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Dynamics of a Higher-Order Three-Dimensional Nonlinear System of Difference Equations

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Abstract: In this paper, we study the semi-cycle analysis of positive solutions and the asymptotic behavior of positive solutions of three-dimensional system of difference equations with a higher order under certain parametric conditions. Furthermore, we show the boundedness and persistence, the rate of convergence of the solutions and the global asymptotic stability of the unique equilibrium point of the proposed system under certain parametric conditions. Finally, for this system, we offer some numerical examples which support our analytical results.

Keywords: system of rational difference equations of order $k+1$; semi-cycle analysis; boundedness and persistence; global asymptotic stability; rate of convergence; sequence analysis

MSC: 39A10; 39A22; 39A30

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1. Introduction

We let $\mathbb{N}$ be a set of all natural numbers, $\mathbb{N}_0$ be a set of non-negative integers, $\mathbb{Z}$ be a set of all integers, $\mathbb{R}$ be a set of all real numbers and for $k \in \mathbb{Z}$ the notation $\mathbb{N}_k$ represent the set of $\{n \in \mathbb{Z} : n \geq k\}$.

Nonlinear difference equations and system of difference equations can be used in modeling problems which arise in physics, finance, engineering, biology, and many other areas (see [1,2]). In fact, by using different discretization methods for continuous problems, we obtain models of difference equations and systems; see, for instance, [3,4]. There has been a lot of studies concerning the periodicity, oscillation behavior, the boundedness, stability of nonlinear difference equations and system of difference equations, see for example [5]. Devault et al., in [6], studied the boundedness, global stability and periodic character of solutions of the following difference equation:

$$x_{n+1} = p + \frac{x_{n-m}}{x_n},\ n \in \mathbb{N}_0,\ m \in \mathbb{N}_2, \tag{1}$$

where $p \in (0, \infty)$, x_{-i}, $i \in \{0, 1, \dots, m\}$ are positive real numbers. Then, in [7], Equation (1) was extended to the following two-dimensional system of difference equations:

$$x_{n+1} = A + \frac{y_{n-m}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{x_n}, \quad n \in \mathbb{N}_0,\ m \in \mathbb{Z}^+, \tag{2}$$

where $A \in (0, \infty)$, x_{-i} and y_{-i}, $i \in \{0, 1, \ldots, m\}$ are positive real numbers. The authors proved that every positive solutions of System (2) is bounded and persist and the unique positive equilibrium point is a global attractor for case $A > 1$. Also, they showed that System (2) has unbounded solutions for the case $0 < A < 1$, and m is odd. Finally, they determined that every positive solution of System (2) is periodic of a prime period two for case $A = 1$ and m is odd, and has no two prime periodic solutions for case $A = 1$ and m is even. Later, Gümüş, in [8], studied the global asymptotic stability of the unique positive equilibrium point under certain parametric conditions and the rate of convergence of positive solutions of System (2). Moreover, the author examined the behavior of positive solutions of System (2) using the semi-cycle analysis method. Finally, Abualrub and Aloqeili, in [9], considered the following difference equations system:

$$x_{n+1} = A + \frac{y_{n-m}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-m}}{x_n}, \quad n \in \mathbb{N}_0,\ m \in \mathbb{Z}^+, \tag{3}$$

where parameters $A, B \in (0, \infty)$, the initial values x_{-i}, y_{-i}, $i \in \{0, 1, \ldots, m\}$ are positive real numbers, which is a natural extension of System (2). They investigated the behavior of positive solutions of System (3) employing the semi-cycle analysis method. Also, they studied the asymptotic behavior of the solutions of System (3) for cases $0 < A < 1$ and $0 < B < 1$. Finally, they showed that the positive solutions of System (3) are boundedness and persistence, and that the unique positive equilibrium point of System (2) is globally asymptotically stable. Other related difference equations and systems of difference equations can be found in references [10–22].

Motivated by aforementioned studies, in this study, we consider the following three-dimensional higher-order system of difference equations:

$$u_{n+1} = X + \frac{v_{n-k}}{v_n}, \quad v_{n+1} = Y + \frac{w_{n-k}}{w_n}, \quad w_{n+1} = Z + \frac{u_{n-k}}{u_n},\ n \in \mathbb{N}_0,\ k \in \mathbb{Z}^+, \tag{4}$$

where parameters $X > 0$, $Y > 0$ and $Z > 0$, and initial conditions u_{-i}, v_{-i}, w_{-i}, $i \in \{0, 1, \ldots, k\}$ are arbitrary positive numbers. It is easy to see that System (4) has a unique positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X + 1, Y + 1, Z + 1)$. By taking $X = Y = Z$, System (4) is reduced to the following system:

$$u_{n+1} = X + \frac{v_{n-k}}{v_n}, \quad v_{n+1} = X + \frac{w_{n-k}}{w_n}, \quad w_{n+1} = X + \frac{u_{n-k}}{u_n},\ n \in \mathbb{N}_0,\ k \in \mathbb{Z}^+, \tag{5}$$

where parameter $X > 0$ and initial conditions u_{-i}, v_{-i}, w_{-i}, $i \in \{0, 1, \ldots, k\}$ are arbitrary positive numbers, which was considered in [16] with $p = 3$. So, we suppose that $X \neq Y \neq Z$. From now on, we investigate the dynamical behavior of System (4) for cases $0 < X, Y, Z < 1$ and $X > 1$, $Y > 1$, $Z > 1$. We also deal with the behavior of the positive solutions of System (4) using the semi-cycle analysis method. Finally, we offer numerical examples representing different types of behavior of solutions to System (4).

Our work completes and generalizes the works mentioned in the literature summary above, and this is our main motivation for the present study.

Our paper is organized as follows: The behavior of positive solutions of System (4) using semi-cycle analysis is studied in Section 2. The asymptotic behavior of positive solutions of System (4) when $0 < X < 1$, $0 < Y < 1$ and $0 < Z < 1$ is investigated in Section 3. In Section 4, we study the boundedness and persistence of System (4) and the global behavior of the unique equilibrium point of System (4), whereas the rate of convergence of System (4) is studied in Section 5. Some numerical examples which support our analytical results are given in Seciton 6.

2. Semi-Cycle Analysis

In this section, we discuss the behavior of positive solutions of System (4) using semi-cycle analysis. It is easy to see that System (4) has a unique positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X + 1, Y + 1, Z + 1)$.

