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New Results on Ulam Stabilities of Nonlinear Integral Equations

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Abstract: This article deals with the study of Hyers–Ulam stability (HU stability) and Hyers–Ulam–Rassias stability (HUR stability) for two classes of nonlinear Volterra integral equations (VIEqs), which are Hammerstein-type integral and Hammerstein-type functional integral equations, respectively. In this article, both the HU stability and HUR stability are obtained for the first integral equation and the HUR stability is obtained for the second integral equation. Among the used techniques, we present fixed point arguments and the Gronwall lemma as a basic tool. Two supporting examples are also provided to demonstrate the applications and effectiveness of the results.

Keywords: Volterra integral equation; HUR stability; HU stability; Gronwall lemma; higher dimensions

MSC: 39B82; 45D05

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1. Introduction

In recent literature, an extensive amount of attention has been focused on the investigation of the existence and uniqueness of solutions, HU stability, and HUR stability of VIEqs, Volterra integro-differential equations (VIDEqs), ordinary differential equations (ODEqs), functional differential equations (FDEqs), etc., for both pure mathematical research and concrete real-world applications. In particular, for some practical and realistic approaches to nonlinear integral equations (IEqs), where researchers are paying great attention to the effects caused by the nonlinearity of dynamical equations in nonlinear science, see, e.g., the books of Burton[1], Corduneanu [2], Wazwaz [3] and the references of these books. Furthermore, potential theory has contributed more than any field to give to nonlinear IEqs. In addition, mathematical physics models, such as diffraction problems, scattering in quantum mechanics, and water waves, also contributed to the creation of nonlinear IEqs. Hence, it is important to investigate the qualitative properties of VIEqs.

As for a comprehensive treatment of the subject with regard to the qualitative behaviors of certain VIEqs, VIDEqs, IDEqs and some others, we refer the readers to the following works: for the existence and Ulam stability of quadratic IEqs by Schauder's fixed point theorem, see Abbas and Benchohra [4]; the existence and asymptotic stability of nonlinear VIEqs by a fixed point theorem, see Banaś and Rzepka [5]; the stability of FDEqs by fixed point theory, see Burton [6]; the HU stability and HUR stability of VIEqs with delay, Hammerstein IEqs and IDEqs, respectively, see Castro and Ramos [7] and Castro and Simões [8,9]; the HU stability for ODEqs and partial differential equations via the Gronwall lemma, see Ciplea et al. [10]; the HUR stability of Volterra–Hammerstein IEqs by the fixed point method, see Ciplea et al. [11] and Tunç and Tunç [12]; the Ulam stabilities of iterative FDEqs of the first order by the fixed point method, see Egri [13]; the HU stability and HUR stability of VIDEqs by the fixed point method, see Tunç and Tunç [14] and Tunç et al. [15,16];

the HU stability and HUR stability of VIEqs by the fixed point method, see Jung [17] and Öğrekçi [18]; the HUR stability of functional equations and fractional differential equations by the fixed point method, respectively, see Jung [19] and Khan et al. [20]; the HU stability for operatorial equations and inclusions via nonself operators, see Petru et al. [21]; the HU stability of ODEqs and differential operators, see Popa and Raşa [22]; the Ulam stability of the linear mapping in Banach spaces, see Rassias [23]; Ulam stability, see Ulam [24]; the stability of IDEqs in the sense of Lyapunov, see Bohner and Tunç [25] and Tunç and Tunç [26]; the stability of mappings of the Hyers–Ulam type, see [27]; the Ulam-type stability, see [28]; and the references of these sources.

We now explain the related results in the reference paper for this work.

In 2011, Lungu [29] considered the following scalar VIEq:

$$u(x) = h(x) + \int_0^x f(x,s,u(s),g(u(s)))ds. \tag{1}$$

In Theorem 4.1, Theorem 4.2 for [29], the author obtained stability results of the HU and HUR types for VIEq (1) by using the Gronwall lemma.

In the same work, Lungu [29] also considered the following scalar functional VIEq in higher dimensions:

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t),f(u(s,t)))dsdt. \tag{2}$$

In [29], Theorem 5.1, the author proved a stability result of the HUR type for the nonlinear functional VIEq (2) by using the Gronwall lemma.

Throughout this paper, let $(B,\ |.|)$ denote a (real or complex) Banach space with the norm $|.|$.

In particular, $C([0,\ a),\beta)$ denotes the space of continuous operators from $[0,\ a)$ in B.

We should mention that without loss of generalization, similar representations are also used for some other operators throughout this paper.

