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Asymptotic Relations in Applied Models of Inhomogeneous Poisson Point Flows

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Abstract: A model of a particle flow forming a copy of some image and the distance between the copy and the image are estimated using a special probability metric. The ability of the flow of balls to cover the surface, when grinding the balls, was investigated using formulas of stochastic geometry. Reconstruction of characteristics of an inhomogeneous Poisson flow by inaccurate observations is analysed using the Poisson flow point colouring theorem. The dependence of the Poisson parameter of the distribution of the number of customers in a queuing system with an infinite number of servers and a deterministic service time on the peak load created by an inhomogeneous input Poisson flow is estimated. All these models consist of an inhomogeneous Poisson flow of points and marks glued to each point of the flow and are characterised by their mass, area, volume, observability (or non-observability), and service time. The presence of an asymptotic power–law relationship between model objective functions and parameters of mark crushing is established. These results may be applied in nanotechnology, powder metallurgy, ecology, and consumer services in the implementation of the "Smart City" program. The proposed approach is phenomenological in nature and is justified by the results of real observations and experiments.

Keywords: Poisson point flow; stochastic geometry; safety margin; linear regression estimate; plan of experiment; prediction algorithms; polynomial

MSC: 60K20

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1. Introduction

The Poisson flow of points is widely used in queuing theory, in reliability theory, in mathematical biology and ecology, in astronomy, and in many other applied disciplines [1–5]. In this paper, we consider four systems described by inhomogeneous-labelled Poisson flows [6,7], to the points of which special types of stamps are attached (glued). The efficiency indicators of the systems and the parameter, characterising the grinding of grades, and the corresponding increase in the intensity of the flow are introduced. The dependence of the efficiency indicator on this parameter is investigated, and its asymptotic behaviour is analysed when the grinding parameter tends to infinity. The fast convergence of the efficiency indicator to zero is proven, which helps, in particular, to draw important conclusions about the simulated systems.

In the first model, the mark of each Poisson flow point on a flat rectangle or three-dimensional parallelepiped is its mass. In this way, a model of the image formed by a printer/copier or a 3D printer is obtained. The accuracy of reproduction by the real density of the flow of points of ideal density is investigated when the mass grinding parameter tends to infinity in a specially selected probabilistic metric [8]. This model is closely related to the tasks of nanotechnology.

In the second model, the mark of a Poisson flow point on a rectangle can be a square/circle, the centre of which is glued to this point. The set of stamps forms the so-called Boolean random set [2,9–13]. This applied model can characterise a surface covered with sprayed powder particles. As an indicator of efficiency, the mathematical expectation of the area of the rectangle part that is not covered with stamps is taken. The rate of convergence of this efficiency indicator to zero is investigated when the grinding parameter of powder particles tends to infinity. This model is applicable to the assessment of the quality of coatings created by the powder metallurgy method.

In the third model, the mark of the Poisson flow point on the plane is its colour red or white. Points coloured red are considered as observable flow points, and points coloured white are considered as unobservable [1]. The probability of flow points being coloured red is an unknown parameter that can be interpreted as interfering/hidden [14–17]. When analysing the flow of observed points, the interfering parameter is eliminated by the transition from the number of flow points in a certain area to the ratio of the number of flow points in a certain area to the number of flow points on the entire plane. Convergence in the probability of the relative frequency of the number of observed points in a certain region is established when the mathematical expectation of the number of observed points on the plane tends to infinity. This model arose when comparing the number of tiger tracks in different years in conditions when some tracks in the snow were invisible [18,19]. As a result, data on the accounting of animal tracks in a given year become industrial statistics and require special methods for their processing.

In the fourth model, the mark of a Poisson flow point on a straight line is a segment of length a, the left end of which is glued to each flow point. In this case, the points that are the right ends of the segments characterise the moments of leaving the queuing system of customers after their service during a deterministic time. This paper considers the case when the intensity of the input Poisson flow of customers into a system with an infinite number of devices has a pronounced maximum. The possibility of smoothing this maximum by reducing the service time is being investigated. This problem arises when analysing consumer service systems and conveyor production systems [20–23]. The task of smoothing the peak load was also considered in the papers [24–26] in relation to PC software testing systems. However, in these papers, the assumption was made about a special relationship between the intensity of the input flow and the intensity of service.

