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The Relationship between Ordinary and Soft Algebras with an Application

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Abstract: This work makes a contribution to the theory of soft sets. It studies the concepts of soft semi-algebras and soft algebras, along with some operations. Then, it examines the relations of soft algebras set to their ordinary (crisp) counterparts. Among other things, we show that every algebra of soft sets induces a collection of ordinary algebras of sets. By using the formulas (in Theorem 7 and Corollary 1), we present a novel construction, allowing us to construct a soft algebra from a system of ordinary algebras of sets. Two examples are presented to show how these formulas can be used in practice. This approach is general enough to be applied to many other (soft) algebraic properties and shows that ordinary algebras contain instruments enabling us to construct soft algebras and to study their properties. As an application, we demonstrate how elements of the generated soft algebra can be used to describe the weather conditions of a region.

Keywords: soft set; soft algebra; soft mapping; probability; soft measure

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1. Introduction

In today's world, the mathematical modeling and manipulation of various kinds of uncertainties has turned into a growing concern in the solution of difficult problems in a variety of fields, such as engineering, environmental science, economics, social sciences, and medicine. Even though probability theory, fuzzy set theory [1], rough set theory [2], and interval mathematics [3], and so on, are well-known and effective tools for dealing with ambiguity and uncertainty, each has its own set of limitations; a common major weakness among these mathematical techniques is the limitation of parametrization tools.

In 1999, Molodtsov [4] originated the soft set theory as a mathematical tool for dealing with uncertainty, which is free of the challenges related to the earlier mentioned theories. Soft sets were presented as a collection of parameterized possibilities of a universe. The characteristics of parameter sets associated with soft sets provide a standardized foundation for modeling uncertainty. This led to the rapid growth of soft set theory and its relevant areas in a short amount of time and provided various applications of soft sets in real life, such as medical diagnosis [5,6], evaluation of nutrition systems [7], decision making [8], analysis of networks [9], and information systems [10,11]. To face more complicated problems, some extensions of soft sets were embraced, such as bipolar soft sets and double-framed soft sets, and their efficiency to address practical problems was demonstrated, as illustrated in [12,13].

Multiple researchers applied soft set theory to various mathematical structures such as soft group theory [14], soft ring theory [15], soft category theory [16], soft topology [17–20], infra-soft topology [21], N-soft topology [22], soft topological soft groups and soft rings [23,24], etc.

An algebra of subsets of a universal (ordinary) set is crucial for the growth of several disciplines, including mathematical analysis, probability theory, and economics and finance. The concept of algebras of soft settings was established by Riaz et al. [25]. Various properties and relations of soft algebras remain untouched. Therefore, we further study this concept and establish a general construction for producing soft algebras. This construction helps us to study the properties of soft algebras by means of the properties of ordinary algebras.

The content of the paper is arranged as follows: We provide a summary of the background on soft set theory and probability theory in Section 2. Section 3 concentrates on the study of soft semi-algebras and soft algebras, with their relationships. Then, it investigates some operations on soft algebras. Section 4 uses two remarkable formulas to demonstrate the relationships between ordinary and soft algebras. Section 5 discusses the application of representing weather conditions using elements of the generated soft algebra. We end with a brief conclusion and discussion (Section 6).

2. Preliminaries

Let X be an initial universe, $\mathbf{Q}$ be a set of parameters, and $\mathcal{P}(X)$ denote the set of all subsets of X.

Definition 1 ([4]). *Let $F : \mathcal{Q} \to \mathcal{P}(X)$ be a set-valued mapping and let $\mathcal{Q} \subseteq \mathbf{Q}$. The collection $(F, \mathcal{Q}) = \{(q, F(q)) : q \in \mathcal{Q}\}$ is called a soft set over X.*

A parameterized class of subsets of X represents what is intended by a soft set over X. The family of all soft subsets of X with the parametric set $\mathbf{Q}$ (resp. $\mathcal{Q}$) is denoted by $S_{\mathbf{Q}}(X)$ (resp. $S_{\mathcal{Q}}(X)$).

Remark 1. *The soft set $(F, \mathcal{Q})$ can indeed be extended to the soft set $(F, \mathbf{Q})$ by giving $F(q) = \emptyset$ for each $q \in \mathbf{Q} - \mathcal{Q}$.*

Definition 2 ([11]). *The soft complement $(F, \mathcal{Q})^c$ of a soft set $(F, \mathcal{Q})$ is a soft set $(F^c, \mathcal{Q})$, where $F^c : E \to \mathcal{P}(X)$ is a mapping having the property that $F^c(q) = X - F(q)$ for all $q \in \mathcal{Q}$.*

Notice that $((F, \mathcal{Q})^c)^c = (F, \mathcal{Q})$.

Definition 3 ([26]). *A soft set $(F, \mathcal{Q})$ over X is called null with respect to $\mathcal{Q}$, $\Phi_{\mathcal{Q}}$, if $F(q) = \emptyset$ for all $q \in \mathcal{Q}$, and is called absolute with respect to $\mathcal{Q}$, $X_{\mathcal{Q}}$, if $F(q) = X$ for all $q \in \mathcal{Q}$. The respective null and absolute soft sets are denoted by $\Phi_{\mathbf{Q}}$ and $X_{\mathbf{Q}}$.*

Evidently, $\Phi^c_{\mathbf{Q}} = X_{\mathbf{Q}}$ and $X^c_{\mathbf{Q}} = \Phi_{\mathbf{Q}}$.

Definition 4 ([27]). *A soft set $(F, \mathcal{Q})$ is called finite (resp. countable) if $F(q)$ is finite (resp. countable) for each $q \in \mathcal{Q}$. Otherwise, it is called infinite (resp. uncountable).*

Definition 5 ([28]). *An ordinary soft point $(\{x\}, \mathcal{Q})$ (or shortly x) is a soft set $(F, \mathcal{Q})$ over X, such that $F(q) = \{x\}$ for all $q \in \mathcal{Q}$, where $x \in X$. It is said $x \tilde{\in} (F, \mathcal{Q})$ if $x \in F(q)$ for all $q \in \mathcal{Q}$.*

Definition 6 ([11,29]). *Let $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathbf{Q}$. A soft set $(F_1, \mathcal{Q}_1)$ is a (soft) subset of $(F_2, \mathcal{Q}_2)$ (written by $(F_1, \mathcal{Q}_1) \tilde{\subseteq} (F_2, \mathcal{Q}_2)$) if $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ and $F_1(q) \subseteq F_2(q)$ for all $q \in \mathcal{Q}_1$. $(F_1, \mathcal{Q}_1)$ is soft equal to $(F_2, \mathcal{Q}_1)$ if $(F_1, \mathcal{Q}_1) \tilde{\subseteq} (F_2, \mathcal{Q}_2)$ and $(F_2, \mathcal{Q}_2) \tilde{\subseteq} (F_1, \mathcal{Q}_1)$.*

Definition 7 ([26,30]). *Let $\{(F_i, \mathcal{Q}) : i \in I\}$ be an indexed family of soft sets over X with an index set I. Then*

1. *the soft intersection of $(F_i, \mathcal{Q})$ is a soft set $(F, \mathcal{Q}) = \tilde{\bigcap}_{i \in I} (F_i, \mathcal{Q})$, where $F(q) = \bigcap_{i \in I} F_i(q)$ for all $q \in \mathcal{Q}$*

2. *the soft union of* $(F_i, \mathcal{Q})$ *is a soft set* $(F, \mathcal{Q}) = \tilde{\bigcup}_{i\in I}(F_i, \mathcal{Q})$, *where* $F(q) = \bigcup_{i\in I} F_i(q)$ *for all* $q \in \mathcal{Q}$.

