mathematics

Review

On the Geometry and Topology of Discrete Groups: An Overview

Renata Grimaldi

Dipartimento di Ingegneria, Universita Degli Studi di Palermo, Piazza Marina n. 61, 90133 Palermo, Italy; renata.grimaldi@unipa.it

Abstract: In this paper, we provide a brief introduction to the main notions of geometric group theory and of asymptotic topology of finitely generated groups. We will start by presenting the basis of discrete groups and of the topology at infinity, then we will state some of the main theorems in these fields. Our aim is to give a sample of how the presence of a group action may affect the geometry of the underlying space and how in many cases topological methods may help the determine solutions of algebraic problems which may appear unrelated.

Keywords: discrete groups; Cayley graph; quasi-isometry; ends; simple connectivity at infinity; Universal Covering Conjecture; topological filtrations; inverse representations

MSC: 20F65; 20F69; 57M10

check for updates

Citation: Grimaldi, R. On the Geometry and Topology of Discrete Groups: An Overview. *Mathematics* **2024**, *12*, 766. https://doi.org/10.3390/math12050766

Academic Editors: Daniele Ettore Otera and Fu-Gui Shi

Received: 19 November 2023
Revised: 22 February 2024
Accepted: 2 March 2024
Published: 4 March 2024

1. Introduction

The central idea of the branch of mathematics called geometric group theory (briefly GGT) is to study group theory from a geometrical viewpoint by means of geometric and topological methods, for example with the notions of the fundamental group and covering space (see [1] for an introduction to the subject).

Geometric group theory focuses in particular on those global geometric and topological invariants which detect the shape at infinity of all universal covers of compact spaces having the same fundamental group.

This research field has its origin in the work of Dehn from the beginning of the last century. With his combinatorial approach, he initiated the study of two closely related research areas: 3-manifolds in topology and infinite groups given by presentations in algebra. His ability to use simple combinatorial diagrams to illustrate the synergy between algebra, geometry and topology has made it clear that most of the topological properties of covering spaces are related to the fundamental group and do not depend on its presentation (and hence on the choice of the space one may associate with the group) (see [2]).

More recently, the field of geometric group theory has undergone impressive development due to the work of Gromov [3–5]. The main novelty he introduced is that, instead of studying groups algebraically, GGT uses both topological and geometric methods, since one can consider group theory as "contained" in the vast area of geometry via the notions of word metric and quasi-isometry.

The underlying idea is that, once a group is chosen, the class of all topological models constructed from it should have some common global geometric conditions (at infinity).

For instance, let us consider the following construction. Start with a finitely generated group Γ. For any such group, there exists a topological space X with $\pi_1(X) = \Gamma$. The group Γ acts effectively on the universal cover $\tilde{X}$ of X, and the quotient space for this action is X itself (i.e., $\tilde{X}/\Gamma = X$). The space X can also be chosen to be a 2-complex, and, if the group is finitely presented, X is compact.

Of course, both X and $\tilde{X}$ are not unique. On the other hand, some of the algebraic properties of the group Γ are transferred into geometric/topological conditions of the space

$\tilde{X}$ (or X), and one can refer to them as asymptotic (or geometric) properties. Vice versa, there are also properties of $\tilde{X}$ (or of X) which depend only on Γ.

One of the first examples that comes to mind when thinking of a topological demonstration of an algebraic result is the well known theorem of Schreier, stating that any subgroup H of a free group F_l on l generators ($l \geq 1$ integer) is itself free (see [1,6]).

Here, we have a pure algebraic problem in a pure algebraic setting. The algebraic proof of this fact involves a meticulous and tedious sequence of special transformations on the subgroup's generating set that reduces its length. On the other hand, the process is made more straightforward with a change in perspective and method. More precisely, consider the free group F_l as the fundamental group of a wedge of circles X_l and the covering space associated with the subgroup H, namely $p : \overline{X} \to X_l$, where $\pi_1(\overline{X}) = H$. The space $\overline{X}$ is actually a graph (since it is a cover of a graph), and hence, by contracting a maximal tree T of $\overline{X}$, one obtains another wedge of circles X_m which is homotopically equivalent to $\overline{X}/T$. Finally, by the van Kampen theorem, one obtains that $H \cong \pi_1(\overline{X}) \cong \pi_1(\overline{X}/T) \cong \pi_1(X_m) \cong F_m$, namely a free group.