In this article, first, instead of VIEq (1), we consider the following nonlinear scalar VIEq:

$$\begin{aligned}\vartheta(x) =&h(x) + q(\vartheta(x)) + r(x,\vartheta(x)) \int_0^x f(x,s,\vartheta(s),g(\vartheta(s)))ds \\ &+ p(x,\vartheta(x),\beta(\vartheta(x))) + \int_0^x \Xi(x,s)\rho(\vartheta(s)))ds,\end{aligned} \tag{3}$$

where $x \in [0,\ a)$; $\vartheta \in C([0,a],B\ )$, i.e., $\vartheta : [0,a] \to B$ is continuous; $h \in C([0,\ a),B)$; $q,g,\beta,\rho \in C([0,a) \times C([0,a)))$; $r \in C([0,a) \times B,B)$; $f \in C\big([0,\ a) \times [0,\ a) \times B^2,B\big)$; $p \in C\big([0,\ a) \times B^2,B\big)$; $\Xi \in C([0,\ a) \times [0,\ a),\mathbb{R})$; and $a \in (0,\ \infty]$.

In this paper, we prove two new results with regard to the HU stability and HUR stability of the scalar VIEq (3) by means of the Gronwall lemma (Lungu [29], Rus [30]). Next, in a particular case of VIEq (3), we give a supporting example to demonstrate the applications and effectiveness of the HU stability and HUR stability results.

Second, instead of VIEq (2), we consider the following nonlinear functional VIEq in higher dimensions:

$$
\begin{aligned}
\vartheta(x,y) =& r(\vartheta)(x,y) + g(x,y,h(\vartheta)(x,y)) \\
&+ q(x,y,p(\vartheta)(x,y)) \int_0^x \int_0^y K(x,y,s,t,\vartheta(s,t),f(\vartheta(s,t)))dsdt,
\end{aligned}
\tag{4}
$$

where $x,\ s \in [0,\ a];\ y,\ t \in [0,\ a];\ \vartheta \in C([0,a] \times [0,a], B\ )$, i.e., $\vartheta : [0,a] \times [0,a] \to B$ is continuous; $\phi \in C\left([0,\ a]^2, \mathbb{R}^+\right);\ \mathbb{R}^+ = [0,\infty);\ r \in C\left([0,\ a]^2, B\right);\ f,\ h,\ p \in C\left([0,a]^2 \times B, B\right);$ $g,\ q \in C\left([0,a]^2 \times B, B\right)$; and $K \in C\left([0,\ a]^4 \times B^2, B\right)$. In this article, we prove a new result related to the HUR stability of the scalar VIEq (4) by means of the Gronwall lemma. Finally, in a particular case of VIEq (4), we establish an example to verify the relevance and effectiveness of the HUR stability result.

As for the motivation of this study, the first essential reference paper and the results are those of Lungu [29], Theorems 4.1, 4.2, 5.1. The next sources are the abovementioned papers and books and the HU and HUR stability results in these works. In fact, Hammerstein-type integral equations, Hammerstein-type integral operators, Hammerstein-type integral inclusions, etc., have scientific importance and applications in the mathematical and engineering literature (see, for example, the monograph of Janczak [31]). In addition, VIEq (1) and VIEq (2) have simple forms and they are also not in the form of Hammerstein-type integral equations. Hence, considering and investigating Ulam-type stabilities of the Hammerstein-type integral Equations (1) and (2) were the motivation to do this study. We aimed to extend and improve the results of Lungu [29],Theorems 4.1, 4.2, 5.1 and to provide new contributions to the related works in the earlier literature. We should also note that in the literature, most of the works related to the HU and HUR stabilities of VIEqs, VIDEqs, etc., were done in light of Banach's fixed point theorem, the Bielecki metric, the Pachpatte's inequality and Picard operator theory. However, according to the results of Lungu [29], Theorems 4.1, 4.2, 5.1 and this paper, we can see that the Gronwall lemma is very effective and a suitable tool to prove the HU stability and HUR stability of nonlinear VIEqs. We would like to attract the attention of scholars to this case.

The remaining sections of this paper are structured as follows: Section 2 includes two new results related to the HU and HUR stabilities of VIEq (3) and an example supporting the applications of these theorems. Section 3 includes a new theorem with regard to the HUR stability of VIEq (4) and an example supporting the application of this theorem. Section 4 consists of the discussion of the results. Finally, the conclusion of the paper is presented in Section 5.

2. The HU and HUR Stabilities of VIEq (3)

We now give the HU stability result with regard to VIEq (3) in Theorem 1.

Theorem 1. *Suppose that we have*
(As1)

$$h \in C([0,\ a], B); q,\ g,\ \beta,\ \rho \in C([0,a] \times C([0,a])); r \in C([0,a] \times B, B);$$

$$f \in C\left([0,\ a] \times [0,\ a] \times B^2, B\right); p \in C\left([0,\ a] \times B^2, B\right); \Xi \in C([0,\ a] \times [0,\ a], \mathbb{R}).$$