Theorem 1. *Suppose that $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ is solutions to System (4). Then, either this solution consists of a single semi-cycle or it oscillates about the equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ with semi-cycle having at most k terms.*

Proof. We assume that $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ possess at least two semi-cycles. That is, there exists $n_0 > -k$ such that either

$$\begin{cases} u_{n_0} < 1 + X \leq u_{n_0+1}, \\ v_{n_0} < 1 + Y \leq v_{n_0+1}, \\ w_{n_0} < 1 + Z \leq w_{n_0+1}, \end{cases} \tag{6}$$

or

$$\begin{cases} u_{n_0} \geq 1 + X > u_{n_0+1}, \\ v_{n_0} \geq 1 + Y > v_{n_0+1}, \\ w_{n_0} \geq 1 + Z > w_{n_0+1}. \end{cases} \tag{7}$$

Here, we consider only the first case; the other can be investigated in a similar way. We suppose that the first semi-cycle starting with term $(u_{n_0+1}, v_{n_0+1}, w_{n_0+1})$ has k terms. In this case, we have

$$\begin{cases} u_{n_0} < 1 + X \leq u_{n_0+1}, u_{n_0+2}, \ldots, u_{n_0+k}, \\ v_{n_0} < 1 + Y \leq v_{n_0+1}, v_{n_0+2}, \ldots, v_{n_0+k}, \\ w_{n_0} < 1 + Z \leq w_{n_0+1}, w_{n_0+2}, \ldots, w_{n_0+k}, \end{cases}$$

which implies that $\frac{u_{n_0}}{u_{n_0+k}} < 1$, $\frac{v_{n_0}}{v_{n_0+k}} < 1$ and $\frac{w_{n_0}}{w_{n_0+k}} < 1$; then, we obtain, from System (4),

$$\begin{cases} u_{n_0+k+1} = X + \frac{v_{n_0}}{v_{n_0+k}} < X + 1, \\ v_{n_0+k+1} = Y + \frac{w_{n_0}}{w_{n_0+k}} < Y + 1, \\ w_{n_0+k+1} = Z + \frac{u_{n_0}}{u_{n_0+k}} < Z + 1, \end{cases}$$

from which the result follows. □

Theorem 2. *Suppose that k is an odd integer and $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ is a solution of System (4) which possesses $k-1$ sequential semi-cycle of length one. Then, every semi-cycle after this point is of length one.*

Proof. We let k be the odd integer and $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ be Solution (4) which possesses $k-1$ sequential semi-cycle of length one. Then, from the definition of a semi-cycle, there exists $n_0 \geq -k$ such that either

$$\begin{cases} u_{n_0}, u_{n_0+2}, \ldots, u_{n_0+k-1} < 1 + X \leq u_{n_0+1}, u_{n_0+3}, \ldots, u_{n_0+k}, \\ v_{n_0}, v_{n_0+2}, \ldots, v_{n_0+k-1} < 1 + Y \leq v_{n_0+1}, v_{n_0+3}, \ldots, v_{n_0+k}, \\ w_{n_0}, w_{n_0+2}, \ldots, w_{n_0+k-1} < 1 + Z \leq w_{n_0+1}, w_{n_0+3}, \ldots, w_{n_0+k} \end{cases}$$

or

$$\begin{cases} u_{n_0}, u_{n_0+2}, \ldots, u_{n_0+k-1} \geq 1 + X > u_{n_0+1}, u_{n_0+3}, \ldots, u_{n_0+k}, \\ v_{n_0}, v_{n_0+2}, \ldots, v_{n_0+k-1} \geq 1 + Y > v_{n_0+1}, v_{n_0+3}, \ldots, v_{n_0+k}, \\ w_{n_0}, w_{n_0+2}, \ldots, w_{n_0+k-1} \geq 1 + Z > w_{n_0+1}, w_{n_0+3}, \ldots, w_{n_0+k}. \end{cases}$$

Here, we consider just the first case since the other can be dealt with similarly. In this case, we can write the following inequalities:

$$\begin{cases} u_{n_0+k+1} = X + \frac{u_{n_0}}{u_{n_0+k}} < X+1, \\ v_{n_0+k+1} = Y + \frac{v_{n_0}}{v_{n_0+k}} < Y+1, \\ w_{n_0+k+1} = Z + \frac{w_{n_0}}{w_{n_0+k}} < Z+1, \end{cases}$$

which means that $(u_{n_0+k}, v_{n_0+k}, w_{n_0+k})$ is the k^{th} semi-cycle of length one. By using the induction method, we can easily show that every semi-cycle after this point is of length one. Hence, the proof is complete. □

Theorem 3. *System* (4) *has no nontrivial k-periodic solution (not necessarily prime period k).*

Proof. We suppose that System (4) possesses k-periodic solution. Then, from System (4), we have $(u_{n-k}, v_{n-k}, w_{n-k}) = (u_n, v_n, w_n)$ for all $n \geq 0$ and so

$$\begin{cases} u_{n+1} = X + \frac{v_{n-k}}{v_n} = X+1, \\ v_{n+1} = Y + \frac{w_{n-k}}{w_n} = Y+1, \\ w_{n+1} = Z + \frac{u_{n-k}}{u_n} = Z+1. \end{cases}$$

Hence, solution $(u_n, v_n, w_n) = (X+1, Y+1, Z+1)$ is the equilibrium solution of System (4). □

Theorem 4. *All non-oscillatory solutions of System* (4) *converge to the equilibrium* $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ *as* $n \to \infty$.

Proof. We suppose that System (4) possesses a non-oscillatory solution as $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$. Here, we just prove the case of a single positive semi-cycle since the other can be achieved in a similar way. Then, from assumption, for all $n \geq -k$, we have $(u_n, v_n, w_n) \geq (X+1, Y+1, Z+1)$ and so

$$\begin{cases} u_{n+1} = X + \frac{v_{n-k}}{v_n} \geq X+1, \\ v_{n+1} = Y + \frac{w_{n-k}}{w_n} \geq Y+1, \\ w_{n+1} = Z + \frac{u_{n-k}}{u_n} \geq Z+1, \end{cases}$$

which implies that $v_{n-k} \geq v_n$, $w_{n-k} \geq w_n$, and $u_{n-k} \geq u_n$. So, we obtain

$$\begin{cases} u_{n-k} \geq u_n \geq u_{n+k} \geq \cdots \geq X+1, \quad n \in \mathbb{N}_0, \\ v_{n-k} \geq v_n \geq v_{n+k} \geq \cdots \geq Y+1, \quad n \in \mathbb{N}_0, \\ w_{n-k} \geq w_n \geq w_{n+k} \geq \cdots \geq Z+1, \quad n \in \mathbb{N}_0, \end{cases}$$

which means that $\{u_n\}, \{v_n\}, \{w_n\}$ have k convergent subsequences

$$\{u_{nk}\}, \{u_{nk+1}\}, \ldots, \{u_{nk+(k-1)}\},$$
$$\{v_{nk}\}, \{v_{nk+1}\}, \ldots, \{v_{nk+(k-1)}\}$$

and

$$\{w_{nk}\}, \{w_{nk+1}\}, \ldots, \{w_{nk+(k-1)}\},$$

since they are decreasing and bounded from below. So, each subsequence is convergent. Hence, for all $j \in \{0, 1, \ldots, k-1\}$ there exist a_i, b_i, c_i such that

$$\lim_{n\to\infty} u_{nk+i} = a_i, \quad \lim_{n\to\infty} v_{nk+i} = b_i, \quad \lim_{n\to\infty} w_{nk+i} = c_i.$$

Thus,

$$(a_0, b_0, c_0), (a_1, b_1, c_1), \ldots, (a_{k-1}, b_{k-1}, c_{k-1})$$

is a k-periodic solution of System (4), which contradicts Theorem 3, because System (4) has no non-trivial periodic solutions of (not necessarily prime) period k. Therefore, the solution tends to the equilibrium. □

3. The Case $0 < X < 1, 0 < Y < 1$ and $0 < Z < 1$

In this section, we discuss the asymptotic behavior of the positive solutions of System (4) when $0 < X < 1, \quad 0 < Y < 1$ and $0 < Z < 1$.