Definition 8 ([31,32])**.** *Let* X, Y *be two different universes parameterized by* $\mathcal{Q}, \mathcal{Q}'$, *respectively, and let* $g : Z \to Y, h : \mathcal{Q} \to \mathcal{Q}'$ *be mappings. The image of a soft set* $(F, \mathcal{Q})\tilde{\subseteq}(X, \mathcal{Q})$ *under* $f_{g,h}$, *or simply* $f : (X, \mathcal{Q}) \to (Y, \mathcal{Q}')$, *is a soft subset* $f(F, \mathcal{Q}) = (f(F), h(q))$ *of* $(Y, \mathcal{Q}')$ *which is given by*

$$f(F)(q') = \begin{cases} \bigcup\limits_{q\in h^{-1}(q')\cap\mathcal{Q}} g(F(q)), & h^{-1}(q') \cap \mathcal{Q} \neq \emptyset \\ \emptyset, & \text{otherwise}, \end{cases}$$

for each $q' \in \mathcal{Q}'$.

The inverse image of a soft set $(G, \mathcal{Q}')\tilde{\subseteq}(Y, \mathcal{Q}')$ *under* f *is a soft subset* $f^{-1}(G, \mathcal{Q}') = (f^{-1}(G), h^{-1}(q'))$, *such that*

$$(f^{-1}(G)(q) = \begin{cases} g^{-1}(G(h(q))), & h(q) \in \mathcal{Q}' \\ \emptyset, & \text{otherwise}, \end{cases}$$

for each $q \in \mathcal{Q}$.

The soft mapping f *is called bijective if the mappings* g, h *are bijective.*

Definition 9. *A subfamily* Σ *of the power set* $\mathcal{P}(X)$ *of a nonempty set* X *is said to be an algebra on* X *if* Σ *meets the following properties:*

1. $\emptyset \in \Sigma$,
2. *if* $F \in \Sigma$, *then* $F^c \in \Sigma$, *and*
3. *if* $F_n \in \Sigma$, *for all* $n = 1, 2, \ldots, k$, *then* $\cup_{n=1}^{k} F_n$ *is in* Σ.

Definition 10 ([33])**.** *Let* P *be a probability function defined on the collection of all possible events* $\mathcal{A}$ *from a sample space* X. *Then* P *is called a distribution if* $\sum_{x\in X} P(x) = 1$. *The set of all elements with non-zero probability is called support of* P.

In this work, if X happens to be a finite set, then $\mathcal{A}$ will be identical to $\mathcal{P}(X)$, and P with the support $\{x_1, x_2, \ldots, x_n\}$ will be represented by

$$P = \frac{P(x_1)}{x_1} + \frac{P(x_2)}{x_2} + \cdots + \frac{P(x_n)}{x_n}.$$

3. Soft Semi-Algebras and Soft Algebras

Definition 11. *A family* $\mathcal{H}\tilde{\subseteq}S_{\mathcal{Q}}(X)$ *is called a soft semi-algebra on* X *if it satisfies the following properties:*

1. $\Phi_{\mathcal{Q}}, X_{\mathcal{Q}}$ *are in* $\mathcal{H}$,
2. *if* $(F, \mathcal{Q}), (G, \mathcal{Q})$ *are in* $\mathcal{H}$, *then* $(F, \mathcal{Q})\tilde{\cap}(G, \mathcal{Q})$ *is in* $\mathcal{H}$, *and*
3. *if* $(F, \mathcal{Q})^c = \tilde{\bigcup}_{i=1}^{n}(F_i, \mathcal{Q})$, *where* $(G_1, \mathcal{Q}), (G_2, \mathcal{Q}), \ldots, (G_n, \mathcal{Q})$ *are in* $\mathcal{H}$ *and* $(G_i, \mathcal{Q})\tilde{\cap}(G_j, \mathcal{Q}) = \Phi_{\mathcal{Q}}$ *for* $i \neq j$, *then* $(F, \mathcal{Q})$ *is in* $\mathcal{H}$.

Let us recall the definition of a soft algebra.

Definition 12 ([25])**.** *A family* $\Sigma\tilde{\subseteq}S_{\mathcal{Q}}(X)$ *is said to be a soft algebra on* X *if* Σ *meets the following properties:*

1. $\Phi_{\mathcal{Q}}$ *is in* Σ,
2. *if* $(F, \mathcal{Q})$ *is in* Σ, *then* $(F, \mathcal{Q})^c$ *is in* Σ, *and*
3. *if* $(F_n, \mathcal{Q})$ *is in* Σ, *for all* $n = 1, 2, \ldots, k$, *then* $\tilde{\bigcup}_{n=1}^{k}(F_n, \mathcal{Q})$ *is in* Σ.

If (3) in the above definition holds true for countably infinite members of Σ, then Σ will be called a soft σ-algebra on X (see [34]).

Example 1. *The collections* $\{\Phi_{\mathcal{Q}}, X_{\mathcal{Q}}\}$ *and* $S_{\mathcal{Q}}(X)$ *are trivially soft algebras. They are, respectively, the smallest and the largest soft algebras.*

Example 2. *For any* $(F,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X)$, $\{\Phi_{\mathcal{Q}}, (F,\mathcal{Q}), (F,\mathcal{Q})^c, X_{\mathcal{Q}}\}$ *is a soft algebra.*

Readily, each soft σ-algebra is a soft algebra and each soft algebra is a soft semi-algebra, but not the converse.

Example 3. *Let* X *be an infinite universal set and* $\mathcal{Q}$ *be a set of parameters. The collection*

$$\Sigma = \{(F,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X) : \text{either } (F,\mathcal{Q}) \text{ or } (F,\mathcal{Q})^c \text{ is finite}\}$$

will be a soft algebra. On the other hand, Σ *is not a soft* σ*-algebra.*

Example 4. *Let* $\mathbb{R}$ *be the set of real numbers and let* $\mathcal{Q} = \{q_1, q_2\}$*. The collection*

$$\mathcal{H} = \{\{(q_1,[a,b)),(q_2,(c,d])\} : a,b,c,d \in \mathbb{R} \text{ with } a<b, c<d\}$$

is a soft semi-algebra but not a soft algebra.