(As2) There exist positive constants r_{M_0}, M, q_L, f_L, p_L, g_L, β_L and ρ_L such that

$$r_{M_0} = \max_{x \in [0,a]} |r(x,\vartheta)|, \forall\ x \in [0,\ a], \forall\ \vartheta \in B,$$

$$|\Xi(x,s)| \leq M, \forall\ x,\ s \in [0,\ a],$$

$$|q(\vartheta) - q(v)| \leq q_L|\vartheta - v|, \forall\ \vartheta,\ v \in B,$$

$$|f(x,s,\vartheta_1,v_1) - f(x,s,\vartheta_2,v_2)| \le f_L|\vartheta_1 - \vartheta_2| + f_L|v_1 - v_2|,$$
$$\forall\, x,\ s \in [0,\ a], \forall\, \vartheta_1,\ \vartheta_2,\ v_1,\ v_2 \in B,$$
$$|p(x,\vartheta_1,v_1) - p(x,\vartheta_2,v_2)| \le p_L|\vartheta_1 - \vartheta_2| + p_L|v_1 - v_2|,$$
$$\forall\, x \in [0,\ a], \forall\, \vartheta_1,\ \vartheta_2,\ v_1,\ v_2 \in B,$$
$$|g(\vartheta) - g(v)| \le g_L|\vartheta - v|, \forall\, \vartheta,\ v \in B,$$
$$|\beta(\vartheta) - \beta(v)| \le \beta_L|\vartheta - v|, \forall\, \vartheta,\ v \in B,$$
$$|\rho(\vartheta) - \rho(v)| \le \rho_L|\vartheta - v|, \forall\, \vartheta,\ v \in B.$$

Then,

(*a*) *VIEq (3) has a unique solution* ϑ^* *in* $C([0,a],B)$*;*
(*b*) *For each* $\varepsilon > 0$*, if* $\vartheta \in C([0,a],B)$ *is a solution of the inequality*

$$\left|\vartheta(x) - h(x) - q(\vartheta(x)) - r(x,\vartheta(x)) \int_0^x f(x,s,\vartheta(s),g(\vartheta(s)))ds \right.$$
$$\left. - p(x,\vartheta(x),\beta(\vartheta(x))) - \int_0^x \Xi(x,s)\rho(\vartheta(s))ds \right| \le \varepsilon, \forall\, x \in [0,\ a], \tag{5}$$

then

$$|\vartheta(x) - \vartheta^*(x)| \le C_f \times \varepsilon, \forall\, x \in [0,\ a],$$

where

$$C_f = \frac{1}{C_0} \exp\left[\left(r_{M_0} f_L(1 + g_L) + \rho_L M\right) C_0^{-1} a\right]$$

with

$$C_0 = 1 - q_L - p_L - p_L\beta_L > 0.$$

Hence, VIEq (3) is HU stable.

Proof.

(*a*) The proof can be easily done as in Rus [32]. We will not give the proof of (a).
(*b*) According to (As1), (As2) and (5) we derive

$$\begin{aligned}
|\vartheta(x) - \vartheta^*(x)| \le &\left|\vartheta(x) - h(x) - q(\vartheta(x)) - r(x,\vartheta(x)) \int_0^x f(x,s,\vartheta(s),g(\vartheta(s)))ds\right. \\
&\left. - p(x,\vartheta(x),\beta(\vartheta(x))) - \int_0^x \Xi(x,s)\rho(\vartheta(s))ds \right| \\
&+ |q(\vartheta(x)) - q(\vartheta^*(x))| \\
&+ |p(x,\vartheta(x),\beta(\vartheta(x))) - p(x,\vartheta^*(x),\beta(\vartheta^*(x)))| \\
&+ \left| r(x,\vartheta(x)) \int_0^x f(x,s,\vartheta(s),g(\vartheta(s))) \right. \\
&\left. - r(x,\vartheta^*(x)) \int_0^x f(x,s,\vartheta^*(s),g(\vartheta^*(s))) \right| ds \\
&+ \left| \int_0^x [\Xi(x,s)\rho(\vartheta(s)) - \Xi(x,s)\rho(\vartheta^*(s))] \right| ds \\
\le &\, \varepsilon + q_L|\vartheta(x) - \vartheta^*(x)| + p_L|\vartheta(x) - \vartheta^*(x)|
\end{aligned}$$

$$
\begin{aligned}
&+ p_L|\beta(\vartheta(x)) - \beta(\vartheta^*(x))| \\
&+ r_{M_0} f_L \int_0^x |\vartheta(s) - \vartheta^*(s)| ds + r_{M_0} f_L \int_0^x |g(\vartheta(s)) - g(\vartheta^*(s))| ds \\
&+ \rho_L \int_0^x |\Xi(x,s)|\, |\vartheta(s) - \vartheta^*(s)| ds \\
\leq & \varepsilon + q_L|\vartheta(x) - \vartheta^*(x)| + p_L|\vartheta(x) - \vartheta^*(x)| \\
&+ p_L \beta_L |\vartheta(x) - \vartheta^*(x)| \\
&+ r_{M_0} f_L \int_0^x |\vartheta(s) - \vartheta^*(s)| ds + r_{M_0} f_L g_L \int_0^x |\vartheta(s) - \vartheta^*(s)| ds \\
&+ \rho_L M \int_0^x |\vartheta(s) - \vartheta^*(s)| ds
\end{aligned} \tag{6}
$$