Theorem 5. *Consider system (4). Let $0 < X < 1$, $0 < Y < 1$, $0 < Z < 1$ and $T = \max\{X, Y, Z\}$. Assume that $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ is a positive solution of system (4). Then, the next statements are true.*

(a) If k is odd and $0 < u_{2m-1} < 1, 0 < v_{2m-1} < 1, 0 < w_{2m-1} < 1, u_{2m} > \frac{1}{1-T}, v_{2m} > \frac{1}{1-T}, w_{2m} > \frac{1}{1-T}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then

$$\lim_{n\to\infty} u_{2n} = \infty, \quad \lim_{n\to\infty} v_{2n} = \infty, \quad \lim_{n\to\infty} w_{2n} = \infty,$$
$$\lim_{n\to\infty} u_{2n+1} = X, \quad \lim_{n\to\infty} v_{2n+1} = Y, \quad \lim_{n\to\infty} w_{2n+1} = Z.$$

(b) If k is odd and $0 < u_{2m} < 1, 0 < v_{2m} < 1, 0 < w_{2m} < 1, u_{2m-1} > \frac{1}{1-T}, v_{2m-1} > \frac{1}{1-T}, w_{2m-1} > \frac{1}{1-T}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then

$$\lim_{n\to\infty} u_{2n+1} = \infty, \quad \lim_{n\to\infty} v_{2n+1} = \infty, \quad \lim_{n\to\infty} w_{2n+1} = \infty,$$
$$\lim_{n\to\infty} u_{2n} = X, \quad \lim_{n\to\infty} v_{2n} = Y, \quad \lim_{n\to\infty} w_{2n} = Z.$$

Proof. We consider only Case (a) since the other case can be proven similarly.

(a) Since $T = \max\{X, Y, Z\}$, we obtain

$$\begin{cases} 0 < u_1 = X + \frac{v_{-k}}{v_0} < X + \frac{1}{v_0} < X + 1 - T \le X + 1 - X = 1, \\ 0 < v_1 = Y + \frac{w_{-k}}{w_0} < Y + \frac{1}{w_0} < Y + 1 - T \le Y + 1 - Y = 1, \\ 0 < w_1 = Z + \frac{u_{-k}}{u_0} < Z + \frac{1}{u_0} < Z + 1 - T \le Z + 1 - Z = 1 \end{cases}$$

and

$$\begin{cases} u_2 = X + \frac{v_{-k+1}}{v_1} > X + v_{-k+1} > v_{-k+1} > \frac{1}{1-T}, \\ v_2 = Y + \frac{w_{-k+1}}{w_1} > Y + w_{-k+1} > w_{-k+1} > \frac{1}{1-T}, \\ w_2 = Z + \frac{u_{-k+1}}{u_1} > Z + u_{-k+1} > u_{-k+1} > \frac{1}{1-T}. \end{cases}$$

By using induction method, we obtain

$$0 < u_{2n-1}, v_{2n-1}, w_{2n-1} < 1, \quad u_{2n}, v_{2n}, w_{2n} > \frac{1}{1-T}, \quad n \in \mathbb{N},$$

and so for $l > \frac{k+3}{2}$

$$\begin{cases} u_{2l} & = X + \frac{v_{2l-(k+1)}}{v_{2l-1}} > X + v_{2l-(k+1)} = X + Y + \frac{w_{2l-(2k+2)}}{w_{2l-k-2}} \\ & > X + Y + w_{2l-(2k+2)} = X + Y + Z + \frac{u_{2l-(3k+3)}}{u_{2l-2k-3}} > X + Y + Z + u_{2l-(3k+3)}, \\ u_{4l} & = X + \frac{v_{4l-(k+1)}}{v_{4l-1}} > X + v_{4l-(k+1)} = X + Y + \frac{w_{4l-(2k+2)}}{w_{4l-k-2}} \\ & > X + Y + w_{4l-(2k+2)} = X + Y + Z + \frac{u_{4l-(3k+3)}}{u_{4l-2k-3}} > X + Y + Z + u_{4l-(3k+3)}. \end{cases}$$

Similarly, one can easily see that inequality $u_{6l} > X + Y + Z + u_{6l-(3k+3)}$ holds. Thus, for all $p \in \mathbb{N}$, $u_{2pl} > X + Y + Z + u_{2pl-3(k+1)}$. If $n = pl$, then $\lim_{n\to\infty} u_{2n} = \infty$, as $p \to \infty$, $n \to \infty$. Similarly, we can obtain $\lim_{n\to\infty} v_{2n} = \infty$, $\lim_{n\to\infty} w_{2n} = \infty$. On the other hand, using the conditions of Case (a) and taking the limit on both sides of each equations of (4) in the following system,

$$u_{2n+1} = X + \frac{v_{2n-k}}{v_{2n}}, \quad v_{2n+1} = Y + \frac{w_{2n-k}}{w_{2n}}, \quad w_{2n+1} = Z + \frac{u_{2n-k}}{u_{2n}},$$

we have

$$\lim_{n\to\infty} u_{2n+1} = X, \quad \lim_{n\to\infty} v_{2n+1} = Y, \quad \lim_{n\to\infty} w_{2n+1} = Z,$$

which is desired. □

4. The Case $X > 1$, $Y > 1$ and $Z > 1$

In this section, we deal with the boundedness and persistence of positive solutions of System (4) when $X > 1$, $Y > 1$ and $Z > 1$. Also, we show that the unique positive equilibrium of System (4) is globally asymptotically stable when $X > 1$, $Y > 1$ and $Z > 1$.

Theorem 6. *Consider System (4). Let $X > 1$, $Y > 1$ and $Z > 1$. Then, for every positive solution of System (4) we have the following inequalities, for $i = \overline{k+3, 3k+3}$ and $l \geq 0$:*

$$\begin{cases} X < u_{(3k+3)l+j} \leq \left(u_j + \frac{XYZ(X+1)+XZ}{1-XYZ}\right)\left(\frac{1}{XYZ}\right)^l + \frac{XYZ(X+1)+XZ}{XYZ-1}, \\ Y < v_{(3k+3)l+j} \leq \left(v_j + \frac{XYZ(Y+1)+XY}{1-XYZ}\right)\left(\frac{1}{XYZ}\right)^l + \frac{XYZ(Y+1)+XY}{XYZ-1}, \\ Z < w_{(3k+3)l+j} \leq \left(w_j + \frac{XYZ(Z+1)+YZ}{1-XYZ}\right)\left(\frac{1}{XYZ}\right)^l + \frac{XYZ(Z+1)+YZ}{XYZ-1}, \end{cases}$$

where $l \geq 0$, $j = \overline{2, 3k+4}$.