Such a statement of the question is caused by numerous problems for specialists in the subject areas when processing and interpreting the data they receive. These include observations of the long-term operation of copiers, in which the quality of duplicated images drops sharply. Apparently, this is due to the sintering of copier powder particles during prolonged heating. The dependence on the particle size was also observed in experiments to protect ball-bearing rings from collisions with balls by including polymer particles in the ball-bearing oil. It turned out that with a decrease in the size of these particles, the uptime of the ball bearing increases markedly. Another task was the observation of geographers for the detection of animal tracks, depending on meteorological conditions. In particular, the possibility of detecting a trace is significantly affected by the thickness of the snow cover. In addition, if such a trace study is carried out over a large area, then the financial conditions for organising observations also come to the fore. Many data-processing problems arise in the "Smart City" program. Traffic jams significantly affect the quality of public services. Therefore, recently, service centres, for example, sports complexes, are switching from hard copies of user subscriptions to electronic cards. Such maintenance allows users not to associate the time of their arrival for service with its periodic start.

The study of such phenomena creates an impetus for the development of mathematical and statistical methods for the analysis of complex systems. Such problems arise from various practical applications, and the use of mathematical models in solving them turns out to be a rather complicated procedure. Therefore, it is more convenient to build not very accurate, but rather easily calculated simulation models, in which the dependence of performance indicators on model parameters is explicit.

The construction of such models requires the combination of several rather heterogeneous mathematical techniques. Here, it is very convenient to model an inhomogeneous

Poisson flow of points with stamps pasted on them, the parameters of which are weight, size, etc., characteristics that are related to the intensity of the flow. The construction of asymptotic relations of the dependence of the efficiency index of such a system on the parameters of the marks requires the use of inequalities commonly used for probabilistic metrics when analysing the convergence rate in the limit theorems of probability theory. Formulas of stochastic geometry can also serve as a convenient tool for constructing such relations, especially in the framework of a Boolean model based on the Poisson flow construction. Another convenient technique for working with point flows of an inhomogeneous Poisson flow is the method of interfering (hidden) parameter. In this case, we can use well-known methods for analysing statistical samples in the presence of an interfering parameter. All these circumstances require, when constructing and studying the systems under consideration, the connection of heterogeneous mathematical elements: the choice of an inhomogeneous Poisson flow of points, the choice of stamps glued to the flow points, the choice of the ratio between the intensity of the flow of points and the characteristic of the brand, the choice of the system efficiency indicator and the choice of a mathematical method for analysing the dependence of the efficiency indicator on the parameter of stamps.

The established effects of the convergence of the system efficiency indicator to zero with the aspiration to infinity of the parameter of grinding marks are new. Methods of the theory of probability metrics [8], methods of the theory of random sets [2], methods of the theory of Poisson flows [1], and integral formulas for the parameter of the Poisson distribution of the number of points of the Poisson flow in a certain segment were used to study them. A characteristic feature of these models is a special choice of the efficiency indicator, the grinding parameter of the stamps glued to the flow points, and the relationship between the efficiency indicator and the grinding parameter. The proposed approach is phenomenological in nature. It is initiated and justified by the results of real observations and experiments.

2. Proximity of the Poisson Flow of Points to the Intensity Function When Grinding the Masses of Points

This mathematical model is based on the notion of a Poisson flow of Π points on r-dimensional Euclidean space E^r. The number of points of the Poisson flow in the Lebesgue-measurable subset $A \subseteq E^r$ does not depend on the number of points in any other Lebesgue-measurable subset $B \subseteq E^r\ :\ \ A \bigcap B = \oslash$ and obeys the Poisson distribution with the parameter $\Lambda(A)$. Next, we will assume that there exists a piecewise continuous function $\lambda(x),\ x \in E^r$ such that for any Lebesgue-measurable set $A \subseteq E^r$, the parameter $\Lambda(A) = \int_A \lambda(x)dx$. The function $\lambda(x),\ x \in E^r$, is called the intensity of the Poisson flow Π.

Consider Poisson point flow Π on r-dimensional rectangle X with a continuous intensity function $\lambda(x)$ such that

$$\Lambda(X) = \int_X \lambda(x)dx < \infty. \tag{1}$$

If X' is a Lebesgue-measurable subset in the rectangle X, then a random number of points $n(X')$ of the Poisson flow Π in the set X' has a Poisson distribution with the parameter

$$\Lambda(X') = \int_{X'} \lambda(x)dx. \tag{2}$$

Now let us denote Π_m Poisson flow on the rectangle X with the intensity function $m\lambda(x)$. Let us put $n_m(X')$ the number of flow points Π_m on the set X' and define random variables $N_m(X') = \dfrac{n_m(X')}{m}$. Here, the random variable $N_m(X')$ characterises the total mass of the flow points Π_m in the set X', assuming that the mass of each point is $1/m$.