Lemma 1. *Let* $\{\Sigma_i : i \in I\}$ *be an indexed family of soft algebras on* X *with an index set* I*. Then* $\tilde{\bigcap}_{i\in I}\Sigma_i$ *is a soft algebra.*

Proof. Since each soft algebra Σ_i contains $\Phi_{\mathcal{Q}}$, so $\tilde{\bigcap}_{i\in I}\Sigma_i$ is not null and it contains $\Phi_{\mathcal{Q}}$. Let $(F,\mathcal{Q})\tilde{\in}\tilde{\bigcap}_{i\in I}\Sigma_i$. Then $(F,\mathcal{Q})\tilde{\in}\Sigma_i$ for each $i \in I$, and therefore, $(F,\mathcal{Q})^c\tilde{\in}\Sigma_i$ for each $i \in I$. Hence, $(F,\mathcal{Q})^c\tilde{\in}\tilde{\bigcap}_{i\in I}\Sigma_i$. For the same reason, $\tilde{\bigcap}_{i\in I}\Sigma_i$ is closed under the finite soft unions. □

On the other hand, the union of two soft algebras need not be a soft algebra.

Example 5. *Let* $X = \{1,2,3\}$ *and let* $\mathcal{Q} = \{q_1, q_2\}$*. Given the following two soft algebras:* $\Sigma_1 = \{\Phi_{\mathcal{Q}}, (F_1,\mathcal{Q}), (F_2,\mathcal{Q}), X_{\mathcal{Q}}\}$ *and* $\Sigma_2 = \{\Phi_{\mathcal{Q}}, (F_3,\mathcal{Q}), (F_4,\mathcal{Q}), X_{\mathcal{Q}}\}$*, where* $(F_1,\mathcal{Q}) = \{(q_1,\{1\}), (q_2,\emptyset)\}$*,* $(F_1,\mathcal{Q}) = \{(q_1,\{2,3\}),(q_2,X)\}$*,* $(F_3,\mathcal{Q}) = \{(q_1,\emptyset),(q_2,\{2,3\})\}$*, and* $(F_4,\mathcal{Q}) = \{(q_1,X), (q_2,\{1\})\}$*. Then* $\Sigma_1\tilde{\cup}\Sigma_2$ *is not a soft algebra.*

However, the next result demonstrates that the union of soft algebras is a soft algebra under certain conditions.

Theorem 1. *Let* $\{\Sigma_n : n \in \mathbb{N}\}$ *be a countable family of soft algebras on* X*. If* $\Sigma_1\tilde{\subseteq}\Sigma_2\tilde{\subseteq}\cdots$*, then* $\tilde{\bigcup}_{n\in\mathbb{N}}\Sigma_n$ *is a soft algebra.*

Proof. Since Σ_n is a soft algebra for all $n \in \mathbb{N}$, then $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma_n$. Thus, $\Phi_{\mathcal{Q}}\tilde{\in}\tilde{\cup}_{n\in\mathbb{N}}\Sigma_n$. Let $(F,\mathcal{Q})\tilde{\in}\tilde{\cup}_{n\in\mathbb{N}}\Sigma_n$. Then $(F,\mathcal{Q})\tilde{\in}\Sigma_n$ for some n and so $(F,\mathcal{Q})^c\tilde{\in}\Sigma_n$ as Σ_n is a soft algebra. Surely, $(F,\mathcal{Q})^c\tilde{\in}\tilde{\cup}_{n\in\mathbb{N}}\Sigma_n$. Suppose $(F,\mathcal{Q}),(G,\mathcal{Q})\tilde{\in}\tilde{\cup}_{n\in\mathbb{N}}\Sigma_n$. Since $\Sigma_1\tilde{\subseteq}\Sigma_2\tilde{\subseteq}\cdots$, then there exists some $n_0 \in \mathbb{N}$ such that $(F,\mathcal{Q}),(G,\mathcal{Q})\tilde{\in}\Sigma_0$. Thus, $(F,\mathcal{Q})\tilde{\cup}(G,\mathcal{Q})\tilde{\in}\Sigma_0$ as Σ_0 is a soft algebra and hence, $(F,\mathcal{Q})\tilde{\cup}(G,\mathcal{Q})\tilde{\in}\tilde{\cup}_{n\in\mathbb{N}}\Sigma_n$. □

It is worth noting that the above result does not hold true for soft σ-algebras. Let $X = \mathbb{N}$ and $\mathcal{Q}$ be a set of parameters. If Σ_n is the collection of all soft subsets of $(F_n,\mathcal{Q})$ with their complements, where $F_n = \{1,2,\ldots,n\}$, then $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing collection of soft algebras. Then $\tilde{\bigcup}_{n\in\mathbb{N}}\Sigma_n$ is a soft algebra mentioned in Example 3, which is not a soft σ-algebra.

Lemma 2. *Let $\mathcal{H}$ be a subcollection of $S_{\mathcal{Q}}(X)$. Then there exists a unique soft algebra Σ on X containing $\mathcal{H}$, in the sense that if Σ^* is any other soft algebra containing $\mathcal{H}$, then $\Sigma\tilde{\subseteq}\Sigma^*$.*

Proof. Since $S_{\mathcal{Q}}(X)$ is a soft algebra containing $\mathcal{H}$, so such a soft algebra always exists. Therefore, the soft algebra Σ obtained in Lemma 1 is the required soft algebra. □

We refer to the above soft algebra as the soft algebra on X generated by $\mathcal{H}$ and denote it by $\mathcal{A}(\mathcal{H})$. If $\mathcal{H}$ is a countable collection, then $\mathcal{A}(\mathcal{H})$ is called the countably generated soft algebra. The soft algebra considered in Example 2 is the soft algebra generated by $\{(F,\mathcal{Q})\}$.

Theorem 2. *Let $\mathcal{H}, \mathcal{H}_i\tilde{\subseteq}S_{\mathcal{Q}}(X)$, for $i = 0,1,2$, and let Σ be any soft algebra on X. The soft algebra $\mathcal{A}(\mathcal{H})$ generated by $\mathcal{H}$ possesses the below properties:*

1. $\mathcal{H}\tilde{\subseteq}\mathcal{A}(\mathcal{H})\tilde{\subseteq}\Sigma$.
2. $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{A}(\mathcal{H})$.
3. *$\mathcal{H}$ is a soft algebra if $\mathcal{H} = \mathcal{A}(\mathcal{H})$.*
4. *$\mathcal{H}_1\tilde{\subseteq}\mathcal{H}_2$ implies $\mathcal{A}(\mathcal{H}_1)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$.*
5. $\mathcal{A}(\mathcal{H}_1), \mathcal{A}(\mathcal{H}_2)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_2)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2) = \mathcal{A}(\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_2))$.
6. *$\mathcal{H}\tilde{\subseteq}\mathcal{H}_0\tilde{\subseteq}\mathcal{A}(\mathcal{H})$ implies $\mathcal{A}(\mathcal{H}_0) = \mathcal{A}(\mathcal{H})$.*

Proof.

(1) It follows from the definition of $\mathcal{A}(\mathcal{H})$.