Proof. We let $\{(u_n, v_n, w_n)\}$ be a positive solution of System (4) and be $X > 1$, $Y > 1$ and $Z > 1$. Since $u_n > 0$, $v_n > 0$ and $w_n > 0$ for all $n \geq -k$, from System (4), we have

$$u_n > X > 1, \quad v_n > Y > 1, \quad w_n > Z > 1, \quad n \geq 1. \tag{8}$$

Further employing (4) and (8), we obtain

$$\begin{cases} u_n = X + \frac{v_{n-k-1}}{v_{n-1}} < X + \frac{1}{Y} u_{n-k-1}, \\ v_n = Y + \frac{w_{n-k-1}}{w_{n-1}} < Y + \frac{1}{Z} v_{n-k-1}, \\ w_n = Z + \frac{u_{n-k-1}}{u_{n-1}} < Z + \frac{1}{X} w_{n-k-1} \end{cases} \tag{9}$$

for all $n \geq 2$. We let $\{(x_n, y_n, z_n)\}$ be the solution of system

$$x_n = X + \frac{1}{Y} y_{n-k-1}, \quad y_n = Y + \frac{1}{Z} z_{n-k-1}, \quad z_n = Z + \frac{1}{X} x_{n-k-1}, \quad n \geq 2, \tag{10}$$

such that

$$x_i = u_i, \quad y_i = v_i, \quad z_i = w_i, \quad i = \overline{-k+1, 1}. \tag{11}$$

Now, we use the induction method to prove

$$u_n < x_n, \quad v_n < y_n, \quad w_n < z_n, \quad n \geq 2. \tag{12}$$

We assume that (12) is true for $n = m \geq 2$. From (9), we have

$$\begin{cases} u_{m+1} < X + \frac{1}{Y}v_{m-k} < X + \frac{1}{Y}y_{m-k} = x_{m+1}, \\ v_{m+1} < Y + \frac{1}{Z}w_{m-k} < Y + \frac{1}{Z}z_{m-k} = y_{m+1}, \\ w_{m+1} < Z + \frac{1}{X}u_{m-k} < Z + \frac{1}{X}x_{m-k} = z_{m+1}. \end{cases} \tag{13}$$

Thus, (12) is true. From (10) and (11), we obtain

$$x_{(3k+3)(l+1)+j} = ax_{(3k+3)l+j} + b, \quad y_{(3k+3)(l+1)+j} = ay_{(3k+3)l+j} + c, \quad z_{(3k+3)(l+1)+j} = az_{(3k+3)l+j} + d, \tag{14}$$

where $l \geq 0$, $j = \overline{2, 3k+4}$, $a = \frac{1}{XYZ}$, $b = X + 1 + \frac{1}{Y}$, $c = Y + 1 + \frac{1}{Z}$ and $d = Z + 1 + \frac{1}{X}$. The general solution to equations in (14) are

$$\begin{aligned} x_{(3k+3)l+j} &= (u_j + \frac{b}{a-1})a^l + \frac{b}{1-a} \\ &= (u_j + \frac{XYZ(X+1) + XZ}{1 - XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(X+1) + XZ}{XYZ - 1}, \end{aligned} \tag{15}$$

$$\begin{aligned} y_{(3k+3)l+j} &= (v_j + \frac{c}{a-1})a^l + \frac{c}{1-a} \\ &= (v_j + \frac{XYZ(Y+1) + XY}{1 - XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(Y+1) + XY}{XYZ - 1}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} z_{(3k+3)l+j} &= (w_j + \frac{d}{a-1})a^l + \frac{d}{1-a} \\ &= (w_j + \frac{XYZ(Z+1) + YZ}{1 - XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(Z+1) + YZ}{XYZ - 1}, \end{aligned} \tag{17}$$

where $l \geq 0$, $j = \overline{2, 3k+4}$. Then, from (8) and (12) and the solutions in (15)–(17), it follows that for all $j = \overline{2, 3k+4}$ and $l \geq 0$

$$\begin{cases} X < u_{(3k+3)l+j} \leq (u_j + \frac{XYZ(X+1)+XZ}{1-XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(X+1)+XZ}{XYZ-1}, \\ Y < v_{(3k+3)l+j} \leq (v_j + \frac{XYZ(Y+1)+XY}{1-XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(Y+1)+XY}{XYZ-1}, \\ Z < w_{(3k+3)l+j} \leq (w_j + \frac{XYZ(Z+1)+YZ}{1-XYZ})(\frac{1}{XYZ})^l + \frac{XYZ(Z+1)+YZ}{XYZ-1}, \end{cases} \tag{18}$$

where $l \geq 0$ and $j = \overline{2, 3k+4}$, which is desired. □

In the following theorem, we show that unique equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ is a global attractor.

Theorem 7. *Let $X > 1$, $Y > 1$ and $Z > 1$. Then, every positive solution of System (4) converges to the equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ as $n \to \infty$.*

Proof. We let $\{(u_n, v_n, w_n)\}$ be a positive solution of System (4) and be $X > 1$, $Y > 1$ and $Z > 1$. We assume that

$$\begin{aligned} u_1 &= \lim_{n\to\infty} sup x_n, \quad u_2 = \lim_{n\to\infty} sup y_n, \quad u_3 = \lim_{n\to\infty} sup z_n, \\ l_1 &= \lim_{n\to\infty} inf x_n, \quad l_2 = \lim_{n\to\infty} inf y_n, \quad l_3 = \lim_{n\to\infty} inf z_n. \end{aligned} \tag{19}$$

From Theorem 6, we can write the next inequalities:

$$0 < X \leq l_1 \leq u_1 < +\infty, \quad 0 < Y \leq l_2 \leq u_2 < +\infty, \quad 0 < Z \leq l_3 \leq u_3 < +\infty. \tag{20}$$

Also, from System (4) and (19), we obtain

$$u_1 \le X + \frac{u_2}{l_2}, \quad u_2 \le Y + \frac{u_3}{l_3}, \quad u_3 \le Z + \frac{u_1}{l_1},$$
$$l_1 \ge X + \frac{l_2}{u_2}, \quad l_2 \ge Y + \frac{l_3}{u_3}, \quad l_3 \ge Z + \frac{l_1}{u_1}, \tag{21}$$

from which it follows that

$$l_1 u_2 \ge X u_2 + l_2, \tag{22}$$
$$u_1 l_2 \le X l_2 + u_2, \tag{23}$$
$$l_2 u_3 \ge Y u_3 + l_3, \tag{24}$$
$$u_2 l_3 \le Y l_3 + u_3, \tag{25}$$
$$l_3 u_1 \ge Z u_1 + l_1, \tag{26}$$

and

$$u_3 l_1 \le Z l_1 + u_1. \tag{27}$$

By multiplying both sides of inequality in (22) by u_3 and both sides of inequality in (27) by u_2, we obtain

$$u_3 l_1 u_2 \ge X u_2 u_3 + l_2 u_3, \quad u_2 u_3 l_1 \le Z u_2 l_1 + u_2 u_1,$$

from which it follows that

$$X u_2 u_3 + l_2 u_3 \le Z u_2 l_1 + u_2 u_1. \tag{28}$$

Similarly, multiplying both sides of inequality in (23) by u_3 and both sides of inequality in (24) by u_1, we obtain

$$u_3 u_1 l_2 \le u_3 X l_2 + u_3 u_2, \quad u_1 l_2 u_3 \ge Y u_3 u_1 + l_3 u_1,$$

from which it follows that

$$Y u_3 u_1 + l_3 u_1 \le u_3 X l_2 + u_3 u_2. \tag{29}$$

Again, similarly, multiplying both sides of inequality in (25) by u_1 and both sides of inequality in (26) by u_2, we have

