepsilon - 2\Delta_g}{2\alpha\Delta_h}, \frac{(1-\xi)[2\alpha - \varepsilon(4+\alpha)]}{2\Delta_h[\alpha(1-\xi)+4]}\right\}.$$

Remark 1. *The condition of the stability Theorem 1 is determined based on the Routh–Hurwitz stability condition of constant coefficient differential equations of second order (see the book Ahmed and Rama [2], pages 89–89).*

Remark 2. *The result of this paper may be arranged for any number of delays. In that case, it is necessary to update the Lyapunov–Krasovskii functional and conditions of the results accordingly.*

Proof. We define an LKF $V(t) = V(X(t), Y(t))$ by

$$2V(t) = \|\alpha X + Y\|^2 + 4\sum_{i=1}^{2}\int_0^1 \langle H_i(\sigma X), X\rangle d\sigma + \|Y\|^2 + 2\lambda\sum_{i=1}^{2}\int_{-r_i(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds$$
$$+ 4b(t)\int_0^1 \langle G(\sigma X), X\rangle d\sigma, \tag{6}$$

where $\lambda > 0$ and will be chosen in advance.

We will show that the LKF (6) allows for the UUB solutions to (4).

Obviously, we have $V(0,0) = 0$.

From (i), it is clear that

$$2\sum_{i=1}^{2}\delta_{h_i}\|X\|^2 \le 4\sum_{i=1}^{2}\int_0^1 \langle H_i(\sigma X), X\rangle d\sigma \le 2\sum_{i=1}^{2}\Delta_{h_i}\|X\|^2.$$

and

$$\delta_g\|X\|^2 \le 2\int_0^1 \langle G(\sigma X), X\rangle d\sigma \le \Delta_g\|X\|^2. \tag{7}$$

By using an elementary inequality, we can write that

$$0 \le \|\alpha X + Y\|^2 \le 2\left\{\alpha^2\|X\|^2 + \|Y\|^2\right\}. \tag{8}$$

Furthermore, it is also notable that

$$0 \le \lambda\sum_{i=1}^{2}\int_{-r_i(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds. \tag{9}$$

Hence, combining (7)–(9) into (6), we obtain

$$\begin{aligned} 2V(t) \geq & 2\sum_{i=1}^{2}\delta_{h_i}\|X\|^2 + \|Y\|^2 + b(t)\delta_g\|X\|^2 \\ \geq & D_2\{\|X\|^2 + \|Y\|^2\}, \end{aligned}$$

where $D_2 = \min\left\{2\sum_{i=1}^{2}\delta h_i, 1, \delta_g\right\}$.

Next, depending upon the conditions of Theorem 1 and some elementary inequalities, we can derive that

$$\begin{aligned} 2V(t) \leq & 2\left\{\alpha^2\|X\|^2 + \|Y\|^2\right\} + 2\sum_{i=1}^{2}\Delta h_i\|X\|^2 + \|Y\|^2 \\ & + 2\lambda\sum_{i=1}^{2}r_i(t)\int\limits_{t-r_i(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta + 2b(t)\Delta_g\|X\|^2 \\ & = 2(b(t)\Delta_g + \alpha^2 + \sum_{i=1}^{2}\Delta h_i)\|X\|^2 + 3\|Y\|^2 + 2\lambda\sum_{i=1}^{2}r_i(t)\int\limits_{t-r_i(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta \\ \leq & D_3\{\|X\|^2 + \|Y\|^2\} + 2\lambda\sum_{i=1}^{2}(\gamma_i)\int\limits_{t-r_i(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta, \end{aligned}$$

where

$$D_3 = \max\left\{2\left(b(t)\Delta_g + \alpha^2 + \sum_{i=1}^{2}\Delta h_i\right), 3\right\}.$$