In light of the above findings, i.e., from (6) we derive

$$
(1 - q_L - p_L - p_L\beta_L)|\vartheta(x) - \vartheta^*(x)| \leq \varepsilon + \left[r_{M_0} f_L (1 + g_L) + \rho_L M\right] \int_0^x |\vartheta(s) - \vartheta^*(s)| ds. \tag{7}
$$

Then, (7) gives

$$
|\vartheta(x) - \vartheta^*(x)| \leq \frac{\varepsilon}{1 - q_L - p_L - p_L\beta_L} + \frac{\left[r_{M_0} f_L (1 + g_L) + \rho_L M\right]}{1 - q_L - p_L - p_L\beta_L} \int_0^x |\vartheta(s) - \vartheta^*(s)| ds. \tag{8}
$$

Using the Gronwall lemma (Lungu [29], Rus [30]), from (8), we obtain

$$
\begin{aligned}
|\vartheta(x) - \vartheta^*(x)| \leq & \frac{\varepsilon}{1 - q_L - p_L - p_L\beta_L} \exp\left[\frac{(r_{M_0} f_L (1 + g_L) + \rho_L M)a}{1 - q_L - p_L - p_L\beta_L}\right] \\
\leq & \frac{\varepsilon}{C_0} \exp\left[\frac{(r_{M_0} f_L (1 + g_L) + \rho_L M)a}{C_0}\right].
\end{aligned} \tag{9}
$$

Let

$$
C_f = \frac{1}{C_0} \exp\left[\frac{(r_{M_0} f_L (1 + g_L) + \rho_L M)a}{C_0}\right]
$$

and

$$
0 < q_L + p_L + p_L\beta_L < 1.
$$

Hence, according to (9), we have

$$
|\vartheta(x) - \vartheta^*(x)| \leq C_f \times \varepsilon, \forall\, x \in [0,\ a].
$$

As a result of the above inequality, according to the conditions of Theorem 1, we conclude that VIEq (3) is HU stable. □

We now provide a supporting example to demonstrate the numerical application of Theorem 1.

Example 1.

$$\vartheta(x) = \frac{1}{1000}\exp(-x^2) + \frac{1}{100}\sin(\vartheta(x)) + \frac{1}{50 + x^2 + \exp(\vartheta^2(x))}\int_0^x \left[\frac{\vartheta(s) + \sin(\vartheta(s))}{1000 + x^2 + s^2}\right]ds$$
$$+ \frac{1}{500 + x^6}\sin(\vartheta(x)) + \frac{1}{10000}\int_0^x \exp[-(x-s)]\sin(\vartheta(s))ds. \tag{10}$$

We note that VIEq (10) is in the form of VIEq (3) with the data as follows:

$$h(x) = \frac{1}{1000}\exp(-x^2),$$
$$r(x, \vartheta) = \frac{1}{50 + x^2 + \exp(\vartheta^2)},$$
$$\Xi(x, s) = \frac{1}{100}\exp[-\ (x - s)], 0 \leq s \leq x \leq a,$$
$$\rho(\vartheta) = \frac{1}{100}\sin(\vartheta),$$
$$f(x, s, \vartheta, g(\vartheta)) = \frac{\vartheta + \sin(\vartheta)}{1000 + x^2 + s^2},$$
$$p(x, \vartheta, \beta(\vartheta)) = \frac{1}{500 + x^6}\sin(\vartheta),$$
$$g(\vartheta) = \sin(\vartheta),$$
$$\beta(\vartheta) = \sin(\vartheta), \text{ where } \vartheta \text{ represents } \vartheta(x).$$

Now, we check the conditions (As1) and (As2) of Theorem 1. To verify that (As1) and (As2) hold, we let $r_{M_0} = 50^{-1}$, $M = 100^{-1}$, $q_L = 100^{-1}$, $f_L = 100^{-1}$, $g_L = 1$, $p_L = 500^{-1}$, $\beta_L = 1$, $\rho_L = 100^{-1}$ *and calculate:*