This example shows how the use of geometry and topology can help to elucidate and visualize the problem and how this can effectively reduce the length and the difficulties of its solution.

In the next sections, we will provide some basic definitions of geometric group theory and of asymptotic topology of groups in order to depict the strong interplay of geometry and topology with group theory, in the spirit of this Special Issue.

2. Discrete Groups and Associated Spaces

For a *discrete group*, we simply mean a countable group with the discrete topology.

Definition 1. *A discrete group Γ is said to be* ***finitely generated*** *if there is a finite set S of generators (which means that any element $g \in \Gamma$ is a product of finitely many powers of some of the generators $s \in S$).*

The group Γ is said to be ***finitely presented***, *$\Gamma = \langle S \mid \mathcal{R} \rangle$, if, additionally, it possesses a finite number of relators $r \in \mathcal{R}$ (i.e., words, made of elements of S, that are equal to the identity e).*

Now, to any finitely generated group Γ with a generating set S, one can define a somehow "natural" metric on it, which is called **the word metric** (for more details, see [1]).

Definition 2. *Given a finitely generated group Γ and its generating set S, the length $l_S(g)$ of any element g of Γ is the smallest integer n such that there exists a sequence $(s_1, s_2, \cdots, s_n)$ of generators in S for which $g = s_1 s_2 \dots s_n$. The distance of two elements a, b of Γ is*

$$d_S(a,b) = l_S(a^{-1}b).$$

Due to this definition, the pair (Γ, d_S), i.e., the group together with the word metric, becomes a metric space. So, the geometry and topology are already known to some extent (even if the so-defined space is discrete).

However, we can do better. To any finitely generated group Γ, we may associate a graph, called the **Cayley graph** of Γ (denoted $Cay(\Gamma, S)$), which depends on the generating set S.

Definition 3. *The vertex set of $Cay(\Gamma, S)$ is Γ. Two vertices g, h are connected by an edge iff $d_S(g,h) = 1$, namely if and only if $h = gs$ or gs^{-1}, for some $s \in S$. Or, equivalently, any vertex g is joined by an edge with all the vertices of the form gs, for $s \in S$.*

Since the group Γ is finitely generated, this graph is locally finite. By construction it is directed and labeled. Moreover, since S generates Γ, the Cayley graph is connected. Finally, one can also define a "natural metric", denoted also by d_S, on $Cay(\Gamma; S)$, as follows:

- One declares any edge to be of length 1;

- The distance $d(x; y)$ of any two points of the Cayley graph can be defined as the length of the shortest path going from x to y.

In this way, the Cayley graph $Cay(\Gamma; S)$ becomes a connected metric space containing (isometrically) Γ. Obviously, it is finite/infinite if and only if Γ is. Furthermore, when restricting this metric to $\Gamma \subset Cay(\Gamma; S)$, one recovers just the word metric of Γ.

If the group Γ is finitely presented, it is possible to improve the construction of the above in order to obtain a locally finite 2-dimensional space associated with it.

Let $\langle S \mid \mathcal{R} \rangle$ be a finite presentation for a finitely presented group Γ.

Definition 4. *The* ***Cayley 2-complex*** *of* $\Gamma = \langle S \mid \mathcal{R} \rangle$ *is obtained by gluing a disk (i.e., a 2-cell) to all the (closed) paths of the Cayley graph which are labeled by relators* $r \in \mathcal{R}$*.*

Remark 1. *Obviously, the Cayley 2-complex is simply connected, because all closed paths in the Cayley graph are labeled by words which are equal to 1 in* Γ*, and, by definition, the set of relators* $\mathcal{R}$ *generates all the relations.*

In most cases, the constructions described above are the most useful ones, but there may be other situations where more of the topology must be determined, and, actually, there is a second, different way to construct the Cayley graph and Cayley 2-complex in a topological vein (see also [2]).