We introduce the distance ρ_m between the point flow Π_m, in which each point has a mass $1/m$, and the intensity function $\lambda(x)$. To do this, we denote $\mathcal{X}$ as the partition of the rectangle X by a finite number $n(\mathcal{X})$ disjoint and Lebesgue-measurable subsets $X'_1, \ldots, X'_{n(\mathcal{X})}$. Let $\mathcal{A}$ be the set of all possible such partitions of $\mathcal{X}$; then, the distance ρ_m is determined by the equality

$$\rho_m = \left(\sup_{\mathcal{X} \in \mathcal{A}} E \sum_{i=1}^{n(\mathcal{X})} (\Lambda(X'_i) - N_m(X'_i))^2 \right)^{1/2}. \quad (3)$$

It follows from the properties of the Poisson flow Π_m that random variables $N_m(X'_i)$, $i = 1, \ldots, n(\mathcal{X})$ are independent. This property is based on the following statements.

In probability, statistics and related fields, a Poisson point process is a type of random mathematical object that consists of points randomly located on a mathematical space with the essential feature that the points occur independently of one another [2]. Consider a collection of disjoint and bounded subregions of the underlying space. By definition, the number of points of a Poisson point process in each bounded subregion will be completely independent of all the others. This property is known under several names such as complete randomness, complete independence [27], or independent scattering [4,6], and it is common to all Poisson point processes. In other words, there is a lack of interaction between different regions and the points in general [28], which motivates the Poisson process, being sometimes called a purely or completely random process [27].

Therefore, the equalities are fulfilled $\Lambda(X'_i) = EN_m(X'_i)$,

$$E \sum_{i=1}^{n(\mathcal{X})} (\Lambda(X'_i) - N_m(X'_i))^2 = \sum_{i=1}^{n(\mathcal{X})} E(\Lambda(X'_i) - N_m(X'_i))^2 =$$

$$= \sum_{i=1}^{n(\mathcal{X})} Var N_m(X'_i) = \frac{1}{m^2} \sum_{i=1}^{n(\mathcal{X})} m\Lambda(X'_i) = \frac{\Lambda(X)}{m},$$

from which, due to Formula (3), we obtain

$$\rho_m = \sqrt{\frac{\Lambda(X)}{m}}. \quad (4)$$

Relation (4) shows how, when splitting the mass of points of the Poisson flow Π_m into m of identical parts, the distance between the flow Π_m and the function $\lambda(x)$ tends to zero at $m \to \infty$.

Remark 1. *This ratio can be used to study the quality of work of* 2*D and* 3*D printers and copiers depending on the mass of individual flow points, which can be interpreted as the mass of flow particles.*

Remark 2. *The results obtained in this section have not been previously used in the analysis of copiers and 2D and 3D printers. They have made it possible to analyse the operation of these devices during heating and sintering, as well as during the grinding of powder particles used in these devices.*

3. Covering Points of the Poisson Flow on the Plane with Cubes of Variable Size

Suppose there is Poisson point flow Π_m with constant intensity $m\overline{\lambda}$, given on the unit square X, whose one side coincides with the axis of the abscissa. Let a square with a side of length $m^{-1/3}$, rotated by a random angle φ (between the axis of the abscissa and the diagonal of the glued square) be glued to each point of the flow Π_m. The angle φ has a uniform distribution on the segment $[0, \pi]$. Thus, the set of squares glued to X forms a random set obeying the Boolean model [2].

Denote S_m as the complement of this random set to the unit square of X. According to Steiner's theorem [2], for the mathematical expectation α_m of the area of a random set S_m, the relation is fulfilled

$$\alpha_m = \exp(-\overline{\lambda} m^{1/3}) \to 0,\ m \to \infty. \tag{5}$$

Formally, we can assume that every square glued to a flow point Π_m on the plane is the side of a cube parallel to the sputtering plane with a volume $1/m$. Then, the mathematical expectation of the total volume of cubes is $\beta_m = \overline{\lambda}$. In the case, when this total volume $\beta_m = \overline{\lambda} m^{-\gamma},\ 0 \le \gamma < 1/3$, it is not difficult to obtain the ratio

$$\alpha_m = \exp(-\overline{\lambda} m^{1/3-\gamma}) \to 0,\ m \to \infty. \tag{6}$$

The obtained relations show that, when the cubes are crushed while maintaining or decreasing their average total volume, the area of the random set S_m decreases to zero.