$$|r(x, \vartheta)| = \frac{1}{50 + x^2 + \exp(\vartheta^2)} \leq \frac{1}{50}, \forall\ x \in [0,\ a], \forall\ \vartheta \in B,$$
$$|\Xi(x, s)| = \frac{1}{100}\exp[-(x - s)] \leq \frac{1}{100}, \forall\ x,\ s \in [0,\ a], 0 \leq s \leq x \leq a,$$
$$|q(\vartheta) - q(v)| = \frac{1}{100}|\sin(\vartheta) - \sin(v)|$$
$$= \frac{1}{50}\left|\frac{\cos(\vartheta + v)}{2}\right|\left|\frac{\sin(\vartheta - v)}{2}\right|$$
$$\leq \frac{1}{100}|\vartheta - v|, \forall \vartheta,\ v \in B,$$
$$f(x, s, \vartheta, g(\vartheta)) = \frac{\vartheta + \sin(\vartheta)}{1000 + x^2 + s^2},$$
$$g(\vartheta) = \sin(\vartheta),$$
$$|f(x, s, \vartheta_1, v_1) - f(x, s, \vartheta_2, v_2)|$$
$$= \frac{1}{1000 + x^2 + s^2}|\vartheta_1 + \sin(v_1) - \vartheta_2 - \sin(v_2)|$$
$$\leq \frac{1}{1000}|\vartheta_1 - \vartheta_2| + \frac{1}{1000}|v_1 - v_2|, \forall\ x,\ s \in [0,\ a], \forall\ \vartheta_1,\ \vartheta_2,\ v_1,\ v_2 \in B,$$
$$|p(x, \vartheta_1, v_1) - p(x, \vartheta_2, v_2)| = \frac{1}{500 + x^6}|\sin(v_1) - \sin(v_2)|$$

$$\leq \frac{1}{500}|v_1 - v_2|, \forall\, x \in [0,\ a], \forall\, \vartheta_1,\ \vartheta_2,\ v_1,\ v_2 \in B,$$

$$|\rho(\vartheta_1) - \rho(\vartheta_2)| = \frac{1}{100}|\sin(\vartheta_1) - \sin(\vartheta_2)| \leq \frac{1}{100}|\vartheta_1 - \vartheta_2|, \forall\, \vartheta_1,\ \vartheta_2 \in B,$$

$$0 < q_L + p_L + p_L\beta_L = \frac{1}{100} + \frac{2}{500} = \frac{7}{500} < 1,$$

$$C_f = C_0^{-1}\exp\Big[\big(r_{M_0} f_L(1 + g_L) + \rho_L M\big) a C_0^{-1}\Big]$$

$$= \frac{500}{493}\exp\left[\left(\frac{1}{2500} + \frac{1}{10000}\right)a\right].$$

Hence, the conditions (As1) and (As2) of Theorem 1 hold. This result shows that VIEq (10) is HU stable. Thus, the application of Theorem 1 is provided by the given example.

In the next Theorem 2, we give the stability result in the sense of HUR for VIEq (3).

Theorem 2. *We assume that (As1) and (As2) of Theorem 1 hold and let $\phi \in C([0,a],\mathbb{R}^+)$, $\mathbb{R}^+ = [0,\infty)$ and ϕ be an increasing function. Then, we have the following results:*

(a) VIEq (3) has a unique solution ϑ^* in $C([0,a],B)$;
(b) If $\vartheta \in C([0,a],B)$ is a solution of the inequality

$$\Bigg|\vartheta(x) - h(x) - q(\vartheta(x)) - r(x,\vartheta(x))\int_0^x f(x,s,\vartheta(s),g(\vartheta(s)))ds$$
$$- p(x,\vartheta(x),\beta(\vartheta(x))) - \int_0^x \Xi(x,s)\rho(\vartheta(s))ds\Bigg| \leq \phi(x), \forall\, x \in [0,\ a], \tag{11}$$

then

$$|\vartheta(x) - \vartheta^*(x)| \leq C_f \times \phi(x),$$
$$\forall\, x \in [0,\ a],$$

where

$$C_f = \frac{1}{1 - q_L - p_L - p_L\beta_L}\exp\left[\frac{\big(r_{M_0} f_L(1+g_L) + \rho_L M\big)a}{1 - q_L - p_L - p_L\beta_L}\right]$$

and

$$q_L + p_L + p_L\beta_L < 1.$$

Hence, VIEq (3) is HUR stable.

Proof. In light of (As1), (As2) and (11), the proof of Theorem 2 can be done analogously to Theorem 1. Hence, we will not give the proof of this theorem. □

Remark 1. *Example 1 can be updated to show that VIEq (10) is also HUR stable. We will not give the details of this discussion for the sake of brevity and to avoid repeating Example 1.*

3. The HUR Stability of the Functional VIEq (4)

In this section, we give the HUR stability result with regard to VIEq (4) in the following theorem.

Theorem 3. *Suppose that we have*

(C1) $r \in C\Big([0,\ a]^2, B\Big)$; $f,\ h,\ p \in C\Big([0,a]^2 \times B, B\Big)$; $g,\ q \in C\Big([0,\ a]^2 \times B, B\Big)$; $K \in C\Big([0,\ a]^4 \times B^2, B\Big)$; $\phi \in C\Big([0,\ a]^2, \mathbb{R}^+\Big)$; *and increasing.*