Consider the **standard 2-complex** $X_{\mathcal{P}}$ associated with the presentation $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ as follows: Start with a bouquet of loops, i.e., the graph with just one vertex v and with $\sharp S$-edge loops at v (one for each $s \in S$), labeled by s. Now, for each relator $r \in \mathcal{R}$ one attaches, along r, a 2-cell with $l(r)$ sides (where $l(g)$ is the length of g) to the bouquet of circles. Obviously, according to the van Kampen theorem, $\pi_1(X_{\mathcal{P}}) = \Gamma$, and its universal covering space $\widetilde{X}_{\mathcal{P}}$ is simply the Cayley 2-complex of Γ, whereas the 1-skeleton of $\widetilde{X}_{\mathcal{P}}$ is the Cayley graph of Γ.

Large-Scale Equivalence

The aforementioned constructions depend on the presentation but not at a "large scale". This is the viewpoint of Gromov [4,5]. If spaces are similar (seen from a long distance), then they should share some common properties that depend on the group that acts on them.

In fact, the word metrics, Cayley graphs and Cayley 2-complexes constructed from distinct presentations of the same group Γ are actually **quasi-isometric** (i.e., geometrically and metrically "similar" in a rough sense).

Definition 5. *A* ***quasi-isometry*** *between two metric spaces* (X, d_X) *and* (Y, d_Y) *is a map* $f : X \to Y$ *such that, for constants* C *and* λ*:*

$$\lambda^{-1} d_X(x_1, x_2) - C \leq d_Y\big(f(x_1), f(x_2)\big) \leq \lambda d_X(x_1, x_2) + C$$

$$\text{and }, \forall y \in Y,\ d_Y\big(y, f(X)\big) \leq C.$$

Roughly speaking, this means that the images of two points which are close (or very far from each others) remain close (or very far), and any point of the target space is uniformly close to the image of some point of the domain. Quasi-isometries do not distinguish small details of the space but rather detect the global geometric behavior.

Since it turns out that an algebraic classification of the class of finitely presented groups is not possible (because the word problem is undecidable), the main goal of geometric group theory is to classify them "geometrically", that is, **up to quasi-isometries**.

From this perspective, one is interested in properties of groups which are quasi-isometry invariants (in fact called *geometric* or *asymptotic* properties).

Remark 2.

- *A quasi-isometry is not necessarily continuous. For instance, real numbers $\mathbb{R}$ and integers $\mathbb{Z}$ are quasi-isometric.*
- *Any finitely generated group Γ (with finite generating set S) is quasi-isometric to its Cayley graph $Cay(\Gamma, S)$ (because the inclusion $(\Gamma, d_S) \hookrightarrow Cay(\Gamma, d_S)$ is a quasi-isometry).*
- *If S and T are two generating sets for the same group G, then the (distinct) metric spaces (G, d_S) and (G, d_T) are quasi-isometric.*
- *As a consequence, given a finitely generated group, one can consider* ***the*** *word metric and* ***the*** *Cayley graph (in the sense that they are well-defined, up to quasi-isometries).*

Let us analyze some basic examples that can elucidate the geometric meaning of quasi-isometries (see [1] for details).

- A metric space is quasi-isometric to a point if and only if it has a finite diameter.
- The group G is finite if and only if its Cayley graph is quasi-isometric to a point. Thus, the *quasi-isometry class* of the trivial group coincides with the set of finite groups. This is why GGT studies only infinite groups.
- The free abelian groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are quasi-isometric if and only if $n = m$ (i.e., they have the same rank).
- The free group of rank 2, F_2, is quasi-isometric to F_k, for any $k \geq 2$.

Now, the obvious question is the following one: when are two groups quasi-isometric? For a far more complete answer, see [1,5].

Definition 6. *A geodesic in a metric space X is a map $f : [a; b] \to X$ s.t. $\forall s; t \in [a, b]$,*

$$d(f(s); f(t)) = |s - t|.$$

A space X is called a geodesic space if any 2 points can be joined by a geodesic. This is equivalent to saying that the distance between any 2 points is the length of the shortest path which joins them.

The space X is said a proper metric space if any closed ball is compact.

An isometric action of a group Γ on a metric space X is said discrete if for any $x \in X$ and $M \in [0; \infty)$, the set $\{g \in \Gamma | d(gx; x) < M\}$ is finite.

The action is said co-compact if the quotient X/Γ is a compact space.