Differentiating the LKF $V(t)$ along the solutions of (4), we obtain

$$\begin{aligned} \dot{V}(t) = & \langle \alpha X + Y, \alpha\dot{X} + \dot{Y}\rangle + 2\frac{d}{dt}\sum_{i=1}^{2}\int\limits_{0}^{1}\langle H_i(\sigma_i X), X\rangle d\sigma_1 + \langle Y, \dot{Y}\rangle \\ & + \lambda\frac{d}{dt}\sum_{i=1}^{2}\int\limits_{-r_i(t)}^{0}\int\limits_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds + 2\frac{d}{dt}b(t)\int\limits_{0}^{1}\langle G(\sigma X), X\rangle d\sigma \\ = & \left\langle \alpha X + Y, \alpha Y - a(t)F(.)Y - b(t)G(X) - \sum_{i=1}^{2}H_i(X) + \sum_{i=1}^{2}\int\limits_{t-r_i(t)}^{t}J_{H_i}(.)Yds + P(.)\right\rangle \\ & + 2\frac{d}{dt}\sum_{i=1}^{2}\int\limits_{0}^{1}\langle H_i(\sigma X), X\rangle d\sigma \\ & + \left\langle Y, -a(t)F(.)Y - b(t)G(X) - \sum_{i=1}^{2}H_i(X) + \sum_{i=1}^{2}\int\limits_{t-r_i(t)}^{t}J_{H_i}(.)Yds + P(.)\right\rangle \\ & + \lambda\frac{d}{dt}\sum_{i=1}^{2}\int\limits_{-r_i(t)}^{0}\int\limits_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds + 2\frac{d}{dt}b(t)\int\limits_{0}^{1}\langle G(\sigma X), X\rangle d\sigma. \end{aligned}$$

Clearly, we have

$$\frac{d}{dt}\sum_{i=1}^{2}\int\limits_{0}^{1}\langle H_i(\sigma X), X\rangle d\sigma = \left\langle \sum_{i=1}^{2}H_i(X), Y\right\rangle.$$

Also, using (iii), we obtain

$$\begin{aligned}\lambda \frac{d}{dt}\sum_{i=1}^{2}\int_{-r_i(t)}^{0}\int_{t+s}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta ds =&\lambda\sum_{i=1}^{2} r_i(t)\|Y(t)\|^2-\lambda\sum_{i=1}^{2}(1-r'_i(t))\int_{t-r_i(t)}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta\\ \leq& 2\lambda\gamma\|Y(t)\|^2-\lambda(1-\xi)\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\|Y(\theta)\|^2 d\theta.\end{aligned}$$

Consequently, from gathering the above results, we find that

$$\begin{aligned}\dot{V}\leq& -\alpha\left\langle X,\sum_{i=1}^{2}H_i(X)\right\rangle-2\langle Y,a(t)F(.)\rangle+\alpha\langle Y,Y\rangle\\ &+\alpha\langle X,(\alpha I-a(t)F(.))Y\rangle+\alpha\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\left\langle X,J_{H_i}(.)Y\right\rangle ds\\ &+2\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\left\langle Y,J_{H_i}(.)Y\right\rangle ds+2\lambda\gamma\|Y\|^2\\ &-\lambda(1-\xi)\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\|Y(\theta)\|^2 d\theta+\Delta_g\|X\|^2+\langle\alpha X+2Y,P(.)\rangle,\end{aligned}$$

where I denotes the identity matrix.

Then, if we apply the conditions (i), (ii) and the inequality $\|\alpha\|\,\|\beta\|\leq 2^{-1}\|\alpha\|^2+2^{-1}\|\beta\|^2$, we obtain

$$\begin{aligned}\dot{V}\leq& -2\alpha\delta_h\|X\|^2-2(\alpha-\varepsilon)\|Y\|^2+\alpha\|Y\|^2+\frac{1}{2}\alpha\varepsilon(\|X\|^2+\|Y\|^2)\\ &+\alpha\Delta_h\gamma(\|X\|^2+\|Y\|^2)+2\Delta_h\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\|Y(s)\|^2 ds\\ &+2\lambda\gamma\|Y\|^2-\lambda(1-\xi)\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\|Y(\theta)\|^2 d\theta\\ &+\|P(.)\|(\alpha\|X\|+2\|Y\|)+\Delta_g\|X\|^2\\ =&-\frac{1}{2}\left(4\alpha\delta_h-\alpha\varepsilon-2\alpha\Delta_h\gamma-2\Delta_g\right)\|X\|^2\\ &-\frac{1}{2}[2\alpha-\varepsilon(4+\alpha)-\gamma(2\alpha\Delta h+4\lambda)]\|Y\|^2\\ &+\frac{1}{2}(4\Delta_h-2\lambda(1-\xi))\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\|Y(\theta)\|^2 d\theta\\ &+\|P(.)\|(\alpha\|X\|+2\|Y\|).\end{aligned}\tag{10}$$

