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Article

Curves Related to the Gergonne Point in an Isotropic Plane

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Abstract: The notion of the Gergonne point of a triangle in the Euclidean plane is very well known, and the study of them in the isotropic setting has already appeared earlier. In this paper, we give two generalizations of the Gergonne point of a triangle in the isotropic plane, and we study several curves related to them. The first generalization is based on the fact that for the triangle ABC and its contact triangle $A_iB_iC_i$, there is a pencil of circles such that each circle k_m from the pencil the lines AA_m, BB_m, CC_m is concurrent at a point G_m, where A_m, B_m, C_m are points on k_m parallel to A_i, B_i, C_i, respectively. To introduce the second generalization of the Gergonne point, we prove that for the triangle ABC, point I and three lines q_1, q_2, q_3 through I there are two points $G_{1,2}$ such that for the points Q_1, Q_2, Q_3 on q_1, q_2, q_3 with $d(I, Q_1) = d(I, Q_2) = d(I, Q_3)$, the lines AQ_1, BQ_2 and CQ_3 are concurrent at $G_{1,2}$. We achieve these results by using the standardization of the triangle in the isotropic plane and simple analytical method.

Keywords: isotropic plane; Gergonne point; generalized Gergonne points

MSC: 51N25

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1. Introduction

An isotropic plane is a projective plane with an absolute figure (f, F) consisting of a real line f and a real point $F \in f$. Isotropic lines are all lines incident with F, and isotropic points are all points incident with f. Two lines intersecting at an isotropic point are called parallel lines. Analogously, any pair of distinct points joined by an isotropic line is said to be parallel. The standard affine model of the isotropic plane is obtained by setting $x_0 = 0$ for the equation of f, and $(0,0,1)$ for the coordinates of F. In this model, the coordinates of points are defined by $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$. The isotropic lines are given by the equations $x = const$. The points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ are parallel if $x_A = x_B$. The isotropic distance $d(A, B)$ of a pair of non-parallel points is defined by $d(A, B) = x_B - x_A$, as explained in [1].

We say that a triangle in the isotropic plane is allowable if all its sides are non-isotropic lines. It was shown in [2] that each allowable triangle can be set in the standard position by choosing an appropriate coordinate system. Such a triangle ABC is inscribed into the circle with the equation $y = x^2$ and has vertices of the form

$$A(a, a^2), \quad B(b, b^2), \quad C(c, c^2), \tag{1}$$

with $a + b + c = 0$.

The following abbreviations

$$p = abc, \quad q = ab + bc + ca, \tag{2}$$

together with their repercussions

$$a^2 + b^2 + c^2 = -2q, \quad a^3 + b^3 + c^3 = 3p, \tag{3}$$

will be useful. In order to prove that some geometric fact is valid for each allowable triangle, it is sufficient to prove it for a standard triangle.

The Gergonne point of a triangle in the isotropic plane was studied in [3], where it was shown that the incircle (excircle) of the standard triangle ABC has the equation

$$k_i \quad \ldots \quad y = \frac{1}{4}x^2 - q, \tag{4}$$

and the contact points are given by

$$A_i(-2a, bc - 2q), \quad B_i(-2b, ca - 2q), \quad C_i(-2c, ab - 2q). \tag{5}$$

The common intersection point

$$\Gamma\left(-\frac{3p}{q}, -\frac{4q}{3}\right) \tag{6}$$

of the lines AA_i, BB_i, and CC_i is called the *Gergonne point* of the triangle ABC.

We study some curves related to the Gergonne point in the isotropic plane, and we present a sort of generalizations of the Gergonne point in the Euclidean case.

2. Materials and Methods

The Gergonne point of the triangle ABC in the Euclidean plane is the intersection point of three lines AA_i, BB_i, CC_i, where A_i, B_i, C_i are the contact points of the triangle and its incircle. In [4], the following generalization is given: let c be a circle concentric to the inscribed circle with the center I and let $\overline{A}_i, \overline{B}_i, \overline{C}_i$ be the intersections of c with IA_i, IB_i, IC_i, respectively. Then, the lines $A\overline{A}_i$, $B\overline{B}_i$, and $C\overline{C}_i$ are concurrent. The analogous situation in the isotropic plane is described in Theorem 1. In order to make the proofs simpler, we use the standardization of triangles. The calculation tool is purely analytical.

