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A New Two-Step Hybrid Block Method for the FitzHugh–Nagumo Model Equation

Mufutau Ajani Rufai [1,*], Athinoula A. Kosti [2], Zacharias A. Anastassi [2] and Bruno Carpentieri [1]

1 Faculty of Engineering, Free University of Bozen-Bolzano, 39100 Bolzano, Italy; bruno.carpentieri@unibz.it
2 Institute of Artificial Intelligence, School of Computer Science and Informatics, De Montfort University, Leicester LE1 9BH, UK; athinoula.kosti@dmu.ac.uk (A.A.K.); zacharias.anastassi@dmu.ac.uk (Z.A.A.)
* Correspondence: mufutauajani.rufai@unibz.it

Abstract: This paper presents an efficient two-step hybrid block method (ETHBM) to obtain an approximate solution to the FitzHugh–Nagumo problem. The considered partial differential equation model problems are semi-discretized, reducing them to equivalent ordinary differential equations using the method of lines. In order to evaluate the effectiveness of the proposed ETHBM, three numerical examples are presented and compared with the results obtained through existing methods. The results demonstrate that the proposed ETHBM produces more efficient results than some other numerical approaches in the literature.

Keywords: hybrid block method; partial differential equations; collocation method; FitzHugh–Nagumo equations; stability analysis

MSC: 65L05; 65L20

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1. Introduction

In this section, we introduce the nonlinear FitzHugh–Nagumo equation (NFNE), which is a well-known model in mathematical biology. The classical form of the NFNE is given by

$$u_t = u_{xx} + u(x,t)\,(\lambda - u(x,t))\,(1 - u(x,t)), \tag{1}$$

and the generalized form of the NFNE is given by

$$u_t = q_1(t)u_{xx} - q_2(t)u_x + q_3(t)u(x,t)\,(\lambda - u(x,t))\,(1 - u(x,t)), \tag{2}$$

where u is the dependent variable, x and t are the space and time variables, respectively, $q_1(t), q_2(t)$, and $q_3(t)$ are arbitrary real-value functions, and $0 \leq \lambda \leq 1$ is a constant. The above equations are subject to the following initial and boundary conditions:

$$u(x,0) = q(x),\ a \leq x \leq b, \tag{3}$$

$$u(a,t) = p_1(t),\ u(b,t) = p_2(t),\ t \geq 0, \tag{4}$$

where $q(x)$, $p_1(t)$, and $p_2(t)$ are the given functions.

The NFNE problems given in Equations (1)–(4) are essential nonlinear reaction–diffusion equations that are used to model the behaviors of excitable systems, such as nerve cells and heart muscle cells. The NFNE has been of interest to theoretical paleontologists, physicists, and applied mathematicians because it has been used to model real-life problems in mathematical biology and computational neuroscience, where FitzHugh–Nagumo equations are used to model the electrical activity of neurons in the brain and the heartbeat of the heart [1–3].

There are several analytical and numerical methods used for solving the NFNE. Some of the most common methods are briefly discussed as follows:

One common analytical method used for integrating FitzHugh–Nagumo equations and related problems is the homotopy perturbation method (HPM). The HPM involves using a homotopy, a continuous deformation of one equation into another, to find the solution of the nonlinear PDE of the form (1). The HPM can be used to obtain analytical solutions with reasonable accuracy. The advantage of HPM is that it can provide accurate solutions with fewer iterations compared to some other analytical methods. The disadvantage is that finding an appropriate initial guess for the solution may be difficult. A detailed description of this method can be found in [4–9].

Another analytical approach used to solve FitzHugh–Nagumo equations and related problems is the Adomian decomposition method (ADM). This method decomposes the solution into Adomian polynomials, which comprise a set of functions. The ADM can be applied to various nonlinear differential problems and provide accurate solutions. The most significant advantage of ADM is its ease of implementation as well as its ability to provide exact solutions to some model problems. Nevertheless, for a satisfactory analytical solution, the series convergence requires more terms to obtain a good analytical solution. More details about ADM and its modifications are reported in [10–12].

Numerical solutions to the NFNE can be obtained by discretizing the spatial and temporal domains and applying finite difference methods. These approaches can yield reliable results for a large range of initial and boundary conditions, but they are computationally expensive and can show numerical instability. For more details about finite difference methods, see [13–16].

Another numerical technique is the finite element method (FEM). The FEM involves discretizing the domain into a finite number of elements and approximating the solution to each element using a polynomial function. The solution is then obtained by combining the element's solution with a global solution. This approach is highly adaptable and handles irregular geometry and complex boundary conditions. However, the FEM can be computationally expensive and require many elements to generate accurate results. For more details about FEM, see [17–20].