Remark 3. *Along with shredding cubes, the proposed approach can be applied to coating a flat surface with shredding balls. In this case, instead of squares, circles are considered, which are projections of balls with a diameter of $m^{-1/3}$ on the plane. Then, the relations are fulfilled*

$$\alpha_m = \exp(-\pi\overline{\lambda} m^{1/3}/4) \to 0,\ m \to \infty,\ \beta_m = \pi\overline{\lambda}/6. \tag{7}$$

Remark 4. *Such limiting ratios can be useful in analysing the quality of the coating obtained by powder metallurgy. Another possible application of these results may be to evaluate the coating of the ice surface with notches in order to reduce the possibility of people sliding on it.*

Remark 5. *The results of this section are based on the analysis of powder metallurgy processes. They show that the grinding of powder particles makes it possible to increase the protective properties of the particle layer on the metal surface. This one is being investigated analytically. The results are very difficult to repeat using the equations of gas dynamics and the processes of gluing solid particles to a metallic surface. The results of this section are the development of well-known results in the field of stochastic geometry, which continue to develop to date [11].*

4. Method of Eliminating the Interfering Parameter in Statistics of Poisson Points Flow

Let the Lebesgue-measurable and disjoint regions be distinguished on the plane $G_k,\ k = 1, \ldots, n$. Poisson point flow Π with continuous intensity $\lambda(x)$ is given, and the relations are fulfilled

$$\overline{\lambda}_k = \int_{G_k} \lambda(x)dx < \infty,\ k = 1, \ldots, r,\ \overline{\lambda} = \sum_{k=1}^{r} \overline{\lambda}_k.$$

Denote $\Lambda_k = \dfrac{\overline{\lambda}_k}{\overline{\lambda}}$ and consider the flow Π_m with the intensity function $m\lambda(x)$. Let each point of the flow Π_m be independent of other points and, from its coordinates with probability p, enter the flow $\overline{\Pi}_m$. Then, the flow $\overline{\Pi}_m$ due to the point colouring theorem of the Poisson flow [1] is Poisson with intensity $pm\lambda(x)$. Therefore, the number n_k of flow points $\overline{\Pi}_m$ in the subdomain G_k has a Poisson distribution with the parameter $pm\overline{\lambda}_k$, and the sum $n = \sum_{k=1}^{r} n_k$ has a Poisson distribution with the parameter $pm\overline{\lambda}$.

Theorem 1. *The convergence in probability of the random variable $N_k = \dfrac{n_k}{n}$ to the parameter Λ_k is valid when $m \to \infty$.*

Proof. Due to the properties of the Poisson distribution, the relations are fulfilled

$$En_k = Var\ n_k = pm\overline{\lambda}_k,\ k = 1, \ldots, r;\ En = Var\ n = pm\overline{\lambda}. \tag{8}$$

It follows from the equalities (8) that

$$Var\,\frac{n_k}{pm\bar{\lambda}_k}=\frac{1}{pm\bar{\lambda}_k},\ Var\,\frac{n}{pm\lambda}=\frac{1}{pm\bar{\lambda}}.$$

From Chebyshev's inequality, we obtain that for any $\varepsilon,\ 0<\varepsilon<1$, and for $m\to\infty$

$$P\left(1-\varepsilon\leq\frac{n_k}{pm\bar{\lambda}_k}\leq 1+\varepsilon\right)\geq 1-\frac{1}{pm\bar{\lambda}_k\varepsilon^2}\to 1,\ k=1,\dots,r,$$

$$P\left(1-\varepsilon\leq\frac{n}{pm\bar{\lambda}}\leq 1+\varepsilon\right)\geq 1-\frac{1}{pm\bar{\lambda}\varepsilon^2}\to 1. \tag{9}$$

Using Formulas (8) and (9), it is not difficult for any $\varepsilon,\ 0<\varepsilon<\frac{1}{2}$, to obtain the inequality

$$P\left(1-2\varepsilon\leq\frac{N_k}{\Lambda_k}\leq 1+4\varepsilon\right)\geq P\left(\frac{1-\varepsilon}{1+\varepsilon}\leq\frac{N_k}{\Lambda_k}\leq\frac{1+\varepsilon}{1-\varepsilon}\right)\geq 1-\frac{1}{pm\bar{\lambda}_k\varepsilon^2}-\frac{1}{pm\bar{\lambda}\varepsilon^2}. \tag{10}$$