Let

$$\lambda=\frac{2\Delta_h}{1-\xi}$$

and

$$\gamma<\min\left\{\frac{4\alpha\delta_h-\alpha\varepsilon-2\Delta_g}{2\alpha\Delta_h},\frac{(1-\xi)[2\alpha-\varepsilon(4+\alpha)]}{2\Delta_h[\alpha(1-\xi)+4]}\right\}.$$

Using (iv), from (10), we derive that

$$
\begin{aligned}
\dot{V} \leq & -K_1\left(\|X\|^2+\|Y\|^2\right)+[D_0+D_1(\|X\|+\|Y\|)][\alpha\|X\|+2\|Y\|] \\
\leq & -K_1\left(\|X\|^2+\|Y\|^2\right)+D_0[\alpha\|X\|+2\|Y\|] \\
& +D_1\left(\frac{2+3\alpha}{2}\right)\|X\|^2+D_1\left(\frac{\alpha+6}{2}\right)\|Y\|^2 \\
\leq & -(K_1-K_2D_1)\left(\|X\|^2+\|Y\|^2\right)+D_0[\alpha\|X\|+2\|Y\|],
\end{aligned}
$$

where $K_1 \in R$, $K_1 > 0$, $K_2 = \max\{\frac{2+3\alpha}{2}, \frac{\alpha+6}{2}\}$.

Letting $D_1 < K_1K_2^{-1}$, it is obvious that there exists some constants $\beta > 0$, $k > 0$ so that

$$
\begin{aligned}
\dot{V} \leq & -\beta\left(\|X\|^2+\|Y\|^2\right)+k\beta(\|X\|+\|Y\|) \\
\leq & -\frac{\beta}{2}\left(\|X\|^2+\|Y\|^2\right)+\beta k^2.
\end{aligned}
$$

Hence, we conclude that all the solutions to the system (4) are UUB. □

4. Numerical Application

As a numerical application of Theorem 1, we give an example with regard to the UUB solutions.

Example 1. *We will deal with the following two-dimensional system that includes two variable delays:*

$$
\begin{aligned}
&\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}+(1+\frac{1}{1+t^2})\begin{pmatrix} 14+e^{-(3x_1^2+4x_2^2)} & 0 \\ 0 & 14+e^{-(3x'^2_1+4x'^2_2)} \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \\
&+(1+\frac{1}{1+2t^2})\begin{pmatrix} 2x_1+\sin x_1 \\ 2x_2+\sin x_2 \end{pmatrix} \\
&+\begin{pmatrix} 3x_1(t-r_1(t))+\sin x_1(t-r_1(t)) \\ 3x_2(t-r_1(t))+\sin x_2(t-r_1(t)) \end{pmatrix}=\begin{pmatrix} \frac{2x_1+3x_1'+4}{25+t^4} \\ \frac{2x_2+3x_2'+4}{25+t^4} \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
a(t) &= 1+\frac{1}{1+t^2} \geq 1, \\
b(t) &= 1+\frac{1}{1+2t^2} \geq 1, \\
b'(t) &= -\frac{4t}{(1+2t^2)^2} \leq 0, \\
r_1(t) &= \frac{1}{40}\sin^2(t).
\end{aligned}
$$

A comparison between the above equation and (4) *shows that*

$$
F(.)=\begin{pmatrix} 14+e^{-(3x_1^2+4x_2^2)} & 0 \\ 0 & 14+e^{-(3x'^2_1+4x'^2_2)} \end{pmatrix},
$$

$$
\sum_{i=1}^{2} H_i(x(t-r_i(t)))=\begin{pmatrix} 3x_1(t-r_1(t))+\sin x_1(t-r_1(t)) \\ 3x_2(t-r_1(t))+\sin x_2(t-r_1(t)) \end{pmatrix},
$$

$$
P(.)=\begin{pmatrix} \frac{2x_1+3x'_1+4}{25+t^4} \\ \frac{2x_2+3x'_2+4}{25+t^4} \end{pmatrix}.
$$