$$u_1 u_2 l_3 \le u_1 Y l_3 + u_1 u_3, \quad u_2 l_3 u_1 \ge u_2 Z u_1 + u_2 l_1,$$

from which it follows that

$$u_2 u_1 Z + u_2 l_1 \le Y u_1 l_3 + u_1 u_3. \tag{30}$$

From (28)–(30), we have

$$X u_2 u_3 + l_2 u_3 + Y u_1 u_3 + l_3 u_1 + u_2 u_1 Z + u_2 l_1 \le u_2 Z l_1 + u_2 u_1 + u_3 X l_2 + u_3 u_2 + u_1 Y l_3 + u_1 u_3, \tag{31}$$

which implies that

$$X u_2 u_3 + l_2 u_3 + Y u_1 u_3 + l_3 u_1 + u_2 u_1 Z + u_2 l_1 - u_2 Z l_1 - u_2 u_1 - u_3 X l_2 - u_3 u_2 - u_1 Y l_3 - u_1 u_3 \le 0 \tag{32}$$

and consequently

$$X u_3 (u_2 - l_2) + Y u_1 (u_3 - l_3) + Z u_2 (u_1 - l_1) - u_3 (u_2 - l_2) - u_1 (u_3 - l_3) - u_2 (u_1 - l_1) \le 0. \tag{33}$$

Since $X - 1 > 0$, $Y - 1 > 0$ and $Z - 1 > 0$, from (33) we have $u_1 = l_1, u_2 = l_2$ and $u_3 = l_3$, from which along with (22)–(27) it follows that

$$\lim_{n \to \infty} u_n = l_1 = u_1 = X + 1, \quad \lim_{n \to \infty} v_n = l_2 = u_2 = Y + 1, \quad \lim_{n \to \infty} w_n = l_3 = u_3 = Z + 1, \tag{34}$$

which completes the proof. □

Lemma 1 ([9]). *Assume that $A > 1$ and $0 < \epsilon < \frac{A-1}{(A+1)(k+1)}$, for $k \in \mathbb{N}$. Then $\frac{2}{(1-(k+1))\epsilon(A+1)} < 1$.*

The following theorem offers us the local stability of the unique equilibrium point of System (4) when $X > 1$, $Y > 1$ and $Z > 1$.

Theorem 8. *If $X > 1$, $Y > 1$ and $Z > 1$, then the unique positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ of System (4) is locally asymptotically stable.*

Proof. The linearized equations of System (4) about the equilibrium point $(X+1, Y+1, Z+1)$ are

$$X_{n+1} = F(X_n),\ n \in \mathbb{N}_0, \tag{35}$$

where $X_n = \left(u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(k+1)}, v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(k+1)}, w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(k+1)}\right)^T$, where

$$\begin{cases} u_n^{(1)} = u_n, u_n^{(2)} = u_{n-1}, \ldots, u_n^{(k+1)} = u_{n-k}, \\ v_n^{(1)} = v_n, v_n^{(2)} = v_{n-1}, \ldots, v_n^{(k+1)} = v_{n-k}, \\ w_n^{(1)} = w_n, w_n^{(2)} = w_{n-1}, \ldots, w_n^{(k+1)} = w_{n-k}, \end{cases} \tag{36}$$

and $F : [0,\infty)^{3k+3} \to [0,\infty)^{3k+3}$ such that for all $T = \left(u^{(1)}, u^{(2)}, \ldots, u^{(k+1)}, v^{(1)}, v^{(2)}, \ldots, v^{(k+1)}, w^{(1)}, w^{(2)}, \ldots, w^{(k+1)}\right) \in [0,\infty)^{3k+3}$, $F(T) = \left(f_1(T), u^{(2)}, \ldots, u^{(k+1)}, f_2(T), v^{(2)}, \ldots, v^{(k+1)}, f_3(T), w^{(2)}, \ldots, w^{(k+1)}\right)$, where

$$f_1(T) = X + \frac{v^{(k+1)}}{v^{(1)}},\ f_2(T) = Y + \frac{w^{(k+1)}}{w^{(1)}},\ f_3(T) = Z + \frac{u^{(k+1)}}{u^{(1)}}.$$

Then, we obtain

$$\begin{cases} \frac{\partial f_1}{\partial v^{(1)}}(T) = -\frac{v^{(k+1)}}{\left(v^{(1)}\right)^2}, & \frac{\partial f_1}{\partial v^{(k+1)}}(T) = \frac{1}{v^{(1)}}, \\ \frac{\partial f_2}{\partial w^{(1)}}(T) = -\frac{w^{(k+1)}}{\left(w^{(1)}\right)^2}, & \frac{\partial f_2}{\partial w^{(k+1)}}(T) = \frac{1}{w^{(1)}}, \\ \frac{\partial f_3}{\partial u^{(1)}}(T) = -\frac{u^{(k+1)}}{\left(u^{(1)}\right)^2}, & \frac{\partial f_3}{\partial u^{(k+1)}}(T) = \frac{1}{u^{(1)}}. \end{cases} \tag{37}$$

$\mathcal{J}_{\mathcal{F}}$ is the Jacobian matrix of F at the equilibrium point $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$, which is given by

$$\mathcal{J}_{\mathcal{F}} = \begin{pmatrix} 0 & 0\cdots0 & 0 & \frac{-1}{Y+1} & 0\cdots0 & \frac{1}{Y+1} & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{-1}{Z+1} & 0\cdots0 & \frac{1}{Z+1} \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \frac{-1}{X+1} & 0\cdots0 & \frac{1}{X+1} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix}. \tag{38}$$

We suppose that $\lambda_1, \lambda_2, \ldots, \lambda_{3k+3}$ are the eigenvalues of matrix $\mathcal{J}_{\mathcal{F}}$ and that $\mathcal{D} = diag(d_1, d_2, \ldots, d_{3k+3})$ is a diagonal matrix, where

$$d_1 = d_{k+2} = d_{2k+3} = 1, \quad d_m = d_{k+1+m} = d_{2k+2+m} = 1 - m\epsilon,$$

for $m \in \{2, 3, \ldots, k+1\}$ and $0 < \epsilon < min\{\frac{X-1}{(X+1)(k+1)}, \frac{Y-1}{(Y+1)(k+1)}, \frac{Z-1}{(Z+1)(k+1)}\}$. From this and taking into account the fact that $1 - m\epsilon > 0$, for all $m \in \{2, 3, \ldots, k+1\}$, we conclude that matrix $\mathcal{D}$ is invertible. Matrix $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$ is given by

$$\begin{pmatrix}
0 & 0\cdots0 & 0 & \frac{-1}{Y+1}\frac{d_1}{d_{k+2}} & 0\cdots0 & \frac{1}{Y+1}\frac{d_1}{d_{2k+2}} & 0 & 0\cdots0 & 0 \\
\frac{d_2}{d_1} & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots\frac{d_{k+1}}{d_k} & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{-1}{Z+1}\frac{d_{k+2}}{d_{2k+3}} & 0\cdots0 & \frac{1}{Z+1}\frac{d_{k+2}}{d_{3k+3}} \\
0 & 0\cdots0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots0 & 0 & 0 & 0\cdots\frac{d_{2k+2}}{d_{2k+1}} & 0 & 0 & 0\cdots0 & 0 \\
\frac{-1}{X+1}\frac{d_{2k+3}}{d_1} & 0\cdots0 & \frac{1}{X+1}\frac{d_{2k+3}}{d_{k+1}} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{d_{2k+4}}{d_{2k+3}} & 0\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots\frac{d_{3k+3}}{d_{3k+2}} & 0
\end{pmatrix}.$$