Many researchers have introduced various analytical and numerical methods to solve problems of the types described in Equations (1)–(4). Those strategies include the analytical method by Li and Guo [21], the variational formulation method presented in [22], the Jacobi–Gauss–Lobatto method by Bhrawy [3], analytical and numerical solutions presented in [23], a numerical approach presented by Chandraker et al. [24], the pseudo-spectral method presented by Daniel and Shizgal [25], exponential and cubic B-spline methods presented in [26,27], optimized numerical methods presented in [28,29], the block method reported in [30], and the differential quadrature method reported by Mittal and Jiwari [31].

Block methods have been applied to solve Burgers' and other equations in the available literature. The previous works have focused on the numerical solutions of various differential equations using different methods. The primary contribution of this study lies in introducing an efficient two-step hybrid block method (ETHBM) designed to provide an approximate solution for both classical and generalized nonlinear FitzHugh–Nagumo equations (NFNE). The results presented in this paper hold substantial significance as they emphasize the efficacy of the ETHBM for solving the NFNE in Equations (1) and (2). The results of numerical examples reported in this paper show that the ETHBM outperforms existing methods used for comparison in accuracy and computational efficiency, thereby establishing its superiority in numerical solution techniques for the considered equations.

2. Problem Discretization and Development of the Proposed Method

In this section of the manuscript, we will focus on the problem discretization and the development of the proposed method.

2.1. Problem Discretization

In order to transform the NFNE in Equations (1) and (2) into a set of ODEs, we will employ the widely utilized method known as the method of lines. This approach involves the discrete partitioning of both the spatial and temporal domains. Firstly, the spatial domain, which ranges from a to b, will be discretized into a collection of N equidistant grid points. These grid points are conveniently labeled as x_i, with each point defined as $x_i = a + ih_x$, where $h_x = \frac{b-a}{N}$. This discretization allows us to break down the spatial dimension into manageable intervals. Similarly, we discretize the temporal domain into a sequence of M equidistant time steps, denoted as $t_n = n\Delta t$, where Δt signifies the size of each time step. This temporal discretization ensures that we accurately capture the system's evolution over time. Then, using finite difference schemes, we approximate the derivatives in the original NFNE given in Equations (1) and (2). The resulting approximation of the classical NFNE takes the following form:

$$\frac{d}{dt}u_i = \frac{1}{h_x^2}(u_{i+1} - 2u_i + u_{i-1}) + u_i(\lambda - u_i)(1 - u_i), \quad i = 1, \ldots, N-1, \tag{5}$$

and the generalized of NFNE takes the following form:

$$\begin{aligned} \frac{d}{dt}u_i = {} & q_1(t)\frac{1}{h_x^2}(u_{i+1} - 2u_i + u_{i-1}) - q_2(t)\frac{u_{i+1}(t) - u_{i-1}(t)}{2h_x} \\ & + q_3(t)u_i(\lambda - u_i)(1 - u_i), \quad i = 1, \ldots, N-1, \end{aligned} \tag{6}$$

where u_i represents the numerical approximation of the function $u(x_i, t_n)$; x_i corresponds to the spatial grid points; and t_n corresponds to the time steps.

2.2. Development of the Proposed Method

In order to obtain an approximate solution for the problems presented in Equations (5) and (6) within a subinterval $[t_n, t_{n+2}]$, a fifth-degree polynomial denoted as $p(t)$ is utilized and represented as

$$u(t) \approx p(t) = \sum_{j=0}^{5} a_j t^j, \tag{7}$$

where the coefficients a_j are real numbers that fulfill specific collocation conditions at designated nodes. We approximate the derivative of $u(t)$ using the following equation:

$$u'(t) \approx p'(t) = \sum_{j=1}^{5} a_j j t^{j-1}. \tag{8}$$

To develop the ETHBM, we select two hybrid points, b_1 and b_2, from the interval $[t_n, t_{n+2}]$. To obtain the system of equations required for the proposed ETHBM, we substitute t_n into Equation (7) and t_n, t_{n+b_1}, t_{n+1}, t_{n+b_2}, and t_{n+2} into Equation (8). This substitution results in a system of equations, which includes the following equations:

$$\begin{aligned} & p(t_n) = u_n, \\ & p'(t_n) = g_n, p'(t_{n+b_1}) = g_{n+b_1}, p'(t_{n+1}) = g_{n+1}, p'(t_{n+b_2}) = g_{n+b_3}, p'(t_{n+2}) = g_{n+2}, \end{aligned}$$

where u_{n+j} and g_{n+j} are approximations of the solution $u(t_{n+j})$ and its derivatives $g(t_{n+j}, u(t_{n+j}))$, respectively.