3. Results

Let $K(m)$ be the pencil of circles k_m with the equation of the form

$$k_m \quad \ldots \quad y = \frac{1}{4}x^2 + m, \tag{7}$$

where $m \in \mathbb{R}$. The inscribed circle k_i belongs to the pencil $K(m)$.

Theorem 1. *Let ABC be the standard triangle, $A_iB_iC_i$ its contact triangle, and k_m a circle of the pencil $K(m)$ given by the Equation (7). Let A_m, B_m, C_m be the points of k_m parallel to A_i, B_i, C_i, respectively. The lines AA_m, BB_m, CC_m are concurrent at a point G_m.*
When the circle k_m runs through the pencil $K(m)$, the points G_m form a special hyperbola.

Proof of Theorem 1. The points $A_m, B_m, C_m \in k_m$ parallel to A_i, B_i, C_i have the coordinates

$$A_m(-2a, a^2 + m), \quad B_m(-2b, b^2 + m), \quad C_m(-2c, c^2 + m).$$

Therefore, the lines AA_m, BB_m, CC_m have the equations

$$y = -\frac{m}{3a}x + a^2 + \frac{m}{3}, \quad y = -\frac{m}{3b}x + b^2 + \frac{m}{3}, \quad y = -\frac{m}{3c}x + c^2 + \frac{m}{3}.$$

They all pass through the point

$$G_m\left(\frac{3p}{m}, -q+\frac{m}{3}\right). \tag{8}$$

Indeed, the calculation $-\frac{m}{3a}\frac{3p}{m}+a^2+\frac{m}{3} = -\frac{abc}{a}+a(-b-c)+\frac{m}{3} = -q+\frac{m}{3}$ gives a proof for the line AA_m.

All points G_m lie on the conic

$$xy+qx-p=0,$$

which is according to [1], a special hyperbola, see Figure 1. □

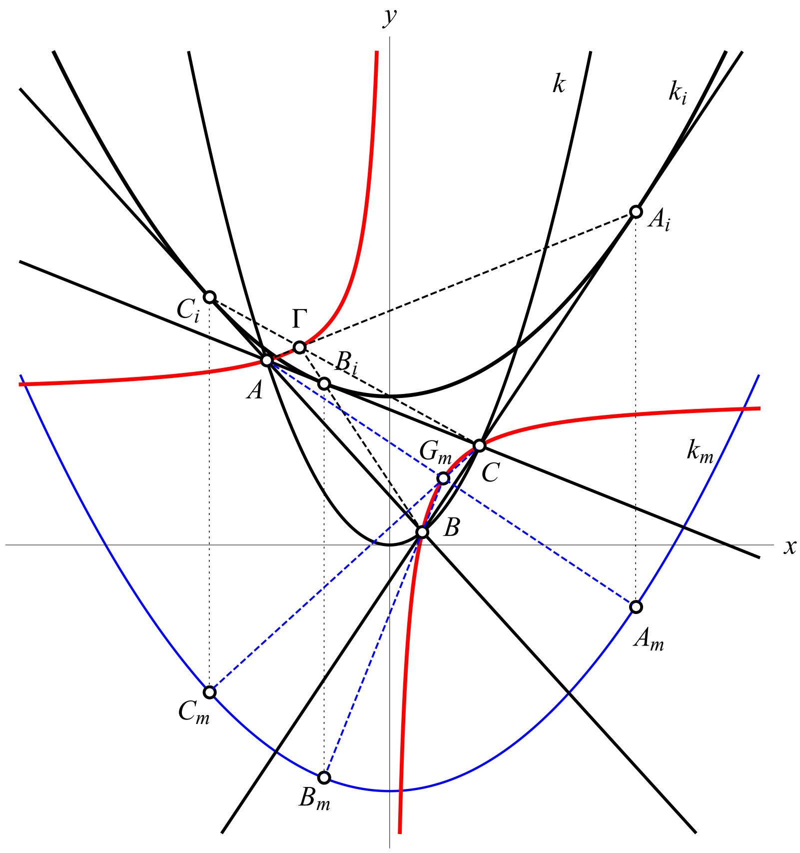

Figure 1. The locus of generalized Gergonne points of the triangle ABC.

The point G_m from Theorem 1 can be called the *generalized Gergonne point* for the triangle ABC and the circle k_m.