(2) The first direction, $\mathcal{A}(\mathcal{H})\tilde{\subseteq}\mathcal{A}(\mathcal{A}(\mathcal{H}))$, follows from (1). Now, we always have $\mathcal{A}(\mathcal{H})\tilde{\subseteq}\mathcal{A}(\mathcal{H})$, so $\mathcal{A}(\mathcal{H})$ is a soft algebra including $\mathcal{A}(\mathcal{H})$. Since $\mathcal{A}(\mathcal{A}(\mathcal{H}))$ is the smallest soft algebra including $\mathcal{A}(\mathcal{H})$. Thus, $\mathcal{A}(\mathcal{A}(\mathcal{H}))\tilde{\subseteq}\mathcal{A}(\mathcal{H})$. Hence, $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{A}(\mathcal{H})$.

(3) Straightforward.

(4) By (1), $\mathcal{H}_2\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$. Since $\mathcal{H}_1\tilde{\subseteq}\mathcal{H}_2$, then $\mathcal{H}_1\tilde{\subseteq}\mathcal{H}_2\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$ and so $\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$ is a soft algebra containing $\mathcal{H}_1$, but $\mathcal{A}(\mathcal{H}_1)$ is the smallest soft algebra containing $\mathcal{H}_1$. Thus, $\mathcal{A}(\mathcal{H}_1)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$.

(5) We only prove the last equality, other inclusions can be concluded easily. Since $\mathcal{H}_1\tilde{\subseteq}\mathcal{H}_1\tilde{\cup}\mathcal{H}_2$, by (4), $\mathcal{A}(\mathcal{H}_1)\tilde{\subseteq}\ \mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2)$. Similarly, $\mathcal{A}(\mathcal{H}_2)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2)$. Therefore, $\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_1)\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2)$. By (4), $\sigma[\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\ \mathcal{A}(\mathcal{H}_1)]\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2)$.
On the other hand, since $\mathcal{H}_1\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1)$ and $\mathcal{H}_2\tilde{\subseteq}\mathcal{A}(\mathcal{H}_2)$, then $\mathcal{H}_1\tilde{\cup}\mathcal{H}_2\tilde{\subseteq}\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_2)$. By (4), $\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2)\tilde{\subseteq}\mathcal{A}[\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_2)]$. Hence, $\mathcal{A}(\mathcal{H}_1\tilde{\cup}\mathcal{H}_2) = \mathcal{A}(\mathcal{A}(\mathcal{H}_1)\tilde{\cup}\mathcal{A}(\mathcal{H}_2))$.

(6) It follows from (2) and (4).

□

Proposition 1. *Let $\mathcal{H}$ be a soft semi-algebra on X. The family*

$$\Sigma = \{(F,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X) : (F,\mathcal{Q}) = \widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q}),\ where,\ (F_i,\mathcal{Q})\tilde{\in}\mathcal{H}\}$$

is a soft algebra on X and $\mathcal{H}\tilde{\subseteq}\Sigma$, where the symbol $\widetilde{\bigsqcup}$ means disjoint soft union.

Proof. The second claim $\mathcal{H}\tilde{\subseteq}\Sigma$ is obvious, because each $(F_i,\mathcal{Q}) = \widetilde{\bigsqcup}^{i}(F_i,\mathcal{Q})\tilde{\in}\Sigma$. It remains to show that Σ is a soft algebra. Clearly $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma$ as $\Phi_{\mathcal{Q}}\tilde{\in}\mathcal{H}\tilde{\subseteq}\Sigma$. Let $(R,\mathcal{Q}),(S,\mathcal{Q})\tilde{\in}\Sigma$. By definition,

$$\begin{aligned}(R,\mathcal{Q}) &= \widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q}),\ \text{where}\ (F_i,\mathcal{Q})\tilde{\in}\mathcal{H}\ \text{for}\ i = 1,2,\ldots,n,\ \text{and}\\ (S,\mathcal{Q}) &= \widetilde{\bigsqcup}_{j=1}^{m}(G_j,\mathcal{Q}),\ \text{where}\ (G_j,\mathcal{Q})\tilde{\in}\mathcal{H}\ \text{for}\ j = 1,2,\ldots,m.\end{aligned}$$

Now

$$\begin{aligned}(R,\mathcal{Q})\tilde{\cap}(S,\mathcal{Q}) &= \left(\widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q})\right)\bigcap\left(\widetilde{\bigsqcup}_{j=1}^{m}(G_j,\mathcal{Q})\right)\\ &= \widetilde{\bigsqcup}_{i=1}^{n}\widetilde{\bigsqcup}_{j=1}^{m}((F_i,\mathcal{Q})\tilde{\cap}(G_j,\mathcal{Q}))\end{aligned}$$

Since $(F_i,\mathcal{Q}),(G_j,\mathcal{Q})\tilde{\in}\mathcal{H}$ for all i and j, and $\mathcal{H}$ is closed under finite soft intersections, therefore, $(R,\mathcal{Q})\tilde{\cap}(S,\mathcal{Q})\tilde{\in}\Sigma$.

Let $(R,\mathcal{Q})\tilde{\in}\Sigma$. There exist $(F_1,\mathcal{Q}),(F_1,\mathcal{Q}),\ldots,(F_n,\mathcal{Q})\tilde{\in}\mathcal{H}$ such that $(R,\mathcal{Q})=\widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q})$. Now

$$(R,\mathcal{Q})^c=\left(\widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q})\right)^c=\tilde{\bigcap}_{i=1}^{n}(F_i,\mathcal{Q})^c=(F_1,\mathcal{Q})^c\tilde{\cap}(F_2,\mathcal{Q})^c\tilde{\cap}\cdots\tilde{\cap}(F_n,\mathcal{Q})^c.$$

By definition of $\mathcal{H}$, each $(F_i,\mathcal{Q})^c=\widetilde{\bigsqcup}_{k=1}^{k_i}(F_{i,k},\mathcal{Q})$, for some $(F_{i,k},\mathcal{Q})\tilde{\in}\mathcal{H}$. Therefore,

$$(R,\mathcal{Q})^c=\tilde{\bigcap}_{i=1}^{n}\left(\widetilde{\bigsqcup}_{k=1}^{k_i}(F_{i,k},\mathcal{Q})\right)=\widetilde{\bigsqcup}_{k=1}^{k_i}\left(\tilde{\bigcap}_{i=1}^{n}(F_{i,k},\mathcal{Q})\right).$$

However, $\mathcal{H}$ is a soft semi-algebra, which implies that $\tilde{\bigcap}_{i=1}^{n}(F_{i,k},\mathcal{Q})\tilde{\in}\mathcal{H}$. This proves that $(R,\mathcal{Q})^c\tilde{\in}\Sigma$. Hence, Σ is a soft algebra. □

Theorem 3. *Let $\mathcal{H}$ be a soft semi-algebra on X and let $\mathcal{A}(\mathcal{H})$ be the soft algebra generated by $\mathcal{H}$. Then, $(F,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H})$ if $(F,\mathcal{Q})=\widetilde{\bigsqcup}_{i=1}^{n}(F_i,\mathcal{Q})$, for some $(F_1,\mathcal{Q}),(F_2,\mathcal{Q}),\ldots,(F_n,\mathcal{Q})\tilde{\in}\mathcal{H}$.*

Proof. From Proposition 1, $\mathcal{H}\tilde{\subseteq}\Sigma$ and, by Theorem 2, $\mathcal{A}(\mathcal{H})\tilde{\subseteq}\Sigma$, since $\mathcal{A}(\mathcal{H})$ is a soft algebra.