After determining the values of a_n for $n = 0, 1, 2, 3, 4$, and 5, we substitute them into Equation (7) and replace the variable t with $t_n + xh$. This substitution results in a continuous approximation using the polynomial in Equation (7), as follows:

$$
\begin{aligned}
p(t_n + xh) = u_n + h\big(\beta_0(x)g_n + \beta_{b_1}(x)g_{n+b_1} + \beta_1(x)g_{n+1} + \beta_{b_2}(x)g_{n+b_2}\big) \\
+ h\big(\beta_2(x)g_{n+2}\big).
\end{aligned}
\tag{9}
$$

where $\{\beta_i(x)\}_{i=0,b_1,1,b_2,2}$ depend on b_1 and b_2. To obtain the proposed ETHBM, we substitute the values of x as $x = \frac{1}{4}, x = 1, x = \frac{7}{4}$, and $x = 2$ into Equation (9). After inserting $b_1 = \frac{1}{4}$ and $b_2 = \frac{7}{4}$, and simplifying, we obtain the following formulas:

$$
\begin{aligned}
u_{n+b_1} &= u_n + \frac{h\big(18692g_{n+b_1} + 1052g_{n+b_2} + 12717g_n - 1708g_{n+1} - 513g_{n+2}\big)}{120960}, \\
u_{n+1} &= u_n + \frac{h\big(2464g_{n+b_1} - 416g_{n+b_2} - 81g_n + 1624g_{n+1} + 189g_{n+2}\big)}{3780}, \\
u_{n+b_3} &= u_n + \frac{7h\big(1316g_{n+b_1} + 956g_{n+b_2} + 81g_n + 2156g_{n+1} - 189g_{n+2}\big)}{17280}, \\
u_{n+2} &= u_n + \frac{1}{945}h\big(512\big(g_{n+b_1} + g_{n+b_2}\big) + 27g_n + 812g_{n+1} + 27g_{n+2}\big).
\end{aligned}
\tag{10}
$$

The formula in Equation (10) is the ETHBM. This method is designed to solve the transformed systems of ODEs given in Equations (5) and (6).

3. Analysis of the ETHBM

In this section of the manuscript, we present the theoretical aspects of the ETHBM. The theoretical analysis presented here provides a rigorous understanding of the ETHBM method and its performance, which is crucial for its practical application.

3.1. Local Truncation Errors, Consistency, and Convergence of the ETHBM

The formulas in Equation (10) can be expressed using the following recurrence relation:

$$
\bar{R}U_n = h\bar{S}U_n', \tag{11}
$$

where the constant matrices $\bar{R}$ and $\bar{S}$ correspond to the coefficients of the left- and right-hand sides of the formulas in Equation (10), and the vectors U_n and U_n' represent the values of the solution and its derivatives at the nodes t_n, t_{n+b_1}, t_{n+1}, t_{n+b_2}, and t_{n+2}.

To analyze the error associated with the ETHBM, we introduce an operator denoted as L, which is linked to the method presented in Equation (10). This operator L takes as its input the function $u(t)$ and the step size h:

$$
L[u(t);h] = \sum_{j\in I}\big[\Theta_j u(t_n + jh) - h\Psi_j u'(t_n + jh)\big], \tag{12}
$$

where Θ_j and Ψ_j are column vectors within the matrices $\bar{R}$ and $\bar{S}$, respectively. Set I consists of $0, b_1, 1, b_2, 2$, representing the indices used in the ETHBM. Assuming that $u(t)$ is sufficiently differentiable, we expand $u(t_n + jh)$ and $u'(t_n + jh)$ in a Taylor series around the point, t_n, where j denotes the index in the set, I. Consequently, we obtain

$$
L[u(t_n);h] = \theta_0 u(t_n) + \theta_1 h u'(t_n) + \theta_2 h^2 u''(t_n) + \cdots + \theta_q h^q u^{(q)}(t_n) + \ldots, \tag{13}
$$

where θ_q is defined as

$$
\theta_q = \frac{1}{q!}\left[\sum_{j\in I}^{k} j^q\Theta_j - q\sum_{j\in I}^{k} j^{q-1}\Psi_j\right],\ q = 0, 1, 2, \ldots. \tag{14}
$$

By utilizing the definitions of order and LTEs as provided in [32,33], we can derive the order (p) and expressions of the LTEs for the obtained formulas. The resulting expressions are presented below:

Table 1 displays the LTEs and order of the obtained formulas. The ETHBM is a two-step method and, therefore, has zero-stability properties. Moreover, the method is consistent since the order p exceeds one. Furthermore, due to its zero-stability and consistency, the ETHBM is convergent.

Table 1. Local truncation error and the order of accuracy for the ETHBM.

Formula	Order	local Truncation Error
u_{n+b_1}	5	$\frac{343h^7u^{(6)}(t_n)}{5{,}898{,}240} + \mathcal{O}(h^7)$
u_{n+1}	5	$-\frac{11h^6u^{(6)}(t_n)}{23{,}040} + \mathcal{O}(h^7)$
u_{n+b_2}	5	$\frac{343h^6u^{(6)}(t_n)}{5{,}898{,}240} + \mathcal{O}(h^7)$
u_{n+2}	6	$\frac{h^7u^{(7)}(t_n)}{20{,}160} + \mathcal{O}(h^8)$