In what follows, and until the end of this section, we will refer to [1,2,5] for terminology.

Lemma 1. *If X, Y are two proper geodesic metric spaces with Γ-actions which are discrete, cocompact and by isometries, then Γ is finitely generated and the spaces X and Y are quasi-isometric. (Consequently, X and Y are quasi-isometric to $Cay(\Gamma; S)$).*

Corollary 1. *The fundamental group of a closed Riemannian manifold is quasi-isometric to its universal covering space.*

Hence, we can deduce that [1]:

- If $H \subset \Gamma$ is a subgroup of a finite index, then H and Γ are quasi-isometric.
- The fundamental group of a closed orientable surface of genus $g \geq 2$ is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$.
- The circle $S^1 = \mathbb{R}/\mathbb{Z}$ has $\pi_1 S^1 = \mathbb{Z}$, and so $\mathbb{Z}$ is quasi-isometric to $\mathbb{R}$.
- The torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ has $\pi_1 T^n = \mathbb{Z}^n$, and so $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$.
- The Euclidean space $\mathbb{R}^n$ is quasi-isometric to $\mathbb{R}^m$ if and only if $n = m$.
- The hyperbolic space $\mathbb{H}^n$ is quasi-isometric to $\mathbb{H}^m$ if and only if $n = m$.
- The hyperbolic space $\mathbb{H}^n$ and the Euclidean space $\mathbb{R}^n$ are not quasi-isometric.

Theorem 1 (Gromov [3], Pansu [7])**.** *A group is quasi-isometric to $\mathbb{R}^n$ if and only if it contains a finite index subgroup isomorphic to $\mathbb{Z}^n$.*

Theorem 2 (Gromov [5], Stallings [8])**.** *If a group is quasi-isometric to the free group in two generators, then it acts properly on some locally finite tree, and hence it is virtually free.*

3. The Topology at Infinity

In this section, we will focus on a more topological aspect of GGT; in particular, we will consider those topological properties of non-compact spaces which depend on some group actions (for an accurate introduction to the subject, see [2]).

The topology at infinity may be defined as the study of global topological properties of complements of compact subsets in open topological spaces. The topological behavior "close to infinity" of non-compact spaces, especially open manifolds, in the presence of a group action, is under study. The idea is to take a space X together with a *filtration* by compact subsets $C_i \subset X$, such that $C_i \subset C_{i+1}$ and $X = \cup_i C_i$, and to look at the topology of $X - C_i$ as i goes to infinity.

3.1. Ends

The simplest topological property at infinity is the condition of being ***one-ended*** (or *connected at infinity*). In other words, being one-ended is equivalent to say that, outside very large compacts, there is only one "way to go to infinity" (for more details, see [1,2,9]).

In fact, with any topological space X, one can associate the so-called **space of ends** (that corresponds, intuitively, to the different ways to go to infinity): it is the set of unbounded connected component of $X - K$ for large compact subspaces K of X.

The next results represent probably the very beginning of geometric group theory.

Theorem 3 (Hopf [10])**.** *Let K be a finite simplicial complex. The number of ends of the universal covering space $\widetilde{K}$ of K depends only on $\pi_1(K)$.*

Hence, it is possible to define the number of ends for a finitely generated group:

Definition 7. *The **number of ends** $e(G)$ **of a group** G is the number of ends of the universal covering space $\widetilde{K}$ of some (equivalently any) finite complex X having G as the fundamental group.*

Remark 3. *The number of ends of the finitely generated group G may also be defined as the number of ends of (one of) its Cayley graph.*

Theorem 4 (Hopf [10])**.** *The number of ends of a group belongs to the set $\{0, 1, 2, \infty\}$.*

Actually, if we have 3 ends, we may consider a compact subset $C \subset \widetilde{K}$, outside of which starts the three different ways to go to infinity e_1, e_2, e_3. Hence, when a non-trivial element of the group $G = \pi_1(X)$ translates C within $\widetilde{K}$, it will belong to only one of these directions, say, e_1. However, outside gC, we must again have three directions to go to infinity. In this new configuration, $\widetilde{K} - gC$, the two directions e_2, e_3 represent the same direction to infinity. However, since $e(G) = 3$ and this number must be homogeneous outside any large compact subset, the direction e_1 should split in two different directions, thus creating a new different end e_4. Using this simple idea, one can prove that, in fact, if $e(G) > 2$, then $e(G)$ must be infinite (see [1,2]).