From the inequality $\Lambda_k\leq 1$, and from Formula (10), we obtain the relation

$$P(-2\varepsilon\leq N_k-\Lambda_k\leq 4\varepsilon)\geq 1-\frac{1}{pm\lambda_k\varepsilon^2}-\frac{1}{pm\lambda\varepsilon^2}\to 1,\ m\to\infty. \tag{11}$$

This proves the statement of Theorem 1. □

Therefore, the relation N_k, for $m\to\infty$, is a consistent estimate of the parameter Λ_k, free of probability p, playing the role of an interfering parameter in this problem.

Remark 6. *Using Inequalities (11), we can assume that $p=p(m)=m^{-\delta},\ 0\leq\delta<1$, and establish convergence by probability N_k to Λ_k at $m\to\infty$.*

Remark 7. *This problem arises when comparing accounts of the number of animal tracks in the snow when the interfering parameter takes on different values caused by different meteorological and economic characteristics in different years.*

Remark 8. *The results obtained in this section show that the analysis of traces of rare animals requires more careful processing of the results obtained. This becomes especially important when traces are investigated over a large area and data collection turns into an industrial statistics procedure.*

5. Peak Loads in the Queuing System $M|D|\infty$

Consider a queuing system with a nonstationary Poisson input flow of intensity $\lambda(t),\ t\geq 0$, with a deterministic service time a and an infinite number of servers. Let the intensity of Poisson input flow $\lambda(t),\ 0\leq t\leq T$, be a continuously differentiable function, and at the point $t_*,\ 0<t_*<T-a$ has a single extremum maximum.

Denote $n(t)$ as the number of customers in the queuing system at time t. It is obvious that the random variable $n(t)$ has a Poisson distribution with the parameter

$$\Lambda(t)=\int_{\max(0,t-a)}^{t}\lambda(u)du. \tag{12}$$

Designate $\lambda(t_*)=\lambda^*,\ \Lambda^*=\sup\limits_{0\leq t\leq T}\Lambda(t)$.

Let us investigate the dependence of the efficiency indicator Λ^* on the parameters $\lambda^*,\ a$. First, focus on the analysis of the smoothing of the peak load determined by the intensity of the input stream $\lambda(t)$, with a decrease in the parameter a. It is obvious that the inequality is true

$$\Lambda^* \le a\lambda^*. \tag{13}$$

Thus, reducing the parameter a leads to a decrease in the value of Λ^*.

Let us now proceed to a more detailed study of the dependence Λ^* on the parameter a. From Formula (12), it follows that the equalities are fulfilled

$$\Lambda(t) = \int_0^t \lambda(u)du \;\; 0 \le t \le a, \; \Lambda(t) = \int_{t-a}^t \lambda(u)du, \;\; a \le t \le T. \tag{14}$$

Now calculate the derivative of twice piecewise continuously differentiable function $\Lambda(t)$ for $t \neq a$:

$$\dot{\Lambda}(t) = \lambda(t), \;\; 0 \le t < a \le T, \; \dot{\Lambda}(t) = \lambda(t) - \lambda(t-a), \;\; a < t \le T. \tag{15}$$

Theorem 2. *The function $\Lambda(t)$ has a single extremum-= maximum t^* on the segment $[0,T]$ and $a \le t^* \le t_* + a$.*

Proof. It follows from Formula (15) that the inequalities are satisfied

$$\dot{\Lambda}(t) \ge 0, \;\; 0 \le t < a; \; \dot{\Lambda}(t) \le 0, \; t_* + a \le t \le T. \tag{16}$$

(1) Assume that $0 \le t_* \le a, \; \lambda(0) \ge \lambda(a)$. From (16), we have that the function $\Lambda(t)$ is non-decreasing on $[0,a]$ and non-increasing on $[t_* + a, T]$. Calculate now

$$\sup_{a<t\le t_*+a} \dot{\Lambda}(t) = \sup_{a<t\le t_*+a} (\lambda(t) - \lambda(t-a)) \le \sup_{a<t\le t_*+a} \lambda(t) - \inf_{a<t\le t_*+a} \lambda(t-a) =$$