3.2. Linear-Stability Analysis

In this section, we follow the guidelines reported by [33] to study the linear stability analysis of the ETHBM. Linear stability analysis is a crucial aspect of numerical methods, as it helps us understand the behavior of the method in the presence of small perturbations. To perform the linear stability analysis, we apply the proposed ETHBM to the differential equation $u' = \nu u$, where $Re(\nu) < 0$. This differential equation is a standard test equation used in the linear stability analysis. By applying ETHBM to this equation, we derive the following recurrence relation:

$$U_{n+1} = M(z)U_n, \quad z = \nu h, \tag{15}$$

where ν is a complex number, and $M(z)$ denotes a stability matrix, with its explicit representation being

$$M(z) = \left(C^0 + zE\right)^{-1}\left(C^1 + zD\right), \tag{16}$$

where the components of C^0, C^1, E, and D are obtained as follows:

$$C^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 + \frac{471}{4480} \\ 0 & 0 & 0 & 1 + \frac{21}{640} \\ 0 & 0 & 0 & 1 - \frac{3}{140} \\ 0 & 0 & 0 & 1 + \frac{1}{35} \end{pmatrix},$$

$$D = \begin{pmatrix} 1 - \frac{4673}{30240} & \frac{61}{4320} & -\frac{263\nu}{30240} & \frac{19}{4480} \\ -\frac{2303}{4320} & -\frac{3773}{4320} & 1 - \frac{1673}{4320} & \frac{49}{640} \\ -\frac{88}{135} & 1 - \frac{58}{135} & \frac{104}{945} & -\frac{1}{20} \\ -\frac{512}{945} & -\frac{116}{135} & -\frac{512}{945} & 1 - \frac{1}{35} \end{pmatrix}.$$

This recurrence relation helps us understand the stability properties of ETHBM. By analyzing the eigenvalues of the matrix $M(z)$, we can determine the stability region of the method. A stable method produces numerical solutions that do not grow unbounded as we increase the time steps. By performing the linear stability analysis, we can gain insights into the behavior of ETHBM and its potential applications in solving real-world problems. This analysis will help us understand the strengths and weaknesses of our method and compare it with other existing numerical methods. The eigenvalues of the stability matrix $M(z)$ for ETHBM are calculated as $\left\{0, 0, 0, \frac{7z^4+85h^3+405z^2+960z+960}{7z^4-85z^3+405z^2-960z+960}\right\}$. These eigenvalues hold particular significance as they directly influence the stability characteristics of the numerical solution. To evaluate the stability of the ETHBM, we introduce the concept of the spectral radius, denoted as $\rho(M(z))$. This parameter offers crucial insights into the method's stability behavior. The region where the absolute value of the spectral radius is less than one, denoted as $\mathbb{S} = \{z \in \mathbb{C} : |\rho(M(z))| < 1\}$, represents the resulting region of absolute stability. Figure 1 visually represents the ETHBM's stability region, demonstrating that the entire left half of the complex plane falls within this region. This observation emphasizes the ETHBM's remarkable quality of being A-stable, signifying its robust capability to solve the considered differential problem effectively.

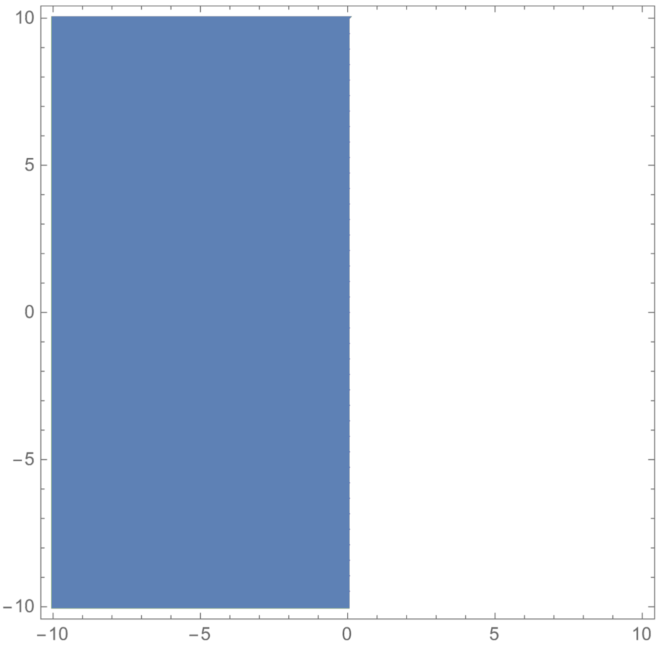

Figure 1. ETHBM's stable region in the complex z-plane.