In order to find the infinite norm of $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$, we show that the sum of absolute values of the entries of each row is less than one. For this, by considering the fact that $d_1 > d_2 > \cdots > d_{k+1}$, $d_{k+2} > d_{k+3} > \cdots > d_{2k+2}$ and $d_{2k+3} > d_{2k+4} > \cdots > d_{3k+3}$, we can write the following inequalities for every $j \in \{1, 2, \ldots, 3k+2\}$:

$$\frac{d_{j+1}}{d_j} < 1, \tag{39}$$

from which, along with $A > 1$ and Lemma 1, it yields

$$\begin{aligned}
\frac{1}{Y+1}\frac{d_1}{d_{k+2}} + \frac{1}{Y+1}\frac{d_1}{d_{2k+2}} &= \frac{1}{Y+1} + \frac{1}{(1-(k+1)\epsilon)(Y+1)} \\
&< \frac{2}{(1-(k+1)\epsilon)(Y+1)} \\
&< 1.
\end{aligned}$$

Similarly,

$$\frac{1}{Z+1}\frac{d_2}{d_{2k+3}} + \frac{1}{Z+1}\frac{d_2}{d_{3k+3}} < 1$$

and

$$\frac{1}{X+1}\frac{d_{2k+3}}{d_1} + \frac{1}{X+1}\frac{d_{2k+3}}{d_{k+1}} < 1.$$

Since $\mathcal{J}_{\mathcal{F}}$ has the same eigenvalue as $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$,

$$\rho(\mathcal{J}_{\mathcal{F}}) = max\{|\lambda_i|\} \leq ||\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}||_{\infty},$$

but

$$
\begin{aligned}
||\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}||_{\infty} = \max\Big\{ & \frac{1}{Y+1} + \frac{1}{(1-(k+1)\epsilon(Y+1)}, \frac{d_2}{d_1}, \frac{d_3}{d_2}, \ldots, \frac{d_{k+1}}{d_k}, \frac{1}{Z+1} + \frac{1}{(1-(k+1)\epsilon(Z+1)}, \\
& \frac{1}{X+1} + \frac{1}{(1-(k+1)\epsilon(X+1)} \Big\} \\
< 1. &
\end{aligned} \tag{40}
$$

Since he modulus of every eigenvalue of $\mathcal{J}_{\mathcal{F}}$ is less than one, the unique equilibrium point $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ of System (4) is locally asymptotically stable. □

Theorem 9. *If $X > 1$, $Y > 1$ and $Z > 1$, then the unique positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ of System (4) is globally asymptotically stable.*

Proof. The result follows immediately from Theorems 7 and 8. □

5. Rate of Convergence

In this section, we study the rate of convergence of a solutions which converges to the equilibrium point $(\bar{u}, \bar{v}, \bar{w}) = (X+1, Y+1, Z+1)$ of System (4) in the region of parameters described by $X > 1, Y > 1$ and $Z > 1$. The following result presents the rate of convergence of solutions of the system of difference equations

$$
\Psi_{n+1} = [M + N(n)]\Psi_n, \tag{41}
$$

where Ψ_n is a $(3k+3)$-dimensional vector, $M \in C^{(3k+3)\times(3k+3)}$ is a constant matrix and $N : \mathbb{Z}^+ \to C^{(3k+3)\times(3k+3)}$ is a matrix function with

$$
\|N(n)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{42}
$$

where $\|.\|$ denotes any matrix norm.

Theorem 10. *(Perron's Theorem, see [18]) Assume that Condition (42) holds. If Ψ_n is a solution of System (41), then either $\Psi_n = 0$ for all large n or*

$$
\vartheta = \lim_{n\to\infty} \frac{||\Psi_{n+1}||}{||\Psi_n||}
$$

or

$$
\vartheta = \lim_{n\to\infty} (||\Psi_n||)^{\frac{1}{n}}
$$

exists and ϑ is equal to the modulus of one of the eigenvalues of matrix M.

Theorem 11. *Assume that $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ is a solution of System (4) such that*

$$
\lim_{n\to\infty} u_n = \bar{u} = X+1, \quad \lim_{n\to\infty} v_n = \bar{v} = Y+1, \quad \lim_{n\to\infty} w_n = \bar{w} = Z+1.
$$

Then, the error vector

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-k}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-k}^2 \\ e_n^3 \\ e_{n-1}^3 \\ \vdots \\ e_{n-k}^3 \end{pmatrix} = \begin{pmatrix} u_n - \bar{u} \\ u_{n-1} - \bar{u} \\ \vdots \\ u_{n-k} - \bar{u} \\ v_n - \bar{v} \\ v_{n-1} - \bar{v} \\ \vdots \\ v_{n-k} - \bar{v} \\ w_n - \bar{w} \\ w_{n-1} - \bar{w} \\ \vdots \\ w_{n-k} - \bar{w} \end{pmatrix}$$

satisfies both of the asymptotic relations for some $i \in \{1, 2, \ldots, k\}$,

$$\vartheta = \lim_{n\to\infty} (||\Psi_n||)^{\frac{1}{n}} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{w})|, \quad \vartheta = \lim_{n\to\infty} \frac{||\Psi_{n+1}||}{||\Psi_n||} = |\lambda_i J_F(\bar{u}, \bar{v}, \bar{w})|, \tag{43}$$

where ϑ *is equal to the modulus of one of the eigenvalues of* $\mathcal{J}_{\mathcal{F}}$ *at the equilibrium point* $(\bar{u}, \bar{v}, \bar{w})$.

Proof 10. We let $\{u_n, v_n, w_n\}_{n=-k}^{\infty}$ be a solution of System (4) such that

$$\lim_{n\to\infty} u_n = \bar{u} = X + 1, \quad \lim_{n\to\infty} v_n = \bar{v} = Y + 1, \quad \lim_{n\to\infty} w_n = \bar{w} = Z + 1. \tag{44}$$