$$= \lambda(a) - \inf_{0<t\le t_*} \lambda(t) = \lambda(a) - \lambda(0) \le 0.$$

Consequently, the function $\Lambda(t)$ has single extremum maximum at the point $t^* = a$.
(2) Assume that $0 \le t_* \le a, \; \lambda(0) \le \lambda(a)$,; then, from (15) and (16), we have that $\dot{\Lambda}(a+0) = \lambda(a) - \lambda(0) \ge 0, \;\; \dot{\Lambda}(t_* + a) \le 0$. Now calculate

$$\sup_{a\le t\le t_*+a} \ddot{\Lambda}(t) = \sup_{a<t\le t_*+a} \dot{\lambda}(t) - \inf_{a<t\le t_*+a} \dot{\lambda}(t) \le 0. \tag{17}$$

Hence, we obtain the existence of a single root of the function $\dot{\Lambda}(t)$ and thus a single extremum maximum t^* for the function $\Lambda(t)$ and the inequality $a \le t^* \le t_* + a$.
(3) If $a \le t_*$, then from Formulas (15) and (16), we have

$$\dot{\Lambda}(t) \ge 0, \; 0 \le t < a; \; \dot{\Lambda}(t) \ge 0, \; a < t \le t_*; \dot{\Lambda}(t) \le 0, \; t_* + a \le t \le T.$$

Analogously with (17), we lead to

$$\dot{\Lambda}(t_*) \ge 0, \; \dot{\Lambda}(t_* + a) \le 0; \; \sup_{t_* \le t \le t_*+a} \ddot{\Lambda}(t) \le 0.$$

Consequently, we obtain the existence of a single extremum maximum t^* for the function $\Lambda(t)$ and the inequality $a \le t_* \le t^* \le t_* + a$, and thus Theorem 2 is proved. □

Let us now proceed to illustrate the results obtained in a numerical example. Suppose that for some $T > a > 0$, the function

$$\lambda(t) = \frac{\exp(-(t-b)^2/(2c))}{\sqrt{2\pi c}}, \; 0 < t < T - a, \; \lambda(t) = 0, \; t \le 0 \text{ or } T - a \le t. \tag{18}$$

Then, for $T = 10, \; b = 5, \; c = 1$, the function $\Lambda(t)$ looks like this.

Figure 1 shows how when the parameter a increases, the maximum Λ^* and the point t^* increase.

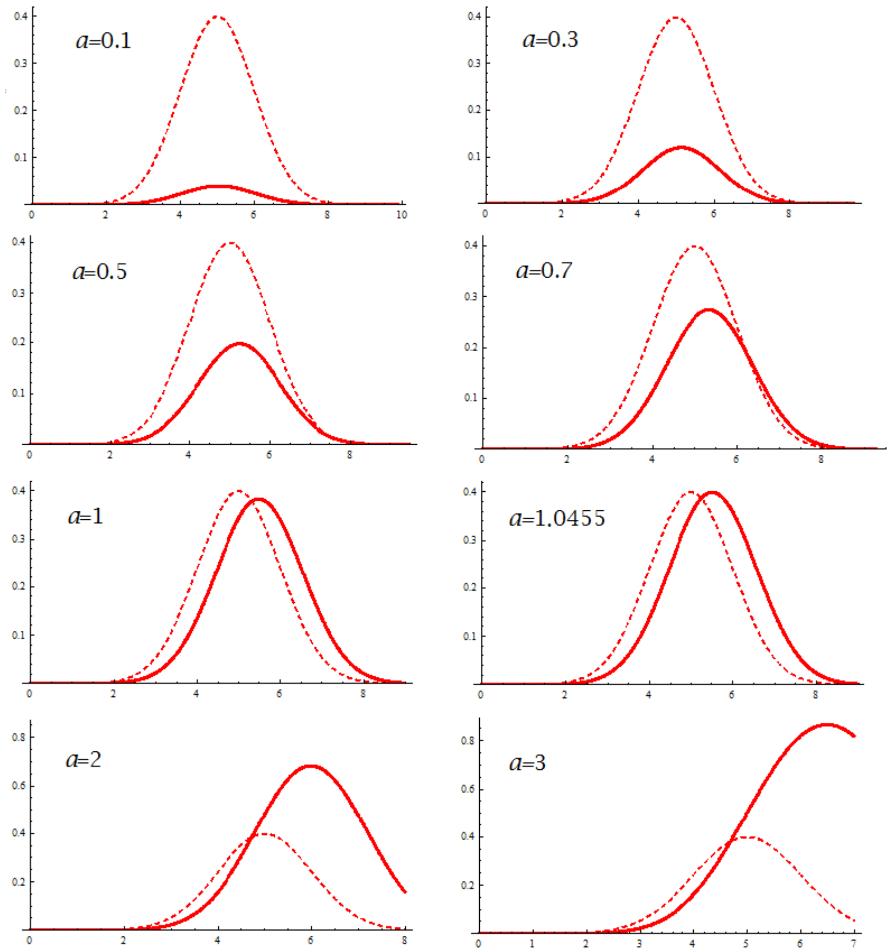

Figure 1. Graphs of functions $\lambda(t)$ (dotted line), $\Lambda(t)$ (solid line).