Then, we have the following relations, respectively:

$$0 \le r_1(t) = \frac{1}{40}\sin^2(t) \le \frac{1}{40} = \gamma,$$

$$i = 1, 2,$$

and

$$r'_1(t) = \frac{1}{20}\sin(t)\cos(t) \le \frac{1}{20} = \xi.$$

The matrix $F(.)$ admits the eigenvalues

$$\lambda_1(F(.)) = 14 + e^{-(3x_1^2+4x_2^2)}$$

and

$$\lambda_2(F(.)) = 14 + e^{-(3x'^2_1+4x'^2_2)}.$$

Hence, we have

$$\delta_f = 14 \le \lambda_i(F(.)) \le \Delta_f = 15.$$

$J_{H_1}(X(t - r_1(t)))$ is given by

$$J_{H_1}(X(t - r_1(t))) = \begin{pmatrix} 3 + \cos x_1(t - r_1(t)) & 0 \\ 0 & 3 + \cos x_2(t - r_1(t)) \end{pmatrix},$$

and its eigenvalues satisfy

$$\delta_{h_i} = 2 \le \lambda_i(J_{H_i}(X)) \le \Delta_{h_i} = 4.$$

$$J_G(X) = \begin{pmatrix} 2 + \cos x_1 & 0 \\ 0 & 2 + \cos x_2 \end{pmatrix},$$

and its eigenvalues satisfy

$$1 = \delta_g \le \lambda_i(J_H(X)) \le \Delta_g = 3.$$

Hence,

$$n = 2, \delta_f = 14, \Delta_f = 15, \delta_{h_i} = 2, \Delta_{h_i} = 4, \varepsilon = 1, \alpha = 13, \gamma = \frac{1}{40}, \xi = \frac{1}{20}, \delta_g = 1, \Delta_g = 3.$$

Therefore, it is clear that

$$0 < \gamma < \min\left\{\frac{4\alpha\delta_h - \alpha\varepsilon - 2\Delta_g}{2\alpha\Delta_h}, \frac{(1-\xi)[2\alpha - \varepsilon(4+\alpha)]}{2\Delta_h[\alpha(1-\xi)+4]}\right\} = \min\left\{\frac{85}{104}, \frac{57}{872}\right\} = \frac{57}{872},$$

$$\|P(.)\| \le (8 + \sqrt{10}\{\|X\| + \|X'\|\}).$$

As a result, the given example meets all the conditions of Theorem 1.

The above system was solved using the fourth- order Runge–Kutta method in MATLAB. Here, the graphs of Figure 1 show the behaviors of paths of solutions.

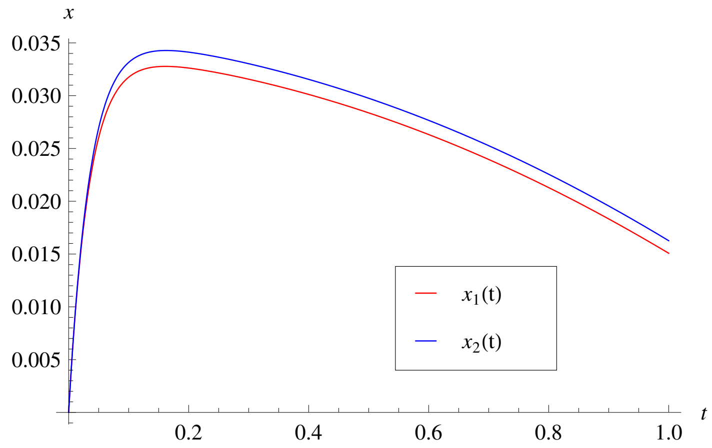

Figure 1. Numerical results of the test problem for $N = 64$ and $\varepsilon = 2^{-2}$.

5. Conclusions

In this article, a system of ODEs of second order with two variable delays is considered. New sufficient conditions are established under which solutions of the considered system are UUB. The technique used in the proof depends on the definition of a new LKF. Our result improves and extends the result of Adeyanju and Tunç ([14], Theorem 3.1) and provides essential contributions with a new result to the relevant literature. The Ulam-type stabilities of the system of the ODEs in the form of (3) and qualitative behaviors of the system of the ODEs (3) with Caputo fractional order and delay(s) can be considered as open problems. For the sake of brevity, we would not like to give proper mathematical models here.

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