The converse is clear, since $(F_i,\mathcal{Q})\tilde{\in}\mathcal{H}\tilde{\subseteq}\mathcal{A}(\mathcal{H})$, for $i=1,2,\ldots,n$, and $\mathcal{A}(\mathcal{H})$ is a soft algebra, so $\tilde{\bigcup}_{i=1}^{n}(F_i,\mathcal{Q})=(F,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H})$. □

Proposition 2. *Let $(Y,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X)-\Phi_{\mathcal{Q}}$ and let Σ be a soft algebra on X. Then*

$$\Sigma\tilde{\cap}(Y,\mathcal{Q})=\{(F,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q}):(F,\mathcal{Q})\tilde{\in}\Sigma\}$$

is a soft algebra on $(Y,\mathcal{Q})$ and is denoted by $\Sigma|_{(Y,\mathcal{Q})}$ or simply $\Sigma|_Y$.

Proof. Since $\Phi_{\mathcal{Q}}\tilde{\cap}(Y,\mathcal{Q})=\Phi_{\mathcal{Q}}$ and $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma$, so $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma|_{(Y,\mathcal{Q})}$. If $(G,\mathcal{Q})\tilde{\in}\Sigma|_{(Y,\mathcal{Q})}$, then $(G,\mathcal{Q})=(F,\mathcal{Q})\tilde{\cap}\,(Y,\mathcal{Q})$ for some $(F,\mathcal{Q})\tilde{\in}\Sigma$. Since $(F,\mathcal{Q})^c\tilde{\in}\Sigma$, we have $(G,\mathcal{Q})^c=(F,\mathcal{Q})^c\,\tilde{\cap}(Y,\mathcal{Q})$. Thus, $(G,\mathcal{Q})^c\tilde{\in}\,\Sigma|_{(Y,\mathcal{Q})}$. Suppose that $(G_1,\mathcal{Q}),(G_2,\mathcal{Q}),\ldots,(G_k,\mathcal{Q})\tilde{\in}\Sigma|_{(Y,\mathcal{Q})}$. Then for each $n=1,2,\ldots,k$, there exists $(F_n,\mathcal{Q})\tilde{\in}\Sigma$ such that $(G_n,\mathcal{Q})=(F_n,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})$. Since $\tilde{\bigcup}_{n=1}^{k}(F_n,\mathcal{Q})\tilde{\in}\Sigma$, so

$$\tilde{\bigcup}_{n=1}^{k}(G_n,\mathcal{Q})=\tilde{\bigcup}_{n=1}^{k}[(F_n,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})]=[\tilde{\bigcup}_{n=1}^{k}(F_n,\mathcal{Q})]\tilde{\cap}(Y,\mathcal{Q}).$$

This implies that $\tilde{\bigcup}_{n=1}^{k}(G_n,\mathcal{Q})\tilde{\in}\Sigma|_{(Y,\mathcal{Q})}$. □

Theorem 4. *Let $\mathcal{H}$ be a subcollection of $S_{\mathcal{Q}}(X)$ and let $(Y,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X)-\Phi_{\mathcal{Q}}$. Then*

$$\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))=\mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q}),$$

where $\mathcal{H}\tilde{\cap}(Y,\mathcal{Q})=\{(F,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q}):(F,\mathcal{Q})\tilde{\in}\mathcal{H}\}$.

Proof. Let $(R,\mathcal{Q})\tilde{\in}\mathcal{H}\tilde{\cap}(Y,\mathcal{Q})$. Then $(R,\mathcal{Q})=(F,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})$ for some $(F,\mathcal{Q})\tilde{\in}\mathcal{H}\tilde{\subseteq}\mathcal{A}(\mathcal{H})$. Therefore, $(R,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q})$ and, hence, $\mathcal{H}\tilde{\cap}(Y,\mathcal{Q})\tilde{\subseteq}\mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q})$. Since by Proposition 2, $\mathcal{A}(\mathcal{H})\,\tilde{\cap}(Y,\mathcal{Q})$ is a soft algebra on Y. Thus, Lemma 2 guarantees that

$\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))\tilde{\subseteq}\mathcal{A}(\mathcal{H})\ \tilde{\cap}(Y,\mathcal{Q})$. To prove the other way of the inclusion, we first need to check that

$$\Sigma = \{(S,\mathcal{Q})\tilde{\in}S_{\mathcal{Q}}(X) : (S,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))\}$$

is a soft algebra. Evidently, $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma$ because $\Phi_{\mathcal{Q}}\tilde{\cap}(Y,\mathcal{Q}) = \Phi_{\mathcal{Q}}\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$. Suppose that $(S,\mathcal{Q})\tilde{\in}\Sigma$. Then $(S,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$. Since $(S,\mathcal{Q})^c\tilde{\cap}(Y,\mathcal{Q}) = (Y,\mathcal{Q}) - [(S,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})]$, this means that $(S,\mathcal{Q})^c\tilde{\in}\Sigma$. Assume that $(H_1,\mathcal{Q}),(H_2,\mathcal{Q}),\cdots,(H_k,\mathcal{Q})\tilde{\in}\Sigma$. Then $(H_1,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q}),(H_2,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q}),\cdots,(H_k,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$. Therefore,

$$[\tilde{\bigcup}_{n=1}^{k}(H_n,\mathcal{Q})]\tilde{\cap}(Y,\mathcal{Q}) = \tilde{\bigcup}_{n=1}^{k}[(H_n,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})]\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q})).$$

Hence, $\tilde{\cup}_{n=1}^{k}(H_n,\mathcal{Q})\tilde{\in}\Sigma$. This proves that Σ is a soft algebra on X. Suppose that $(F,\mathcal{Q})\tilde{\in}\mathcal{H}$. Then $(F,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})\tilde{\in}\ \mathcal{H}\tilde{\cap}(Y,\mathcal{Q})\tilde{\subseteq}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$, and so, $(F,\mathcal{Q})\tilde{\in}\Sigma$. Thus, $\mathcal{H}\tilde{\subseteq}\Sigma$ and, by Lemma 2, $\mathcal{A}(\mathcal{H})\tilde{\subseteq}\Sigma$. Now, if $(R,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q})$, then $(R,\mathcal{Q}) = (F,\mathcal{Q})\tilde{\cap}(Y,\mathcal{Q})$ for some $(F,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H})\ \tilde{\subseteq}\Sigma$ and, hence, $(R,\mathcal{Q})\tilde{\in}\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$. This concludes that $\mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q})\tilde{\subseteq}\ \mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q}))$. Thus, $\mathcal{A}(\mathcal{H}\tilde{\cap}(Y,\mathcal{Q})) = \mathcal{A}(\mathcal{H})\tilde{\cap}(Y,\mathcal{Q})$. □