In the latter result, we have seen how the presence of a group, and hence of a group action, puts very strong constraints on the topological behavior at infinity.

Theorem 5 (Gromov [5])**.** *The number of ends of a group is a quasi-isometry invariant.*

Remark 4.

- *The two last theorems are not true for general open manifolds.*
- *A group has* 0 *ends if and only if it is finite.*
- *The number of ends of* $\mathbb{Z}$ *is* 2*, while* $e(\mathbb{Z}^n)$*, for* $n \geq 2$*, is* 1*.*
- *The free group of rank* 2 *has infinitely many ends.*

At this point, one may wonder whether it is possible to catch some algebraic condition as from the topological notion of number of ends. This is the key problem in geometric group theory: relating geometric properties of a group and its algebraic structure.

Theorem 6 (Hopf [10])**.** *A group has* 2 *ends if and only if it has an infinite cyclic subgroup (i.e.,* $\mathbb{Z}$*) of finite index.*

This result was then generalized by Stallings in the 1970s [8] (but see also [6]). He provided a structure theorem for infinitely ended groups, and, as a result, Dunwoody in [11] was able to prove the famous Wall's Conjecture [12] for finitely presented groups by giving a complete characterization of them starting from finite groups and one-ended groups via a finite number of natural operations (called amalgamated free products and HNN extensions over finite subgroups).

Theorem 7 (Stallings [8])**.** *Let G be a finitely generated group with infinitely many ends.*

- *If G is torsion-free, then G is a non-trivial free product;*
- *Or G is a non-trivial free product with amalgamation, with finite amalgamated subgroup.*

3.2. The Simple Connectivity at Infinity

Having in mind Stallings' theorem and the fact that one-ended groups are the basic pieces for constructing all discrete groups [11], one is led to the study of groups with one end. Furthermore, the first topological notion needed in order to obtain a well-behaved topology is the simple connectivity. Hence, one usually focuses on the behavior at infinity of manifolds and groups with "simply connected ends". Furthermore, for one-ended spaces, the easiest and strongest topological "tameness" condition at infinity is the so-called **simple connectivity at infinity** (see [2,9]). The simple connectivity at infinity tells us approximately that loops which are very far away (i.e., "at infinity") should bound disks which are at a sufficient distance (i.e., "near the infinity").

Definition 8. *A connected, locally compact, topological space* X *with* $\pi_1(X) = 0$ *is* **simply connected at infinity** (SCI) *if for any compact* $k \subseteq X$ *there exists a larger compact* $k \subseteq K \subseteq X$ *such that any closed loop in* $X - K$ *is null homotopic in* $X - k$*.*

Why is this condition so interesting and powerful? Because it turns out that, for $n \geq 3$, the simple connectivity at infinity just features Euclidean spaces $\mathbb{R}^n$ among open contractible n-manifolds, as proven by the following theorem (that is actually a sum of several extensive results of different authors).

Theorem 8 (Stallings [13]; Freedman [14]; Perelman [15])**.**

1. *In dimension* $n \geq 5$*, any differential manifold which is open, contractible and simply connected at infinity is diffeomorphic to the Euclidean space* $\mathbb{R}^n$*.*
2. *In dimension* $n = 4$*, the result is true only for topological manifolds.*
3. *Finally, in dimension* $n = 3$*, the same result holds for both topological and differential manifolds.*

Remark 5.

- *As a corollary, one obtains that Euclidean space* $\mathbb{R}^n$*, for* $n \neq 4$*, admits a unique differential structure.*

- *On the other hand, $\mathbb{R}^4$ supports infinitely many different differential structures (Donaldson [16]).*

Now, what can we say about discrete groups?

Definition 9. *A finitely presented group Γ is simply connected at infinity if the universal covering space $\tilde{X}$ of some compact complex X, having Γ as fundamental group, is* SCI.

Theorem 9.