Remark 9. *Such a task arises when modelling sports complexes, cinemas, and production conveyor systems.*

Remark 10. *The results obtained in this section arose when comparing the previous one, based on hard copy subscriptions with periodic access of users to the service system, and the new one, using electronic cards and allowing users at any time that was convenient for them. The known methods of calculating queuing systems with variable input flow intensity [29,30] are quite complex, and their application for the $M|D|\infty$ system seems superfluous.*

6. Discussion

As possible extensions of the problems considered in this paper, there may be a transition from the Poisson flow model of points to more complex and, perhaps, more adequate models of point flows. The construction of asymptotic relations for the dependence of the flow intensity on the parameter of stamps is not always convenient. It uses very strict requirements for a small (large) parameter characterising the stamp glued to the point (mass, size, probability of detection, time of service of stamps). In the application plan, it may be sufficient to relax these requirements. Such a weakening may lead to a transition from the asymptotic analysis of these dependencies to numerical calculations. There may also be a situation when the presence of a small (large) parameter in the models may not be completely appropriate to the model under consideration. For example, in nanotechnology models, the aspiration to zero of the size or mass of particles contradicts physical conditions.

This circumstance requires greater caution when constructing a mathematical model of the analysed system and, as a consequence, greater selectivity when choosing a mathematical research method. The results obtained in this paper are applied to the analysis of new technical systems. These systems periodically arise in various applied fields and become a source of new problems at the junction of mathematical modelling and system analysis.

7. Conclusions

A model of the flow of particles forming a copy of some image was considered, and it is shown how the distance between the image formed by them and the original image decreases when the particles are crushed. The ability of flow particles to close the surface protected by them during their grinding was investigated. A model for restoring the characteristics of an inhomogeneous Poisson flow with inaccurate observation of flow points and with increasing flow intensity was analysed. The dependence of the Poisson parameter of the distribution of the customer number in a queuing system with an infinite number of servers and a deterministic service time on the peak load created by the Poisson input flow was estimated.

The obtained results indicate the presence of power–law upper bounds for various efficiency indicators, depending on the parameter characterising the grinding of stamps glued to the flow points. They were confirmed by observations of the studied systems. The established effects of the convergence of the system efficiency indicator to zero with the aspiration to infinity of the parameter of grinding marks were obtained. These results were established using methods of the theory of probability metrics, methods of the theory of random sets, methods of the theory of Poisson flows and their colouring points, and on integral formulas for the parameter of the Poisson distribution of the point numbers in the Poisson flow on a certain segment. A characteristic feature of these models is a special choice of the efficiency indicator, the grinding parameter of the stamps glued to the flow points, and the relationship between the efficiency indicator and the grinding parameter. The results presented in this paper may be applied in nanotechnology, in powder metallurgy, in ecology, and in consumer services during the implementation of the "Smart City" program. The proposed approach is phenomenological in nature and is justified by the results of real observations and experiments.

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References

1. Kingman, J.F.C. *Poisson Processes*; Oxford Science Publications: Oxford, UK, 1993.
2. Chiu, S.N.; Stoyan, D.; Kendall, W.S.; Mecke, J. *Stochastic Geometry and Its Applications*; John Wiley and Sons: New York, NY, USA, 2013.
3. Cox, D.R.; Isham, V. *Point Processes*; Chapman and Hall: London, UK, 1980.
4. Streit, R.L. *Poisson Point Processes: Imaging, Tracking, and Sensing*; Metron Inc.: Reston, VA, USA. 2010.
5. Tijms, H.C. *A First Course in Stochastic Models*; John Wiley and Sons: New York, NY, USA, 2003.
6. Moller, J; Waagepetersen, R.P. *Statistical Inference and Simulation for Spatial Point Processes*; Chapman and Hall: London, UK, 2003.
7. Haenggi, M. *Stochastic Geometry for Wireless Networks*; Cambridge University Press: Cambridge, UK, 2013.
8. Zolotarev, V. *Modern Theory of Summation of Random Variables*; VSP: Utrecht, The Netherlands, 1997.
9. Stoyan, D. On some qualitative properties of the Boolean model of stochastic geometry. *ZAMM-J. Appl. Math. Mech./Zeitschrift für Angewandte Mathematik und Mechanik* **1979**, *59*, 447–454. [CrossRef]
10. Stoyan, D. Stereological formulae for size distribution through marked point processes. *Probab. Math. Stat.* **1982**, *2*, 161–166.