Lemma 3. *Let $f : (X,\mathcal{Q}) \longrightarrow (Y,\mathcal{Q}')$ be a soft mapping. Then*

1. *if Σ is a soft algebra on X, then $\{(G,\mathcal{Q}')\tilde{\subseteq}(Y,\mathcal{Q}') : f^{-1}(G,\mathcal{Q}')\tilde{\in}\Sigma\}$ is a soft algebra on Y.*
2. *If Σ' is a soft algebra on Y, the set $f^{-1}(\Sigma')$ is a soft algebra on X.*

Proof. Both (1) and (2) follow from the fact that $f^{-1}(\Phi_{\mathcal{Q}'}) = \Phi_{\mathcal{Q}}, f^{-1}(Y_{\mathcal{Q}'} - (G,\mathcal{Q}')) = X_{\mathcal{Q}} - f^{-1}(G,\mathcal{Q}')$ for each soft set $(G,\mathcal{Q}')$ over Y, and $f^{-1}(\bigcup_{n=1}^{k}(G_n,\mathcal{Q}')) = \bigcup_{n=1}^{k} f^{-1}(G_n,\mathcal{Q}')$ for each collection $\{(G_n,\mathcal{Q}') : n = 1,2,\ldots,k\}$ of soft sets over Y (see Theorem 14 in [31]). □

We shall remark that the direct image of a soft algebra under a soft mapping need not be a soft algebra.

Example 6. *Let $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2\}$, and $\mathcal{Q} = \mathcal{Q}' = \{q_1,q_2\}$. Define the mapping $f_{g,h}$ by*

$$g(x) = \begin{cases} y_1, \text{ if } x \neq x_3; \\ y_2, \text{ if } x = x_3, \end{cases}$$

and $h(q) = q$, for all $q \in \mathcal{Q}$. Let $\Sigma = \{\Phi_{\mathcal{Q}},(F_1,\mathcal{Q}),(F_2,\mathcal{Q}),X_{\mathcal{Q}}\}$, where $(F_1,\mathcal{Q}) = \{(q_1,\{x_1\}),(q_2,\emptyset)\}$ and $(F_2,\mathcal{Q}) = \{(q_1,\{x_2,x_3\}),(q_2,X)\}$. The image of the soft algebra Σ is $f_{g,h}(\Sigma) = \{\Phi_{\mathcal{Q}},(G_1,\mathcal{Q}),\tilde{Y}\}$, where $(G_1,\mathcal{Q}) = \{(q_1,\{y_1\}),(q_2,\emptyset)\}$, which is not a soft algebra on Y.

Theorem 5. *Let $f : (X,\mathcal{Q}) \longrightarrow (Y,\mathcal{Q}')$ be a soft mapping. Then*

$$f^{-1}(\mathcal{A}(\mathcal{H})) = \mathcal{A}(f^{-1}(\mathcal{H})),$$

for each collection $\mathcal{H}$ of soft sets over Y.

Proof. By Lemma 3 (2), $f^{-1}(\mathcal{A}(\mathcal{H}))$ is a soft algebra and $f^{-1}(\mathcal{H})\tilde{\subseteq}f^{-1}(\mathcal{A}(\mathcal{H}))$, and this implies that $\mathcal{A}(f^{-1}(\mathcal{H}))\tilde{\subseteq}\ f^{-1}(\mathcal{A}(\mathcal{H}))$. However, by Lemma 3 (1), the set $\Sigma = \{(G,\mathcal{Q}')\tilde{\subseteq}(Y,\mathcal{Q}') : f^{-1}(G,\mathcal{Q}')\tilde{\in}\mathcal{A}(f^{-1}(\mathcal{H}))\}$ is a soft algebra, and it contains $\mathcal{H}$, hence $\mathcal{A}(\mathcal{H})\tilde{\subseteq}\Sigma$. Therefore, $f^{-1}(\mathcal{A}(\mathcal{H}))\ \tilde{\subseteq}f^{-1}(\Sigma) = \{(F,\mathcal{Q})\tilde{\subseteq}(X,\mathcal{Q}) : (F,\mathcal{Q}) = f^{-1}(G,\mathcal{Q}')$ for some$(G,\mathcal{Q}')\tilde{\in}\Sigma\}\tilde{\subseteq}\ \mathcal{A}(f^{-1}(\mathcal{H}))$. This shows that $f^{-1}(\mathcal{A}(\mathcal{H})) = \mathcal{A}(f^{-1}(\mathcal{H}))$. □

4. Ordinary and Soft Algebras

Proposition 3. *Let Σ be a soft algebra on X, parameterized by $\mathcal{Q}$. Then $\Sigma_q = \{F(q) : (F,\mathcal{Q})\tilde{\in}\Sigma\}$ is an ordinary algebra on X for each $q \in \mathcal{Q}$.*

Proof. Since $\Phi_{\mathcal{Q}}\tilde{\in}\Sigma$, then $\emptyset \in \Sigma_q$ for each $q \in \mathcal{Q}$. Let $F(q) \in \Sigma_q$. Then $(F,\mathcal{Q})\tilde{\in}\Sigma$ and, since Σ is a soft algebra, so $(F,\mathcal{Q})^c\tilde{\in}\Sigma$. However, $(F,\mathcal{Q})^c = \{(q,X - F(q)) : q \in \mathcal{Q}\}$ and so, $F^c(q) = X - F(q) \in \Sigma_q$. Let $\{F_n(q) : n = 1, 2, \ldots, k\}$ be a family of sets in Σ_q. Then, for each n, $(F_n, \mathcal{Q}) \tilde{\in} \Sigma$, and thus, $\tilde{\cup}_{n=1}^{k} (F_n, \mathcal{Q}) \tilde{\in} \Sigma$. Hence, $\cup_{n=1}^{k} F_n(q) \in \Sigma_q$. □

The converse of this lemma is not always true, particularly when $|\mathcal{Q}| > 1$, and the counterexample can be concluded from Example 5 in [34]. The scenario is different when $|\mathcal{Q}| = 1$.

Theorem 6. *Let $\mathcal{Q} = \{q\}$ and let $\mathcal{F}$ be a family of subsets of X. Then $\Sigma = \{(q, F(q)) : F(q) \in \mathcal{F}\}$ is a soft algebra if $\Sigma_q = \{F(q) : (q, F(q)) \tilde{\in} \Sigma\}$ is an ordinary algebra on X.*

Proof. The first direction follows from Proposition 3.