We have

$$u_{n+1} - \bar{u} = X + \frac{v_{n-k}}{v_n} - X - 1 = \frac{v_{n-k} - v_n}{v_n} = \frac{v_{n-k} - \bar{v}}{v_n} - \frac{v_n - \bar{v}}{v_n}, \tag{45}$$

$$v_{n+1} - \bar{v} = Y + \frac{w_{n-k}}{w_n} - Y - 1 = \frac{w_{n-k} - w_n}{w_n} = \frac{w_{n-k} - \bar{w}}{w_n} - \frac{w_n - \bar{w}}{w_n}, \tag{46}$$

$$w_{n+1} - \bar{w} = Z + \frac{u_{n-k}}{u_n} - Z - 1 = \frac{u_{n-k} - u_n}{u_n} = \frac{u_{n-k} - \bar{u}}{u_n} - \frac{u_n - \bar{u}}{u_n}, \tag{47}$$

using the fact that

$$e_n^1 = u_n - \bar{u}, \quad e_n^2 = v_n - \bar{v}, \quad e_n^3 = w_n - \bar{w},$$

then, the equations in (45)–(47) can be rewritten in the following form:

$$e_{n+1}^1 = \frac{e_{n-k}^2}{v_n} - \frac{e_n^2}{v_n}, \tag{48}$$

$$e_{n+1}^2 = \frac{e_{n-k}^3}{w_n} - \frac{e_n^3}{w_n}, \tag{49}$$

$$e_{n+1}^3 = \frac{e_{n-k}^1}{u_n} - \frac{e_n^1}{u_n}, . \tag{50}$$

Now, we let $A_i = C_i = D_i = E_i = M_i = N_i = 0$, for $i \in \{0, 1, \ldots, k\}$, $B_0 = -\frac{1}{v_n}$, $B_i = 0$ for $i \in \{1, 2, \ldots, k-1\}$, $B_k = \frac{1}{v_n}$, $F_0 = -\frac{1}{w_n}$, $F_i = 0$ for $i \in \{1, 2, \ldots, k-1\}$, $F_k = \frac{1}{w_n}$, $L_0 = -\frac{1}{u_n}$, $L_i = 0$ for $i \in \{1, 2, \ldots, k-1\}$ and $L_k = \frac{1}{u_n}$. Then, the equations in (48)–(50) take the form of

$$e^1_{n+1} = \sum_{i=0}^{k} A_i e^1_{n-i} + \sum_{i=0}^{k} B_i e^2_{n-i} + \sum_{i=0}^{k} C_i e^3_{n-i}, \tag{51}$$

$$e^2_{n+1} = \sum_{i=0}^{k} D_i e^1_{n-i} + \sum_{i=0}^{k} E_i e^2_{n-i} + \sum_{i=0}^{k} F_i e^3_{n-i}, \tag{52}$$

$$e^3_{n+1} = \sum_{i=0}^{k} L_i e^1_{n-i} + \sum_{i=0}^{k} M_i e^2_{n-i} + \sum_{i=0}^{k} N_i e^3_{n-i}. \tag{53}$$

We have

$$\begin{cases} \lim_{n\to\infty} A_i = \lim_{n\to\infty} C_i = \lim_{n\to\infty} D_i = \lim_{n\to\infty} E_i = \lim_{n\to\infty} M_i = \lim_{n\to\infty} N_i = 0, \text{ for } i \in \{0,1,\ldots,k\}, \\ \lim_{n\to\infty} B_i = \lim_{n\to\infty} F_i = \lim_{n\to\infty} L_i = 0, \quad \text{for } i \in \{0,1,\ldots,k-1\}, \\ \lim_{n\to\infty} B_0 = -\frac{1}{\bar{v}}, \quad \lim_{n\to\infty} F_0 = -\frac{1}{\bar{w}}, \quad \lim_{n\to\infty} L_0 = -\frac{1}{\bar{v}}, \\ \lim_{n\to\infty} B_k = \frac{1}{\bar{v}}, \quad \lim_{n\to\infty} F_k = \frac{1}{\bar{w}}, \quad \lim_{n\to\infty} L_k = \frac{1}{\bar{u}}. \end{cases} \tag{54}$$

That is,

$$\begin{cases} B_0 = -\frac{1}{\bar{v}} + a_n, \quad B_k = \frac{1}{\bar{v}} + b_n, \\ E_0 = -\frac{1}{\bar{w}} + \alpha_n, \quad E_k = \frac{1}{\bar{w}} + \beta_n, \\ L_0 = -\frac{1}{\bar{u}} + \gamma_n, \quad L_k = \frac{1}{\bar{u}} + \delta_n, \end{cases} \tag{55}$$

where $a_n \to 0,\ b_n \to 0,\ \alpha_n \to 0,\ \beta_n \to 0,\ \gamma_n \to 0,\ \delta_n \to 0$ for $n \to \infty$. Then, we possess the next system of the form (41)

$$\mathcal{E}_{n+1} = (M + N(n))\mathcal{E}_n, \tag{56}$$

where $\mathcal{E}_n = (e^1_n, e^1_{n-1}, \ldots, e^1_{n-k}, e^2_n, e^2_{n-1}, \ldots, e^2_{n-k}, e^3_n, e^3_{n-1}, \ldots, e^3_{n-k})^T$ and

$$M = \begin{pmatrix} 0 & 0\cdots0 & 0 & \frac{-1}{Y+1} & 0\cdots0 & \frac{1}{Y+1} & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{-1}{Z+1} & 0\cdots0 & \frac{1}{Z+1} \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \frac{-1}{X+1} & 0\cdots0 & \frac{1}{X+1} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix}, \tag{57}$$

$$N(n) = \begin{pmatrix} 0 & 0\cdots 0 & 0 & a_n & 0\cdots 0 & b_n & 0 & 0\cdots 0 & 0 \\ 1 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 1\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 1 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & \alpha_n\cdots 0 & \beta_n \\ 0 & 0\cdots 0 & 0 & 1 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 1\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 1 & 0 & 0 & 0\cdots 0 & 0 \\ \gamma_n & 0\cdots 0 & \delta_n & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 1 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 1\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 1 & 0 \end{pmatrix}, \tag{58}$$

where $\|N(n)\| \to 0$ as $n \to \infty$. Matrix M is equal to $\mathcal{J}_{\mathcal{F}}$. So, by applying Theorem 10 to System (4), the result holds. $\square$

6. Numerical Examples

In this section, we provide numerical examples which demonstrate different types of the behavior of solutions to System (4).

Example 1. *Consider the next system:*

$$u_n = X + \frac{v_{n-3}}{v_{n-1}}, \quad v_n = Y + \frac{w_{n-3}}{w_{n-1}}, \quad w_n = Z + \frac{u_{n-3}}{u_{n-1}}, \quad n \in \mathbb{N}_0, \tag{59}$$

with initial values $u_{-3} = 0.51$, $u_{-2} = 8.47$, $u_{-1} = 2.55$, $v_{-3} = 2.08$, $v_{-2} = 3.71$, $v_{-1} = 20.79$, $w_{-3} = 12.28$, $w_{-2} = 0.38$, $w_{-1} = 2.41$ *and parameters* $X = 1.7$, $Y = 2.5$, $Z = 1.9$. *Then, the equilibrium point* $\Gamma_1 = (\bar{u}, \bar{v}, \bar{w}) = (2.7, 3.5, 2.9)$ *of System* (59) *is globally asymptotically stable. That is, since the parametric conditions in Theorems 8 and 9 are satisfied, Figure 1 shows that the equilibrium point* $\Gamma_1 = (\bar{u}, \bar{v}, \bar{w}) = (2.7, 3.5, 2.9)$ *of System* (59) *is globally asymptotically stable (see Figure 1, Theorem 9).*

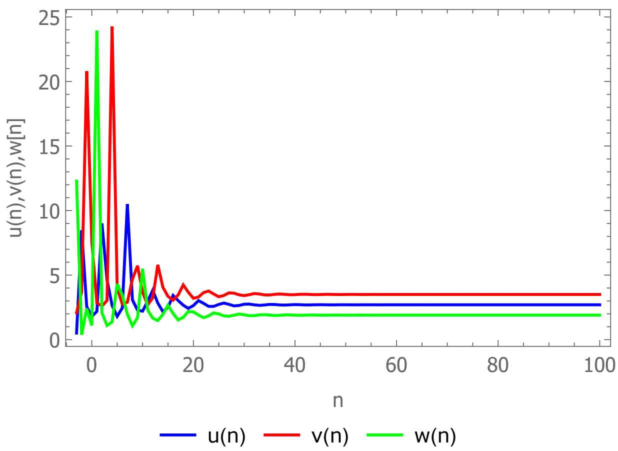

Figure 1. The plot of System (59) with $X > 1$, $Y > 1$ and $Z > 1$.