(C2) There exists positive constants q_M, r_L, g_L, h_L, K_L and f_L such that

$$q_M = \max_{x\in[0,a]} |q(x,y,\vartheta)|, \forall\, x,\ y \in [0,\ a], \forall\, \vartheta \in B,$$

$$|r(\vartheta) - r(v)| \le r_L|\vartheta - v|, \forall\, \vartheta,\ v \in B,$$

$$|g(x,y,\vartheta) - g(x,y,v)| \le g_L|\vartheta - v|, \forall\, x,\ y \in [0,\ a], \forall\, \vartheta,\ v \in B,$$

$$|h(\vartheta) - h(v)| \le h_L|\vartheta - v|, \forall\, \vartheta,\ v \in B,$$

$$|K(x,y,s,t,\vartheta_1,v_1) - K(x,y,s,t,\vartheta_2,v_2)| \le K_L|\vartheta_1 - \vartheta_2| + K_L|v_1 - v_2|,$$

$$\forall\, x,\ s \in [0,\ a], \forall\, y,\ t \in [0,\ a], \forall\, \vartheta_1,\ \vartheta_2,\ v_1,\ v_2 \in B,$$

$$|f(\vartheta) - f(v)| \le f_L|\vartheta - v|, \forall\, \vartheta,\ v \in B\,.$$

Then, we have the following results:

(a) VIEq (4) has a unique solution ϑ^* in $C([0,a]\times[0,a],B)$;

(b) If $\vartheta \in C([0,a]\times[0,a],B)$ is a solution of the inequality

$$\Bigg|\vartheta(x,y) - r(\vartheta)(x,y) - g(x,y,h(\vartheta)(x,y)) - q(x,y,p(\vartheta)(x,y))\int_0^x\int_0^y K(x,y,s,t,\vartheta(s,t),f(\vartheta(s,t)))dsdt\Bigg| \le \phi(x,y), \tag{12}$$

$$\forall\, x,\ y,\ s,\ t \in [0,\ a],$$

then

$$|\vartheta(x,y) - \vartheta^*(x,y)| \le C_{K_L,L_f,g_L,g_M,r_L} \times \phi(x,y), \forall\, x,y \in [0,\ a],$$

where

$$C_{K_L,f_L,g_L,h_L,q_M,r_L} = \frac{1}{1 - r_L - g_Lh_L}\exp\left[\frac{q_MK_L(1+f_L)a^2}{1 - r_L - g_Lh_L}\right]$$

and

$$0 < r_L + g_Lh_L < 1.$$

Hence, VIEq (4) is HUR stable.

Proof.

(*a*) The proof of this theorem can be easily done (see, Lungu [29]). Therefore, we will not give the proof of this theorem.

(*b*) According to the conditions (C1), (C2) and (12), we derive

$$\begin{aligned}\Big|\vartheta(x,y) - \vartheta^*(x,y)\Big| \le &\Bigg|\vartheta(x,y) - r(\vartheta)(x,y) - g(x,y,h(\vartheta)(x,y)) \\ &- q(x,y,p(\vartheta)(x,y))\int_0^x\int_0^y K(x,y,s,t,\vartheta(s,t),f(\vartheta(s,t)))dsdt\Bigg| \\ &+ \Big|r(\vartheta)(x,y) - r(\vartheta^*)(x,y)\Big| + \Big|g(x,y,h(\vartheta)(x,y)) - g(x,y,h(\vartheta^*)(x,y))\Big| \\ &+ \Bigg|q(x,y,p(\vartheta)(x,y))\int_0^x\int_0^y K(x,y,s,t,\vartheta(s,t),f(\vartheta(s,t)))dsdt\end{aligned}$$

$$
\begin{aligned}
&- q(x,y,p(\vartheta^*)(x,y)) \int_0^x \int_0^y K(x,y,s,t,\vartheta^*(s,t), f(\vartheta^*(s,t)))dsdt \Bigg| \\
\leq & \phi(x,y) + r_L|\vartheta(x,y) - \vartheta^*(x,y)| + g_L|h(\vartheta(x,y)) - h(\vartheta^*(x,y))| \\
&+ q_M K_L \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt \\
&+ q_M K_L \int_0^x \int_0^y |f(\vartheta(s,t)) - f(\vartheta^*(s,t))|dsdt \\
\leq & \phi(x,y) + r_L|\vartheta(x,y) - \vartheta^*(x,y)| \\
&+ g_L h_L|\vartheta(x,y) - \vartheta^*(x,y)| \\
&+ q_M K_L \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt \\
&+ q_M K_L f_L \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt.
\end{aligned} \tag{13}
$$

Hence, it follows from (13) that

$$
\begin{aligned}
|\vartheta(x,y) - \vartheta^*(x,y)| \leq & \phi(x,y) + (r_L + g_L h_L)|\vartheta(x,y) - \vartheta^*(x,y)| \\
&+ q_M K_L(1 + f_L) \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt.
\end{aligned} \tag{14}
$$

Then, from (14), it is clear that

$$
\begin{aligned}
(1 - g_L h_L - r_L)|\vartheta(x,y) - \vartheta^*(x,y)| &\leq \phi(x,y) \\
&+ q_M K_L(1 + f_L) \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt.
\end{aligned}
$$