4. Implementation of the Proposed ETHBM

Using approximations of u_{xx} and u_x at (x_i, t), Equations (1) and (2) are discretized to obtain ODE systems in Equations (5) and (6), and then the formulas in Equation (10) are reformulated as $\boldsymbol{G(u)} = \mathbf{0}$, to represent the discretized problem. The unknowns are represented by $\tilde{\mathbf{U}}$, as follows:

$$\tilde{\mathbf{U}} = \left(u_{1,n+b_1}, u_{2,n+b_1}, \ldots, u_{N-2,n+b_1}, u_{1,n+1}, u_{2,n+1}, \ldots, u_{N-2,n+b_2}, u_{1,n+1}, u_{2,n+1}, \ldots, u_{N-2,n+1}\right).$$

The implicit nature of the proposed ETHBM method results in nonlinear equation systems that must be solved at each time step using a nonlinear solver. The nonlinear equations are then solved using the Newton method (NM). The NM uses iterative computations to find the solution to a linear system until convergence, starting with an initial solution guess and updating it until convergence. In the ETHBM scheme, Newton's method adopts the following formulation:

$$\tilde{\mathbf{U}}^{\mathbf{i}+\mathbf{1}} = \tilde{\mathbf{U}}^{\mathbf{i}} - (\mathbf{J_0})^{-\mathbf{1}}\mathbf{G}^{\mathbf{i}},$$

where $\tilde{\mathbf{U}}^{\mathbf{i}}$ represents the current approximation of the solution, $\mathbf{G}^{\mathbf{i}}$ denotes the nonlinear function computed at $\tilde{\mathbf{U}}^{\mathbf{i}}$, and $\mathbf{J_0}$ indicates the fixed Jacobian matrix of G evaluated at $\tilde{\mathbf{U}}^{\mathbf{i}}$. In order to initiate the NM iterations, the solution can be approximated using values from the previous time step. In the ETHBM scheme, known values u_n and g_n are utilized to determine initial values for unknowns in each iteration. These initial values are calculated as $u_{n+j} = u_n + (jh)g_n$, where h represents the time step size and $j = b_1, 1, b_2, 2$ corresponds to four different sets of unknowns within the system. By employing these initial values, the convergence of the NM iterations is accelerated, leading to a reduction in the computational cost.

5. Numerical Examples and Comparisons

In this section, our focus is on presenting the numerical solutions for a real-world model, which is commonly known as the nonlinear FitzHugh–Nagumo in Equations (1) and (2). We utilize our proposed ETHBM to obtain numerical results.

Moreover, we go a step further and compare the results of ETHBM with other existing numerical methods that can be found in the literature. This comparison will help us understand the strengths and weaknesses of our proposed method compared to other methods. By doing so, we hope to provide the performance of ETHBM and its potential applications in solving real-world problems under consideration. The maximum error norm L_∞ is calculated using the following formula:

$$L_\infty = \max_{0\le i\le N, 0\le j\le M} \left| u(x_i, t_j) - U(x_i, t_j) \right|_\infty,$$

where $u(x_i, t_j)$ and $U(x_i, t_j)$ represent the exact solution and the solution obtained through ETHBM, respectively, at the discrete space point, x_i, and time point, t_j. This formula helps us determine the maximum difference between the exact solution and the solution obtained through our proposed method. By utilizing the formulas, we can evaluate the accuracy and efficiency of our proposed method and compare it with other existing numerical methods.

In the tables and figures presented in this paper, we use abbreviations to represent different methods and performance measures. These abbreviations are essential in presenting the numerical results concisely and clearly.

ETHBM denotes the efficient two-step hybrid block method proposed in this manuscript. This method is the focus of our study, and we use it to obtain numerical solutions for Equations (1) and (2).

The polynomial differential quadrature method is represented by PDQM and is cited from [34]. This method is a numerical method used in solving differential equations. It involves approximating the derivatives of the solution using polynomial interpolation.

The compact finite difference block technique is denoted by CFDBT and is described in [35]. This method involves approximating the derivatives of time-dependent partial differential equations using finite difference approximations in combination with the block method.

The discontinuous Galerkin method is referred to as DGM and is reported in [36]. This method is a numerical method used in solving partial differential equations. It involves approximating the solution using piecewise polynomials and discontinuous Galerkin methods. The nonstandard finite difference method is referred to as NFDM and is reported in [37]. The proposed method is implemented using the composed codes in Mathematica 13 on an i7-configured computer running a 64-bit operating system.

5.1. Numerical Example 1

As a first numerical example, we consider the NFNE in Equation (1). The equation is subject to the following initial condition, as presented in [34,35]:

$$u(x, 0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right), \quad x \in [-10, 10].$$

This initial condition represents the system's initial state at time $t = 0$. We can express the solution to this problem in analytical form as follows:

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} - \frac{1}{4}(2\lambda - 1)t\right).$$

This analytical solution helps us understand the system's behavior and provides a benchmark for evaluating the accuracy of our numerical method. The boundary conditions for Example 1 are obtained from the analytical solution. These boundary conditions ensure that the solution remains bounded and well-behaved as we increase the number of time steps. By solving this problem using our proposed method, we can obtain numerical solutions that approximate the exact solution.