Conversely, suppose Σ_q is an algebra. Clearly, $\emptyset \in \Sigma_q$, and so, $(q, \emptyset) = \Phi_{\mathcal{Q}} \tilde{\in} \Sigma$. Let $(F, \mathcal{Q}) \tilde{\in} \Sigma$. Then, $(F, \mathcal{Q}) = (q, F(q))$ and $F(q) \in \Sigma_q$. Since Σ_q is an algebra, so $F^c(q) \in \Sigma_q$. Therefore, $(F, \mathcal{Q})^c = (q, F^c(q)) \tilde{\in} \Sigma$. Let $\{(F_n, \mathcal{Q}) : n = 1, 2, \ldots, k\}$ be a collection of soft sets in Σ. This implies that $(F_n, \mathcal{Q}) = (q, F_n(q))$ for each $n = 1, 2, \ldots, k$, and thus, $\cup_{n=1}^{k} F_n(q) \in \Sigma_q$, as Σ_q is an algebra on X. Therefore,

$$\tilde{\bigcup}_{n=1}^{k} (F_n, \mathcal{Q}) = \tilde{\bigcup}_{n=1}^{k} (q, F_n(q)) = (q, \bigcup_{n=1}^{k} F_n(q)) \tilde{\in} \Sigma.$$

This proves that Σ is a soft algebra on X. □

Theorem 7. *Let $\mathcal{A}$ be an (ordinary) algebra on X and $\mathcal{Q}$ be a set of parameters. Then*

1. *the family $\Sigma(\mathcal{A})$ of soft sets $(F, \mathcal{Q})$ forms a soft algebra on X, where $(F, \mathcal{Q}) = \{(q, F(q)) : F(q) \in \mathcal{A}$ for each $q \in \mathcal{Q}\}$.*
2. *The family $\widehat{\Sigma}(\mathcal{A})$ of soft sets $(F, \mathcal{Q})$ forms a soft algebra on X, where $(F, \mathcal{Q}) = \{(q, F(q)) : F(q) = F(q') \in \mathcal{A}$ for each $q, q' \in \mathcal{Q}\}$.*

Proof. The proof is quite comparable to the second part of the proof for the preceding result. □

Remark 2. *The above formula holds true for any family of subsets of a given universe. Namely, we generate a collection of soft sets from an ordinary family of sets.*

The soft algebra $\Sigma(\mathcal{A})$ is called a soft algebra on X generated by $\mathcal{A}$, and the soft algebra $\widehat{\Sigma}(\mathcal{A})$ is called an extended soft algebra on X via $\mathcal{A}$.

Observe that $\Sigma_q = \widehat{\Sigma}_q = \mathcal{A}$ for each $q \in \mathcal{Q}$.

The following examples show how the process in Theorem 7 can be used:

Example 7. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{Q} = \{q_1, q_2\}$. Consider the following ordinary algebra on X: $\mathcal{A} = \{\emptyset, \{x_1, x_2\}, \{x_3, x_4\}, X\}$. Applying the first formula, we conclude the following soft algebra on X:*

$$\Sigma(\mathcal{A}) = \Big\{\Phi_{\mathcal{Q}}, (F_1, \mathcal{Q}), (F_2, \mathcal{Q}), \cdots, (F_{14}, \mathcal{Q}), X_{\mathcal{Q}}\Big\},$$

where

$(F_1, \mathcal{Q}) = \{(q_1, \emptyset), (q_2, \{x_1, x_2\})\}$,
$(F_2, \mathcal{Q}) = \{(q_1, \emptyset), (q_2, \{x_3, x_4\})\}$,
$(F_3, \mathcal{Q}) = \{(q_1, \emptyset), (q_2, X)\}$,
$(F_4, \mathcal{Q}) = \{(q_1, \{x_1, x_2\}), (q_2, \emptyset)\}$,
$(F_5, \mathcal{Q}) = \{(q_1, \{x_1, x_2\}), (q_2, \{x_1, x_2\})\}$,
$(F_6, \mathcal{Q}) = \{(q_1, \{x_1, x_2\}), (q_2, \{x_3, x_4\})\}$,
$(F_7, \mathcal{Q}) = \{(q_1, \{x_1, x_2\}), (q_2, X)\}$,
$(F_8, \mathcal{Q}) = \{(q_1, \{x_3, x_4\}), (q_2, \emptyset)\}$,
$(F_9, \mathcal{Q}) = \{(q_1, \{x_3, x_4\}), (q_2, \{x_1, x_2\})\}$,
$(F_{10}, \mathcal{Q}) = \{(q_1, \{x_3, x_4\}), (q_2, \{x_3, x_4\})\}$,
$(F_{11}, \mathcal{Q}) = \{(q_1, \{x_3, x_4\}), (q_2, X)\}$,
$(F_{12}, \mathcal{Q}) = \{(q_1, X), (q_2, \emptyset)\}$,
$(F_{13}, \mathcal{Q}) = \{(q_1, X), (q_2, \{x_1, x_2\})\}$, *and*
$(F_{14}, \mathcal{Q}) = \{(q_1, X), (q_2, \{x_3, x_4\})\}$.

Applying the second formula, we conclude the following soft algebra on X:

$$\widehat{\Sigma}(\mathcal{A}) = \Big\{\Phi_{\mathcal{Q}}, (G_1, \mathcal{Q}), (G_2, \mathcal{Q}), X_{\mathcal{Q}}\Big\},$$

where
$(G_1, \mathcal{Q}) = \{(q_1, \{x_1, x_2\}), (q_2, \{x_1, x_2\})\}$ *and*
$(G_2, \mathcal{Q}) = \{(q_1, \{x_3, x_4\}), (q_2, \{x_3, x_4\})\}$.

A less trivial example is

Example 8. *Let $\mathcal{A}_f$ be the finite–cofinite algebra on X. That is,*

$$\mathcal{A}_f = \big\{F \subseteq X : F \text{ is finite or } F^c \text{ is finite}\big\}.$$

If X is finite, $\mathcal{A}_f = \mathcal{P}(X)$. The soft algebra $\Sigma(\mathcal{A}_f)$ on X generated by $\mathcal{A}_f$ is given by the following family:

$$\Sigma(\mathcal{A}_f) = \Big\{\{(q, F(q)) : q \in \mathcal{Q}\}\tilde{\in} S_{\mathcal{Q}}(X) : F(q) \text{ or } F(q)^c \text{ is finite, for each } q \in \mathcal{Q}\Big\}.$$

When we need to define the finite–cofinite algebra Σ_f on X in soft settings, we normally do so as in Example 3, i.e.,

$$\Sigma_f = \Big\{(F, \mathcal{Q})\tilde{\in} S_{\mathcal{Q}}(X) : (F, \mathcal{Q}) \text{ or } (F, \mathcal{Q})^c \text{ is finite}\Big\}.$$

However, both $\Sigma_f = \Sigma(\mathcal{A}_f)$ are equivalent. This example demonstrates how effectively our formulas produce soft algebras. We shall call Σ_f the finite–cofinite soft algebra on X.

It is worth noting that the general formula for constructing soft algebras provided by Theorem 7 can be improved by the use of several ordinary algebras.