Thus, we derive

$$
\begin{aligned}
|\vartheta(x,y) - \vartheta^*(x,y)| \leq & \frac{1}{1 - r_L - g_L h_L}\phi(x,y) \\
&+ \frac{q_M K_L(1 + f_L)}{1 - r_L - g_L h_L} \int_0^x \int_0^y |\vartheta(s,t) - \vartheta^*(s,t)|dsdt.
\end{aligned} \tag{15}
$$

Using the Gronwall lemma (Lungu [29], Rus [30]), from (15), we obtain that

$$
|\vartheta(x,y) - \vartheta^*(x,y)| \leq \frac{1}{1 - r_L - g_L h_L} \exp\left[\frac{q_M K_L\left(1 + L_f\right)a^2}{1 - r_L - g_L h_L}\right]\phi(x,y). \tag{16}
$$

Let

$$
C_{K_L, f_L, g_L, h_L, q_M, r_L} = \frac{1}{1 - r_L - g_L h_L} \exp\left[\frac{q_M K_L(1 + f_L)a^2}{1 - r_L - g_L h_L}\right].
$$

Hence, it follows from (16) that

$$
|\vartheta(x,y) - \vartheta^*(x,y)| \leq C_{K_L, f_L, g_L, h_L, q_M, r_L} \times \phi(x,y). \tag{17}
$$

According to the above data, (C1) and (C2) of Theorem 3 hold. The above outcomes and (17) imply that VIEq (4) is HUR stable. Thus, the proof of Theorem 3 is completed. □

We now provide the second supporting example to demonstrate the numerical application of Theorem 3.

Example 2. *Consider the following VIEq:*

$$\begin{aligned}\vartheta(x,y) =& \frac{1}{1000}\sin(\vartheta(x,y)) + \frac{\sin(\vartheta(x,y))}{2000+x^4+y^4} \\ &+ \frac{\sin(u(x,y))}{1000(1+\exp(x^2+y^2))}\int_0^x\int_0^y \frac{u(s,t)+\sin(u(s,t))}{500+x^4+y^4+s^2+t^2}dsdt, \end{aligned} \tag{18}$$

where $0 \le s \le x \le a$ *and* $0 \le t \le y \le a$.

We note that VIEq (18) is in the form of VIEq (4) with the data as follows:

$$\begin{aligned} r(\vartheta)(x,y) &= \frac{1}{1000}\sin(\vartheta(x,y)), \\ g(x,y,h(\vartheta)(x,y)) &= \frac{\sin(\vartheta(x,y))}{2000+x^4+y^4}, \\ h(\vartheta)(x,y) &= \sin(\vartheta(x,y)), \\ q(x,y,p(\vartheta)(x,y)) &= \frac{\sin(\vartheta(x,y))}{1000(1+\exp(x^2+y^2))}, \\ p(\vartheta)(x,y) &= \sin(\vartheta(x,y)), \\ K(.) = K(x,y,s,t,\vartheta(s,t),f(\vartheta(s,t))) &= \frac{\vartheta(s,t)+\sin(\vartheta(s,t))}{500+x^4+y^4+s^2+t^2}, \\ f(\vartheta(s,t)) &= \sin(\vartheta(s,t)). \end{aligned}$$

We now check the conditions (C1) and (C2) of Theorem 3. To show that (C1) and (C2) hold, we let $q_M = 1000^{-1}$, $r_L = 1000^{-1}$, $g_L = 2000^{-1}$, $h_L = 1$, $K_L = 500^{-1}$, $f_L = 1$ *and calculate:*

$$|q(x,y,p(\vartheta)(x,y))| = \frac{|\sin(\vartheta(x,y))|}{1000(1+\exp(x^2+y^2))} \le \frac{1}{1000}, \forall\, x,\ y \in [0,\ a], \forall\, \vartheta \in B,$$

$$\begin{aligned} |r(\vartheta(x,y)) - r(v(x,y))| &= \frac{1}{1000}|\sin(\vartheta(x,y)) - \sin(v(x,y))| \\ &\le \frac{1}{1000}|\vartheta(x,y) - v(x,y)|, \forall\, x,\ y \in [0,\ a], \forall\, \vartheta,\ v \in B, \\ |g(x,y,h(\vartheta)(x,y)) - g(x,y,h(v)(x,y))| &= \frac{|\sin(\vartheta(x,y)) - \sin(v(x,y))|}{2000+x^4+y^4} \\ &\le \frac{1}{2000}|\sin(\vartheta(x,y)) - \sin(v(x,y))| \\ &\le \frac{1}{2000}|\vartheta(x,y) - v(x,y)|, \forall\, x,\ y \in [0,\ a] \in B, \\ |K(.) - K(.)| &\le \frac{1}{500}|\vartheta(s,t) - v(s,t)| + \frac{1}{500}|\sin(\vartheta(s,t)) - \sin(v(s,t))| \\ &\le \frac{1}{250}|\vartheta(s,t) - v(s,t)|, \forall\, x,\ y,\ s,\ t \in [0,\ a], \forall\, \vartheta,\ v \in B, \\ r_L + g_L h_L &= \frac{1}{500} + \frac{1}{2000} = \frac{1}{400}, 0 < r_L + g_L h_L = \frac{1}{400} < 1, \end{aligned}$$