- *The simple connectivity at infinity is a well-defined property for finitely presented groups (in the sense that it does not depend on the presentation) [17].*
- *The simple connectivity at infinity is also a quasi-isometry invariant of finitely presented groups [18].*

3.3. The Universal Covering Conjecture

Another interesting implication of the simple connectivity at infinity comes from its connection with the so-called *Universal Covering Conjecture*. Since the 1960s, topologists have studied the behavior at infinity of contractible universal covering spaces of closed 3-manifolds and proposed the following problem/conjecture (for a more historical panoramic view, see [9,19]):

Conjecture 1 (Universal Covering Conjecture)**.** *The universal covering space of a (connected, orientable) closed, aspherical (i.e., with a contractible universal cover) 3-manifold is simply connected at infinity. If the manifold is also irreducible, then the universal cover is $\mathbb{R}^3$.*

- This conjecture is now a theorem due to Perelman's recent proof of Thurston *Geometrization Conjecture* [15].
- On the other hand, the Universal Covering Conjecture fails in a higher dimension, as proved by Davis in the 1980s.

Theorem 10 (Davis [20])**.** *For any $n \geq 4$, there exist closed, aspherical n-manifolds whose universal covers are not homeomorphic to Euclidean spaces (in particular, they are not SCI).*

So we are left with the following natural, interesting and difficult question:

Question 1. *Are there topological conditions which characterize the class of contractible universal covering spaces of closed manifolds?*

3.4. Topological Filtrations

In the 1980s, Poénaru partially solved the Universal Covering Conjecture, for those 3-manifolds whose fundamental groups satisfy some geometric or topological conditions (see, e.g., [21]), by "approximating" the universal cover with a filtration of compact and simply connected 3-manifolds.

Definition 10. *A topological space X is* ***weakly geometrically simply connected*** *(briefly* WGSC *[22]) if it can be written as an ascending union of compact, connected and simply connected subspaces. Namely, X is* WGSC *if it admits a* ***filtration****, $X = \cup_i K_i$, with $K_i \subset K_{i+1}$ and such that K_i are compact, connected and with $\pi_1 K_i = 0$.*

Definition 11. *A simply connected complex X is* **quasi-simply filtered** (QSF) *if for any compact sub-complex $C \subset X$ there exists a simply connected compact complex K and a PL-map $f : K \to X$ so that $C \subset f(K)$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a PL-homeomorphism.*

The latter condition simply means that every compact subset C of X can be included (homeomorphically) inside of the image of an abstract compact and simply connected complex that is equipped with a simplicial map into X, whose set of double points lies outside the compact C we started with. In other words, a topological space is QSF if it admits a **quasi-simple filtration**, i.e., a filtration which can be "approximated" by finite, simply connected complexes.

This topological notion has interesting group-theoretical ramifications, as testified by the next results:

Theorem 11 (Brick–Mihalik [23]; Funar–Otera [22])**.**

- *If K_1, K_2 are two presentation complexes for the same finitely presented group Γ, then $\tilde{K}_1$ is QSF $\iff$ $\tilde{K}_2$ is QSF. (Ths implies the that QSF property is well-defined fro finitely presented groups).*
- *Many finitely presented groups are* QSF *(see Remark 7).*

The main reason for using this notion lies in the fact that, since for open 3-manifolds, being simply connected at infinity is equivalent to being WGSC, in order to prove the Universal Covering Conjecture, one simply needs a method which yields a filtration of the universal cover of a closed 3-manifold.

Theorem 12 (Poénaru [24])**.** *An open* QSF *3-manifold is* WGSC *and hence simply connected at infinity.*

Remark 6. *Thus, in order to verify the simple connectivity at infinity of the universal cover of a closed 3-manifold, it suffices to construct a quasi-filtration of it (and this is much easier than obtaining a whole* WGSC *filtration).*

Theorem 13 (Poénaru [21])**.** *Let M^3 be a closed 3-manifold, and assume that $\Gamma = \pi_1(M^3)$ satisfies some "nice geometric condition", then $\widetilde{M}^3$ is* QSF *(and hence simply connected at infinity).*

Remark 7. *The set of "nice geometric conditions" includes: Gromov hyperbolicity, Cannon almost-convexity, automaticity and combability (in the sense of Thurston), geometric simple connectivity, etc. In particular, the class of groups with a "nice geometry" is quite large (see, e.g., [9,25,26]).*