11. Brandel1, M.; Brands, D.; Maike, S.; Rheinbach, O.; Schroder, J.; Schwarz, A.; Stoyan, D. Effective hyperelastic material parameters from microstructures constructed using the planar Boolean model. *Comput. Mech.* **2022**, *69*, 1295–1321. [CrossRef]
12. Hmamouche, Y. *Applications of Stochastic Geometry in the Modelling and Analysis of Wireless Networks. Networking and Internet Architecture [cs.NI]* ; Ecole Nationale Superieure Mines-Telecom Atlantique: Paris, France, 2020; ffNNT: 2020IMTA0212ff. fftel-03199210.
13. Hmamouche, Y.; Benjillali, M.; Saoudi, S.; Yanikomeroglu, H.; Di Renzo, M. New Trends in Stochastic Geometry for Wireless Networks: A Tutorial and Survey. *Proc. IEEE* **2021**, *109*, 1200–1252. [CrossRef]
14. Cox, D.R.; Hinkley, D.V. *Theoretical Statistics*; Chapman and Hall: London, UK, 1979.
15. Young, G.A.; Smith, R.L. *Essentials of Statistical Inference*; Cambridge University Press: Cambridge, UK, 2005.
16. Holevo, A.S. *Quantum Systems, Channels, Information*; De Gruyter: Berlin, Germany, 2013.
17. Holevo, A.S. Quantum noise as noncommutative stationary random process. *Int. J. Mod. Phys.* **2022**, *A37*, 2243011. [CrossRef]
18. Bocharnikov, V.N.; Krasnopeev, S.M. World Forest monitoring by network of indigenous peoples. In Proceedings of the 18th ESRI Annual User Conference, San Diego, CA, USA, 24–28 July 1998.
19. Murzin, A.A. Construction of the Amur tiger population dynamics model and forecast calculations. Its number for the period from 2015 to 2022. *Int. J. Appl. Fundam. Res.* **2018**, *11*, 333–340. (In Russian)
20. Boyle, D.E.; Yates, D.C.; Yeatman E.M. Urban Sensor Data Streams: London 2013. *IEEE Internet Comput.* **2013**, *17*, 12–20. [CrossRef]
21. Greenfield, A. *Against the Smart City*; Verso: London, UK, 2013.
22. Porshnev, S.V.; Karelin, I.A. Investigation of the features of a non-stationary single-channel queuing system in the context of the number of applications served. *Cloud Sci.* **2017**, *4*, 366–374. (In Russian)
23. Vasilyev, S.A.; Tsareva , G.O. Analysis of Queueing Systems with an Infinite Number of Servers and a Small Parameter. *RUDN J. MIPh* **2018**, *26*, 167-–175. [CrossRef]
24. Pham, H. *Software Reliability*; Springer: Singapore, 2000.
25. Yamada, S. *Software Reliability Modelling, Fundamentals and Applications*; Springer: Tokyo, Japan, 2014.
26. Minamino, Y.; Makita, Y.; Inoue, S.; Yamada, S. Efficiency evaluation of software faults correction based on queuing simulation. *Mathematics* **2022**, *10*, 1438. [CrossRef]
27. Daley, D.J.; Vere-Jones, D. *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure*; Springer: Berlin/Heidelberg, Germany, 2007.
28. Feller, W. *An Introduction to Probability Theory and Its Applications, Volume 2*; John Wiley and Sons: Hoboken, NJ, USA, 1971.
29. Ivnitsky, V. *Theory of Queuing Networks*; Fizmatlit: Moscow, Russia, 2004. (In Russian)
30. Katrakhov, V; Ryzhkov, D. *Introduction to the Functional-Analytical Method in the Dynamic Theory of Queuing*; DVGU Publishing House: Vladivostok, Vladivostok, 2004. (In Russian)

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