Corollary 1. *Let $\mathscr{A} = \{\mathcal{A}_{\mathcal{Q}} : q \in \mathcal{Q}\}$ be a collection of (ordinary) algebras on X indexed by a set of parameters $\mathcal{Q}$. The family $\Sigma(\mathscr{A})$ of all soft sets $(F, \mathcal{Q})$ forms a soft algebra on X, where $(F, \mathcal{Q}) = \{(q, F(q)) : F(q) \in \mathcal{A}_{\mathcal{Q}}$ for each $q \in \mathcal{Q}\}$.*

Notice that, if for each $q, q' \in \mathcal{Q}$, $\mathcal{A}_{\mathcal{Q}} = \mathcal{A}_{\mathcal{Q}'} = \mathcal{A}$, then $\Sigma(\mathscr{A}) = \Sigma(\mathcal{A})$.

We have seen that each soft algebra produces a family of (ordinary) algebras of size $\leq|\mathcal{Q}|$, so we obtain the following observation:

Lemma 4. *Let $\mathscr{A} = \{\mathcal{A}_q : q \in \mathcal{Q}\}$ be the family of all ordinary algebras on X from a soft algebra Σ. Then*

$$\Sigma \tilde{\subseteq} \Sigma(\mathscr{A}).$$

Proof. It can be concluded from the definition of soft sets and the soft algebra generated by $\mathscr{A}$. □

Theorem 8. *Let $\{\mathcal{A}_q : q \in \mathcal{Q}\}$ be a family of ordinary algebras on X indexed by $\mathcal{Q}$. Then*

$$\Sigma(\tilde{\bigcap}\mathcal{A}_q) = \tilde{\bigcap}\Sigma(\mathcal{A}_q).$$

Proof. Let $(F, \mathcal{Q}) \tilde{\in} \Sigma(\tilde{\cap}\mathcal{A}_q)$. Then

$$\begin{aligned}
&\Longrightarrow \quad F(q) \in \bigcap \mathcal{A}_q \\
&\Longrightarrow \quad F(q) \in \mathcal{A}_q, \text{ for each } q \in \mathcal{Q} \\
&\Longrightarrow \quad \{(q, F(q)) : q \in \mathcal{Q}\} \in \Sigma(\mathcal{A}_q), \text{ for each } q \in \mathcal{Q} \\
&\Longrightarrow \quad (F, \mathcal{Q}) \tilde{\in} \tilde{\bigcap}\Sigma(\mathcal{A}_q).
\end{aligned}$$

Hence, $\Sigma(\tilde{\cap}\mathcal{A}_q) \tilde{\subseteq} \tilde{\cap}\Sigma(\mathcal{A}_q)$. The converse can be followed by reversing the above steps. □

Consequently, we have the following corollary:

Corollary 2. *Let $\mathcal{E}$ be a subcollection of $S_{\mathcal{Q}}(X)$, such that $\Phi_{\mathcal{Q}}, X_{\mathcal{Q}} \tilde{\in} \mathcal{E}$. Then*

$$\Sigma(\mathcal{A}(\mathcal{E})) = \mathcal{A}(\Sigma(\mathcal{E})),$$

where $\Sigma(\mathcal{E})$ is the family of all soft sets $\{(q, F(q)) : q \in \mathcal{Q}\}$, such that $F(q) \in \mathcal{E}$ for each $q \in \mathcal{Q}$.

5. Weather Forecasting Application

As an application, elements in a soft algebra can represent weather forecasting. Consider the weather conditions for a specific week of a region **R**. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be our universe, where $x_1 = \textit{sunny}$, $x_2 = \textit{rainy}$, $x_3 = \textit{cloudy}$, $x_4 = \textit{thunderstorms}$, $x_5 = \textit{windy}$, and $x_6 = \textit{snowy}$. Suppose the set of parameters A is the seven days of the week. The set of all possible events is $S_{\mathbf{Q}}(X)$, which is equivalent to the soft algebra generated by the ordinary algebra $\mathcal{P}(X)$ with respect to A. Assume that the probability distribution on weather predictions of this week is given in the following Table 1:

Table 1. The probability distribution of weather predictions.

$X_{\mathcal{Q}}$	x_1	x_2	x_3	x_4	x_5	x_6
Sunday	0.7	0	0.3	0	0	0
Monday	0.1	0	0.9	0	0	0
Tuesday	0	0.8	0.2	0	0	0
Wednesday	0.6	0	0.3	0	0.1	0
Thursday	0	0	0.8	0	0.2	0
Friday	0	0.3	0.3	0.1	0.1	0.2
Saturday	0	0.3	0	0	0.2	0.5

The weather predictions for working weekdays $\mathcal{Q}_0$ can be described by the element $(F, \mathcal{Q}_0)$ in the soft algebra $S_{\mathbf{Q}}(X)$ as

$$\begin{aligned} F(sunday) &= \frac{0.7}{x_1} + \frac{0.3}{x_3}, \\ F(monday) &= \frac{0.1}{x_1} + \frac{0.9}{x_3}, \\ F(tuesday) &= \frac{0.8}{x_2} + \frac{0.2}{x_3}, \\ F(wednesday) &= \frac{0.6}{x_1} + \frac{0.3}{x_3} + \frac{0.1}{x_5}, \\ F(thursday) &= \frac{0.8}{x_3} + \frac{0.2}{x_5}. \end{aligned}$$

The soft set $(F, \mathcal{Q}_0)$ indicates that the region **R** will be partly cloudy on Sunday and mostly cloudy on Monday. Tuesday is expected to be cloudy. Furthermore, **R** has remained cold, but warming is estimated on Wednesday, with mostly cloudy conditions on Thursday.

If $\mathcal{Q}_1$ contains the weekend days, then the soft set $(G, \mathcal{Q}_1)$ represents the weather as

$$\begin{aligned} F(friday) &= \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5} + \frac{0.2}{x_6}, \\ F(saturday) &= \frac{0.3}{x_2} + \frac{0.2}{x_5} + \frac{0.5}{x_6}. \end{aligned}$$

The soft set $(G, \mathcal{Q}_1)$ forecasts a cold wave for the region **R**. The weather on Friday will be rainy, cloudy, and possibly windy. People may hear thunder in the distance. A snowfall is also expected on Friday, but the most snowfall will be expected on Saturday, with a noticeable wind chill.

6. Conclusions

This paper contributes to the field of soft set theory by investigating the concept of soft semi-algebras and soft algebras. We have discussed how soft algebras perform when used for specific operations such as restricting to a soft set, taking soft unions and soft intersections, generating by a collection of soft sets, and taking images or preimages under certain soft mappings. We have shown that each element of a soft algebra can be represented by a finite soft union of disjoint elements of a soft semi-algebra. We have seen that a soft algebra produces a family of ordinary algebras. Then, we have established two formulas by which one can construct a soft algebra from an ordinary algebra or a family of ordinary algebras. This construction provides a general framework for studying soft algebras and investigating their properties in relation to their counterparts in ordinary algebras. In particular, one can study probability theory, measure theory, etc., in the context of soft set theory without defining all the related terminologies. As an application to the generated soft algebras, we have shown that their elements can describe the weather conditions for some area over time.

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