The computational order of convergence (OC) in Table 2 is obtained using the following formula:

$$OC = -\log_2\left(\frac{L_\infty(h)}{L_\infty(2h)}\right).$$

Table 2. Order of convergence for Example 1 with $t \in [0,1], \lambda = 0.5$.

h	Method	L_∞	*ROC*
$\frac{1}{10}$	ETHBM	2.1011×10^{-10}	
$\frac{1}{20}$	ETHBM	5.1773×10^{-12}	5.3428
$\frac{1}{40}$	ETHBM	1.3484×10^{-13}	5.2629
$\frac{1}{80}$	ETHBM	4.9511×10^{-15}	4.7674

For Example 1, we present the numerical results obtained using the proposed ETHBM method for $\lambda = 0.75$ and $N = 100$ at different time values in Table 2. We also compare these results with the numerical results obtained using the CFDBT and the PDQM methods. The data in Table 2 demonstrate that the L_∞ error of the proposed ETHBM is smaller than the L_∞ errors obtained using the CFDBT and PDQM methods, demonstrating the effectiveness and robustness of the proposed ETHBM method in solving the NFNE problem in Equation (1). Furthermore, we provide a plot of the absolute error at $t = 5$ and $N = 100$ for Example 1 in Figure 2, and the CPU time in seconds that the computer used to obtain the Figure 2 using the ETHBM is 4.3125. This figure shows that the proposed ETHBM method yields significantly smaller errors than the CFDBT and PDQM methods.

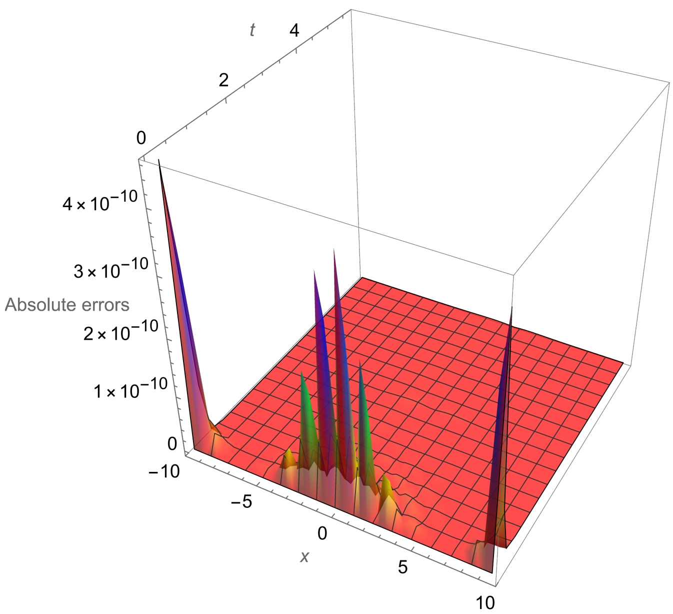

Figure 2. Plot of absolute error with $N = 100$ on $(x,t) \in [-10,10] \times [0,5]$ for Example 1.

5.2. Numerical Example 2

Consider a class of NFNE in Equation (1), which belongs to the class of stiff test problems as reported by [35,37]. The problem is given by

$$u_t = u_{xx} + \eta\, u(x,t)\,(\lambda - u(x,t))\,(1 - u(x,t)), \quad x \in [-10, 10],$$

where the parameter $\lambda \in (0,1)$ plays a critical role in characterizing the overall dynamics of the equation, while $\eta > 0$ represents the natural growth rate. This stiff problem is subject to the following initial condition:

$$u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\eta}x}{2\sqrt{2}}\right), \quad x \in [-10, 10].$$

This problem is an example of a stiff differential equation, which is a type of differential equation that is difficult to solve numerically due to the presence of rapidly varying and slowly varying components. The solution to this problem can be expressed in analytical form as follows:

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\eta}\left(x + t\left(\sqrt{\frac{\eta}{2}}(2\lambda - 1)\right)\right)}{2\sqrt{2}}\right).$$

We use the above exact solution to provide the boundary conditions for Example 2, guaranteeing that the solution remains well-behaved and bounded as we increase the number of time steps.

As an illustration, we present the numerical results obtained using the proposed ETHBM technique for $\lambda = 0.2, t = 0.5$, and $N = 100$ at various η values in Table 3. Additionally, we compare these results with the numerical results obtained using the CFDBT and the NFDM methods. The data in Table 3 show that the proposed ETHBM method has a smaller L_∞ error than the L_∞ errors obtained using the CFDBT and NFDM methods. This result highlights the effectiveness and robustness of the proposed ETHBM method in solving the stiff type of the NFNE in Equation (1). Furthermore, we provide the plots of the absolute errors at $t = 0.5, N = 100, \eta = 2$ and $t = 0.5, N = 100, \eta = 10$ for Example 2 in Figures 3 and 4, respectively; the CPU times in seconds that the computer used to obtain these figures using the ETHBM are 2.0582 and 2.2344, respectively. These figures show that the proposed ETHBM method yields significantly smaller errors than the CFDBT and PDQM methods.