Example 2. *Consider the next system:*

$$u_n = X + \frac{v_{n-4}}{v_{n-1}}, \quad v_n = Y + \frac{w_{n-4}}{w_{n-1}}, \quad w_n = Z + \frac{u_{n-4}}{u_{n-1}}, \quad n \in \mathbb{N}_0, \tag{60}$$

with initial values $u_{-4} = 3.13$, $u_{-3} = 1.51$, $u_{-2} = 0.71$, $u_{-1} = 1.12$, $v_{-4} = 1.27$, $v_{-3} = 0.08$, $v_{-2} = 1.42$, $v_{-1} = 2.23$, $w_{-4} = 0.77$, $w_{-3} = 0.28$, $w_{-2} = 0.18$, $w_{-1} = 1.21$ *and parameters* $X = 0.78$, $Y = 0.43$, $Z = 0.97$. *Then, the equilibrium point* $\Gamma_2 = (\bar{u}, \bar{v}, \bar{w}) = (1.78, 1.43, 1.97)$ *of System* (60) *is not globally asymptotically stable. Moreover, System* (60) *has unbounded solution. More precisely, from Thereom* 5, *since* $X > 1$, $Y > 1$ *and* $Z > 1$, *System* (60) *has unbounded solutions. Furthermore, Figure* 2 *implies that the equilibrium point* $\Gamma_2 = (\bar{u}, \bar{v}, \bar{w}) = (1.78, 1.43, 1.97)$ *of System* (60) *is unstable (see Figure* 2, *Theorem* 5*).*

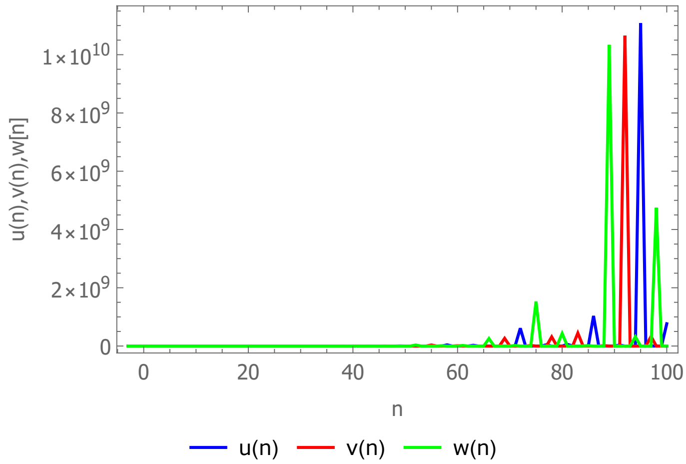

Figure 2. The plot of System (60) with $X < 1$, $Y < 1$ and $Z < 1$.

Example 3. *Consider the next system:*

$$u_n = X + \frac{v_{n-5}}{v_{n-1}}, \quad v_n = Y + \frac{w_{n-5}}{w_{n-1}}, \quad w_n = Z + \frac{u_{n-5}}{u_{n-1}}, \quad n \in \mathbb{N}_0, \tag{61}$$

with initial values $u_{-5} = 2.14$, $u_{-4} = 1.44$, $u_{-3} = 0.92$, $u_{-2} = 0.53$, $u_{-1} = 0.01$, $v_{-5} = 0.13$, $v_{-4} = 2.4$, $v_{-3} = 1.56$, $v_{-2} = 3.27$, $v_{-1} = 0.03$, $w_{-5} = 0.25$, $w_{-4} = 1.96$, $w_{-3} = 2.25$, $w_{-2} = 3.05$, $w_{-1} = 0.31$ *and parameters* $X = Y = Z = 1$. *Then, the solution of System* (61) *oscillates about the equilibrium point* $(\bar{u}, \bar{v}, \bar{w}) = (2, 2, 2)$ *of System* (61) *with semi-cycles having at most five terms. Then, the equilibrium point* $(\bar{u}, \bar{v}, \bar{w}) = (2, 2, 2)$ *of System* (61) *is not globally asymptotically stable. Further, System* (60) *possesses an unbounded solution (see Figure* 3, *Theorem* 9*).*

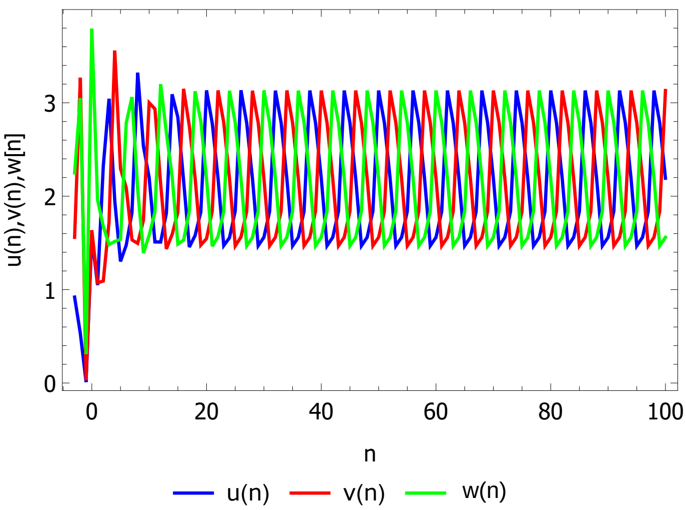

Figure 3. The plot of System (61) with $X = Y = Z = 1$.

7. Conclusions

This study represents a contribution to the analysis of three-dimensional concrete nonlinear system of difference equations with arbitrary constant and different parameters. This paper mainly discusses the dynamic properties of a class of higher-order system of difference equations by utilizing semi-cycle analysis, stability theory and rate of convergence. The main results are as follows.

(i) From semi-cycle analysis of System (4), it is determined that System (4) has no non-oscillatory negative solutions, no decreasing non-oscillatory solutions, no nontrivial periodic solutions of period k. It is also determined that the solution of System (4) is either non-oscillatory solution or it oscillates about the equilibrium point of System (4), with semi-cycles having $k+1$ terms.

(ii) When $X > 1$, $Y > 1$ and $Z > 1$, the positive solution of System (4) is bounded and persists.

(iii) When $X > 1, Y > 1$ and $Z > 1$, every positive solutions of System (4) converges to the equilibrium $(X+1, Y+1, Z+1)$.

(iv) When $X > 1, Y > 1$ and $Z > 1$, the unique equilibrium point of System (4) is globally asymptotically stable.

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