$$
\begin{aligned}
C_{K_L,f_L,g_L,h_L,q_M,r_L} &= \frac{1}{1-r_L-g_Lh_L}\exp\left[\frac{q_MK_L(1+f_L)a^2}{1-r_L-g_Lh_L}\right] \\
&= \frac{400}{399}\exp\left(\frac{400}{399}\times\frac{a^2}{250000}\right) \\
&= \frac{400}{399}\exp\left(\frac{a^2}{249375}\right), \\
|\vartheta(x,y)-\vartheta^*(x,y)| &\leq C_{K_L,L_f,g_L,g_M,r_L}\times\phi(x,y) \\
&= \frac{400}{399}\exp\left(\frac{a^2}{249375}\right)\times\phi(x,y), \forall\, x,y\in[0,\ a].
\end{aligned}
$$

Hence, (C1) and (C2) hold. Thus, VIEq (4) is HUR stable. Therefore, the application of Theorem 3 is valid.

4. Discussion

We now provide some discussion to compare the results of this paper with the related ones that can be found in the literature.

(1) It should be noted that VIEq (3) and VIEq (4) are new mathematical models given as a Hammerstein-type integral equation and Hammerstein-type functional integral equation, respectively. According to data from the relevant literature and in the references of this paper, there is no result with regard to the HU stability and HUR stability of VIEq (3) and VIEq (4). The results of this work are novel regarding the HU stability and HUR stability of nonlinear VIEqs.

(2) To the best of our knowledge, in the relevant literature, nearly all of the results with regard to the HU stability, HUR stability, etc., of various VIEqs and VIDEqs were obtained with the help of Banach's fixed point theorem, the Bielecki metric, the Pachpatte's inequality and the Picard operator theory. However, we found the paper of Lungu [29], where the Gronwall lemma was used as a basic tool to prove the HU stability and HUR stability of nonlinear VIEqs. It can be seen that the Gronwall lemma plays a very effective role when investigating the HU stability and HUR stability of nonlinear VIEqs. In this article, we benefit from the advantages of the Gronwall lemma to prove our main results. Furthermore, the Gronwall lemma can lead to less restrictive conditions with regard to the HU stability and HUR stability of nonlinear VIEqs.

(3) If $q(\vartheta(x))=0$, $r(x,\vartheta(x))=0$, $p(x,\vartheta(x),\beta(\vartheta(x)))=0$ and $\int\limits_0^x \Xi(x,s)\rho(\vartheta(s))ds=0$, then VIEq (3) is reduced to the VIEq (1) investigated by Lungu [29]. Hence, VIEq (3) extends and improves VIEq (1) from Lungu [29].

(4) If $r(\vartheta)(x,y)=0$ and $q(x,y,p(\vartheta)(x,y))=1$, then VIEq (4) is reduced to the VIEq (2) investigated by Lungu [29]. Hence, VIEq (4) extends and improves VIEq (2) from Lungu [29].

(5) In light of (3) and (4), Theorems 1 , 2 and Theorem 3 generalize and improve the results of Lungu [29], Theorems 4.1, 4.2 and Lungu [29], Theorem 5.1, respectively. Furthermore, for the above choices, the conditions of our main results, namely, Theorems 1–3, reduce to those of Lungu [29], Theorems 4.1, 4.2 and Lungu [29], Theorem 5.1, respectively.

(6) In this paper, two supporting examples are given to demonstrate the relevance and effectiveness of the HU stability and HUR stability results of Theorems 1, 2 and the HUR stability result of Theorem 3, respectively. However, in Lungu [29], Theorems 4.1, 4.2 and Lungu [29], Theorem 5.1, respectively, there is no example for the applications and illustrations of these results. Example 1 and Example 2 are considered as additional and effective contributions provided by this paper.

5. Conclusions

In this study, we investigated the HU stability and HUR stability of a nonlinear VIEq and nonlinear functional VIEq in higher dimensions. We proved three new results, which have sufficient conditions with regard to these qualitative concepts, and the technique of the proofs is based on the use of the Gronwall lemma. As numerical applications of these results, two supporting examples are provided to demonstrate the applications and effectiveness of the results. Comparisons between our results and those found in the literature are provided. As future suggestions, the HU stability and HUR stability of nonlinear Caputo fractional VIDEqs, Caputo–Hadamard fractional VIDEqs, Riemann–Liouville fractional VIDEqs, etc., can be considered as open problems.

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Abbreviations

IEq	Integral equation
VIEq	Volterra integral equation
IDEq	Integro-differential equation
VIDEq	Volterra integro-differential equation
HU	Hyers–Ulam
HUR	Hyers–Ulam–Rassias
ODEq	Ordinary differential equation
FDEq	Functional differential equation

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