4. Inverse Representations

The main tool for proving the last theorems of the previous section was the following notion, invented and developed by Poénaru in [27] and thereafter utilized in his scientific work (see [28] but also [19,26,29]):

Definition 12. *Let M^3 be a 3-manifold. A topological **inverse representation** for M^3 is a non-degenerate simplicial map*

$$f : X^2 \longrightarrow M^3 \text{ such that:}$$

- *X^2 is a simplicial 2-complex, which is* QSF*;*
- *The map f is "essentially surjective", which means that M^3 can be obtained from the closure $\overline{f(X^2)}$ with the addition of cells of dimensions $\lambda = 2$ and $\lambda = 3$;*
- *The map f is "zippable" (one can pass from X^2 to $f(X^2)$ by an infinite sequence of "simple" quotient maps f_i of very special type, and this has a strong control over the singularities of f).*

This exotic notion seems to be suited for the world of 3-manifolds, but it turns out that it can be used very well for discrete groups too. The necessary adjustment is to look at groups as 3-dimensional objects. However, of course, not all groups are 3-manifold groups; hence, one has to allow manifolds to have singularities.

Lemma 2. *Any finitely presented group $G = \langle S|R\rangle$ can be seen as the fundamental group of a compact but singular 3-manifold $M^3(G)$ associated with G.*

This is proved in [26,30]. Here, we can simply state that $M^3(G)$ is obtained by attaching $|R|$ handles of index 2 to a handlebody of genus $|S|$.

Definition 13. *A topological inverse representation for a finitely presented group G is a topological inverse representation of the 3-manifold* $\widetilde{M}^3(G)$.

In general, the image

$$\mathrm{Im}(f) \subset M^3$$

and the set of double points of f,

$$M_2(f) = \{x \in X^2 \mid \sharp\{f^{-1}(f(x))\} > 1\} \subset X^2,$$

are not closed subsets, and this is one of the main difficulties when dealing with inverse representations [28]. Furthermore, as a result, the following definitions arise naturally:

Definition 14. *A topological inverse representation is* ***easy*** *if* $f(X^2)$ *and* $M_2(f)$ *are closed.*

Definition 15. *An* ***easy group*** *is a finitely presented group G admitting an easy inverse-representation; this is a non-degenerate, zippable, quasi-surjective, simplicial map* $f : X^2 \to \widetilde{M}^3(G)$, *from a* QSF *complex* X^2, *for which* $f(X^2)$ *and* $M_2(f)$ *are closed subsets.*

Here, below, we summarize the recent developments concerning this interesting property of groups and manifolds.

Theorem 14 (Otera–Poénaru [30])**.** *Easy groups are* QSF*.*

Theorem 15 (Otera–Poénaru, [25])**.** *Groups admitting Lipschitz and tame 0-combings are easy.*

Theorem 16 (Otera–Poénaru, [31])**.** *Given a finitely presented* QSF *group* Γ, *one can construct a 2-dimensional* WGSC *topological inverse representation, which is both easy and equivariant.*

Conjecture 2 (Poénaru)**.** *All finitely presented groups are easy.*

5. Conclusions

In this short essay, we intended to give an elementary idea of the close basic connections between geometry, topology and group theory, following the underlying idea of this Special Issue. In particular, we have focused on two aspects, one quite geometric (geometric group theory) and the other more topological (asymptotic topology).

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. de la Harpe, P. *Topics in Geometric Group Theory*; Chicago Lectures in Mathematics; Chicago University Press: Chicago, IL, USA, 2000.
2. Geoghegan, R. *Topological Methods in Group Theory*; Graduate Texts in Mathematics 243; Springer: Berlin/Heidelberg, Germany, 2008.
3. Gromov, M. Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Etud. Sci.* **1981**, *53*, 53–73. [CrossRef]
4. Gromov, M. Hyperbolic groups. In *Essays in Group Theory*; Mathematical Sciences Research Institute Publications; Springer: New York, NY, USA, 1987; Volume 8, pp. 75–263.