Table 3. Comparison of L_∞ for Example 1 with $\lambda = 0.75, N = 100$.

t	L_∞ for ETHBM	L_∞ for CFDBT	L_∞ for PDQM
0.2	1.0658×10^{-12}	5.4996×10^{-8}	4.7416×10^{-5}
0.5	1.8416×10^{-11}	1.0653×10^{-7}	1.2312×10^{-4}
1.0	4.8221×10^{-11}	1.5822×10^{-7}	2.6261×10^{-4}
1.5	5.3832×10^{-11}	1.9190×10^{-7}	4.2096×10^{-4}
2.0	5.5785×10^{-11}	2.1710×10^{-7}	5.9999×10^{-4}
3.0	8.9709×10^{-11}	2.5517×10^{-7}	1.0324×10^{-3}
5.0	2.8277×10^{-10}	3.1252×10^{-7}	2.3050×10^{-3}

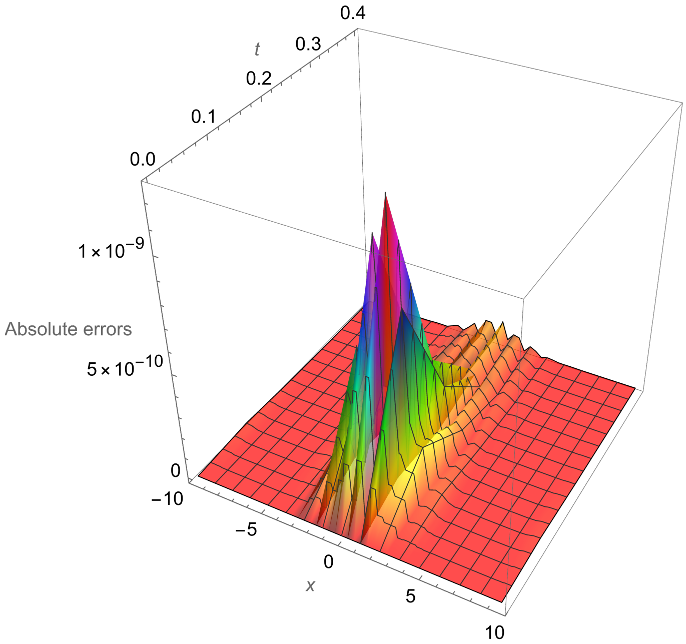

Figure 3. Plot of absolute errors with $N = 100, \eta = 2$ on $(x, t) \in [-10, 10] \times [0, 0.5]$ for Example 2.

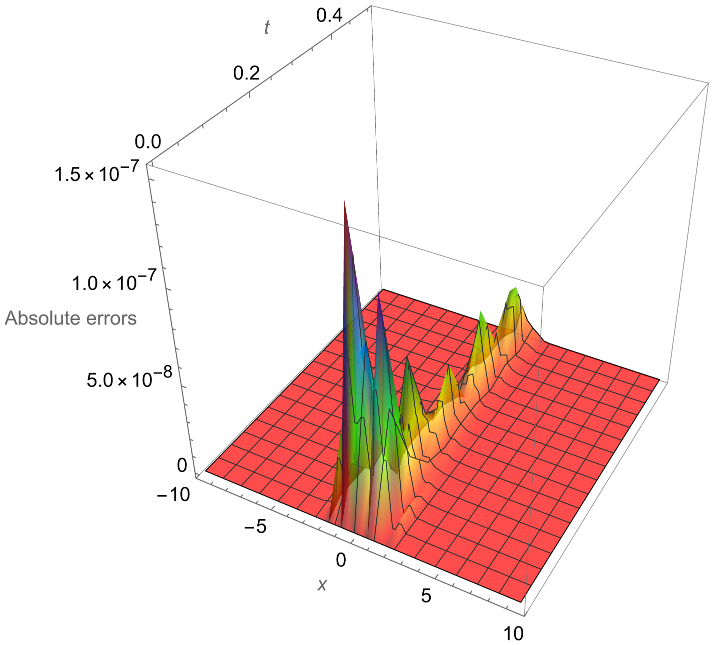

Figure 4. Plot of absolute errors with $N = 100, \eta = 10$ on $(x, t) \in [-10, 10] \times [0, 0.5]$ for Example 2.

5.3. *Numerical Example* 3

We apply the proposed method in the third numerical example to solve the generalized NFNE presented in Equation (2). This equation is subject to a specific initial condition, as reported in [34,36]. The initial condition represents the system's initial state at time $t = 0$ and is given by

$$u(x, 0) = \frac{\lambda}{2} + \frac{\lambda}{2} \tanh\left(\frac{x}{2}\right), \quad x \in [-10, 10].$$

The above initial condition provides a starting point for solving the NFNE problem in Equation (2). The analytical solution to the NFNE problem in Equation (2) is given by the following expression:

$$u(x,t) = \frac{\lambda}{2} + \frac{1}{2}\lambda \tanh\left(\frac{1}{2}\lambda(x - (3-\lambda)\sin(t))\right),$$

where the parameter $\lambda \in (0,1)$. This analytical solution is a useful reference for evaluating the accuracy and efficiency of numerical methods for solving the NFNE problem in Equation (2). The boundary conditions of this problem are obtained from the exact solution, ensuring that the solution remains well-behaved and bounded as we increase the number of time steps.