5. Gromov, M. Asymptotic invariants of infinite groups. In *Geometric Group Theory, Volume 2*; London Mathematical Society Lecture Note Series 182; Cambridge University Press: Cambridge, UK, 1993; pp. 1–295.
6. Serre, J.P. *Trees*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2022.
7. Pansu, P. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergod. Theory Dynam. Syst.* **1983**, *3*, 415–445. [CrossRef]
8. Stallings, J.R. On torsion-free groups with infinitely many ends. *Ann. Math.* **1968**, *88*, 312–334. [CrossRef]
9. Otera, D.E. Topological tameness conditions of groups. Results and developments. *Lith. Math. J.* **2016**, *3*, 357–376. [CrossRef]
10. Hopf, H. Enden offener räume und unendliche diskontinuierliche gruppen. *Comment. Math. Helv.* **1944**, *16*, 81–100. [CrossRef]
11. Dunwoody, M.J. The accessibility of finitely presented groups. *Invent. Math.* **1985**, *81*, 449–457. [CrossRef]
12. Wall, C.T.C. Pairs of relative cohomological dimension one. *J. Pure Appl. Algebra* **1971**, *1*, 141–154. [CrossRef]
13. Stallings, J.R. The piece-wise linear structure of Euclidean spaces. *Proc. Cambr. Math. Soc.* **1962**, *58*, 481–488. [CrossRef]
14. Freedman, M. The Topology of four-dimensional manifolds. *J. Diff. Geom.* **1983**, *17*, 279–315. [CrossRef]
15. Kleiner, B.; Lott, J. Notes on Perelman's papers. *Geom. Topol.* **2008**, *12*, 2587–2855. [CrossRef]
16. Donaldson, S.K. An application of gauge theory to four-dimensional topology. *J. DIff. Geom.* **1983**, *18*, 278–315. [CrossRef]
17. Tanasi, C. Sui gruppi semplicemente connessi all'infinito. *Rend. Ist. Mat. Univ. Trieste* **1999**, *31*, 61–78.
18. Funar, L.; Otera, D.E. A refinement of the simple connectivity at infinity of groups. *Arch. Math.* **2003**, *81*, 360–368. [CrossRef]
19. Otera, D.E. On simplicial resolutions of groups. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2022**, *16*, 138. [CrossRef]
20. Davis, M.W. Groups generated by reflections and aspherical manifolds not covered by Euclidean spaces. *Ann. Math.* **1983**, *117*, 293–324. [CrossRef]
21. Poénaru, V. Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. *J. Diff. Geom.* **1992**, *35*, 103–130.
22. Funar, L.; Otera, D.E. On the WGSC and QSF tameness conditions for finitely presented groups. *Groups Geom. Dyn.* **2010**, *4*, 549–596. [CrossRef]
23. Brick, S.G.; Mihalik, M.L. The QSF property for groups and spaces. *Math. Z.* **1995**, *220*, 207–217. [CrossRef]
24. Poénaru, V. Killing handles of index one stably and π_1^∞. *Duke Math. J.* **1991**, *63*, 431–447. [CrossRef]
25. Otera, D.E.; Poénaru, V. Tame combings and easy groups. *Forum Math.* **2017**, *29*, 665–680. [CrossRef]
26. Otera, D.E.; Poénaru, V. Topics in Geometric Group Theory, Part I. In *Handbook of Group Actions, Volume V, Advanced Lectures in Mathematics Volume 48*; International Press of Boston, Inc.: Boston, MA, USA, 2020; pp. 307–353.
27. Poénaru, V. The collapsible pseudo-spine representation theorem. *Topology* **1992**, *31*, 625–636. [CrossRef]
28. Poénaru, V. Topics in Geometric Group Theory, Part II. In *Handbook of Group Actions, Volume V, Advanced Lectures in Mathematics Volume 48*; International Press of Boston, Inc.: Boston, MA, USA, 2020; pp. 347–398.
29. Otera, D.E. On Poénaru's inverse representations. *Quaes. Math.* **2017**, *40*, 277–293. [CrossRef]
30. Otera, D.E.; Poénaru, V. "Easy" Representations and the QSF property for groups. *Bull. Belgian Math. Soc.-Simon Stevin* **2012**, *19*, 385–398. [CrossRef]
31. Otera, D.E.; Poénaru, V. Finitely presented groups and the Whitehead nightmare. *Groups Geom. Dyn.* **2017**, *11*, 291–310. [CrossRef]