Table 4 compares the L_∞ errors for Example 3 at $t = 0.5$ with $\lambda = 0.75$ and $N = 100$ using three different numerical methods: ETHBM, CFDBT, and NFDM. This table shows the L_∞ errors for each method at different time values. From this table, we can see that the ETHBM method has the smallest L_∞ error for all values of times, followed by DGM and PDQM, indicating that the ETHBM method provides the most accurate approximation of the solution for Example 3. Furthermore, we can observe that as the value of time increases, the L_∞ error also increases for all three methods. This suggests that a shorter time is necessary for a more accurate solution approximation. Overall, the results in Table 5 demonstrate the effectiveness of the ETHBM method in providing accurate approximations of solutions to the generalized NFNE problem in Equation (2). Also, we provide a plot of the absolute errors at $t = 0.2$ and $N = 100$ for Example 3 in Figure 4; the CPU time in seconds that the computer used to obtain this figure using the ETHBM is 1.7343. This figure shows that the proposed ETHBM method yields significantly smaller errors.

Table 4. Comparison of L_∞ for Example 2 at $t = 0.5$ with $\lambda = 0.2, N = 100$.

η	L_∞ **for ETHBM**	L_∞ **for CFDBT**	L_∞ **for NFDM**
0.5	5.0804×10^{-12}	1.6401×10^{-8}	4.1322×10^{-6}
1.0	2.6857×10^{-11}	1.0738×10^{-7}	2.4227×10^{-5}
2.0	1.2253×10^{-9}	6.4360×10^{-7}	1.5759×10^{-4}

Table 5. Comparison of L_∞ for Example 3 with $\lambda = 0.75, N = 100$.

t	L_∞ **for ETHBM**	L_∞ **for DGM**	L_∞ **for PDQM**
0.2	7.0474×10^{-11}	1.1228×10^{-5}	1.2350×10^{-5}
0.5	8.0531×10^{-11}	7.5845×10^{-5}	5.1986×10^{-4}
1.0	1.4813×10^{-10}	9.1278×10^{-5}	6.3283×10^{-4}
1.5	2.9014×10^{-10}	9.2423×10^{-5}	8.5383×10^{-4}
2.0	5.5413×10^{-10}	1.7724×10^{-5}	9.9123×10^{-4}
3.0	7.5484×10^{-10}	1.4609×10^{-4}	1.7904×10^{-3}

6. Conclusions

This paper presents an efficient two-step hybrid block method (ETHBM) to obtain an approximate solution to non-linear FitzHugh–Nagumo equations (NFNE). The proposed method's efficiency is demonstrated by its successful application to three different NFNEs, including the stiff type of NFNE problem. ETHBM results are compared with some existing numerical methods to determine its effectiveness. A comparison of numerical results in Tables 2–4 and the plots of absolute errors in Figures 2–5 show that the proposed ETHBM method outperforms several other numerical methods. These results highlight the effectiveness of ETHBM in accurately approximating solutions to NFNE presented in Equations (1) and (2).

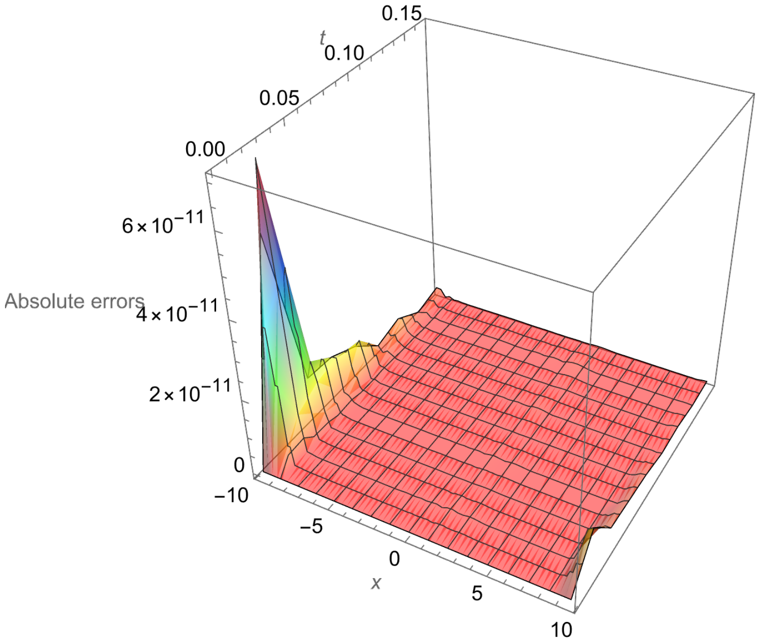

Figure 5. Plot of absolute errors with $N = 100$ on $(x, t) \in [-10, 10] \times [0, 0.2]$ for Example 3.

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