The Gergonne point Γ of the triangle ABC is identical to G_{-q}.

The locus of generalized Gergonne points also passes through the vertices of the triangle ABC since $G_{3bc}=A$, $G_{3ca}=B$, and $G_{3ab}=C$.

In [5,6], the authors gave some further generalizations of the concept of Gergonne point in the Euclidean case. Here, we study some analogues of these results in the isotropic case.

Theorem 2. *Let ABC be the standard triangle, I a point in the isotropic plane and q_1, q_2, q_3 three lines through I. There are at most two values $d \in \mathbb{R}\backslash\{0\}$ such that for points Q_1, Q_2, Q_3 on q_1, q_2, q_3 with $d(I,Q_1)=d(I,Q_2)=d(I,Q_3)=d$, the lines AQ_1, BQ_2 and CQ_3 are concurrent.*

Proof of Theorem 2. Let I be given by the coordinates $(\overline{x}, \overline{y})$, and let q_i have the equations $y = k_i(x - \overline{x}) + \overline{y}$, $i = 1, 2, 3$. All points T such that $d(I, T) = d$ lie on the isotropic line with the equation $x = \overline{x} + d$. Therefore, points Q_i have coordinates $(d + \overline{x}, k_i d + \overline{y})$, see Figure 2.

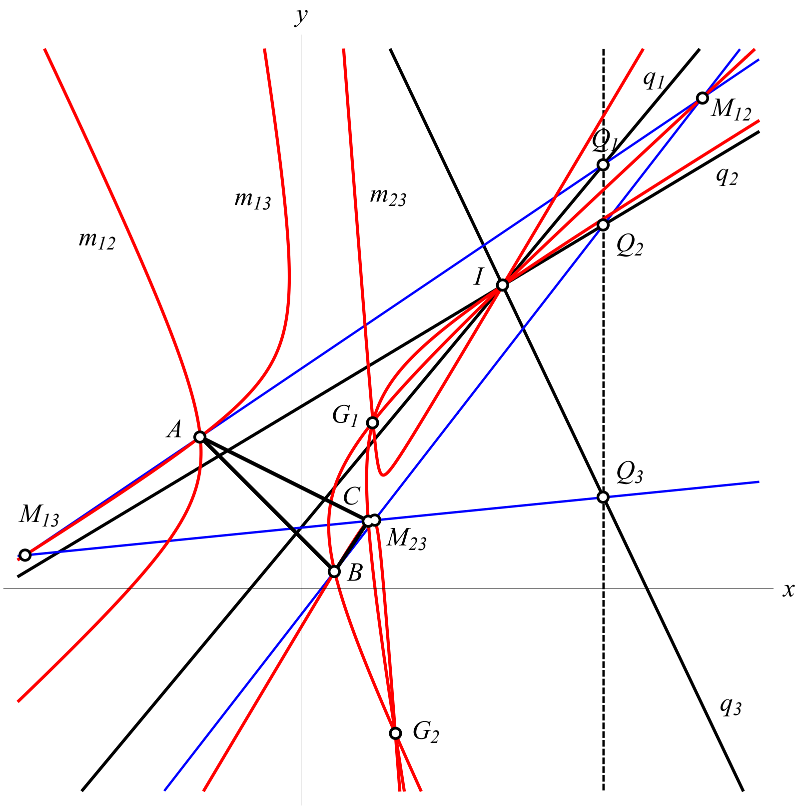

Figure 2. Generalized Gergonne points G_1, G_2 for the triangle ABC, point I and lines q_1, q_2, q_3 through I.

Thus,

$$\begin{aligned} AQ_1 \quad \ldots \quad y &= \frac{k_1 d + \overline{y} - a}{d + \overline{x} - a}(x - a) + a^2, \\ BQ_2 \quad \ldots \quad y &= \frac{k_2 d + \overline{y} - b}{d + \overline{x} - b}(x - b) + b^2, \\ CQ_3 \quad \ldots \quad y &= \frac{k_3 d + \overline{y} - c}{d + \overline{x} - c}(x - c) + c^2. \end{aligned} \tag{9}$$

Let $M_{12} = AQ_1 \cap BQ_2$, $M_{23} = BQ_2 \cap CQ_3$, $M_{13} = AQ_1 \cap CQ_3$. Some trivial but long calculations deliver the following values of d for which these three points coincide

$$d_{1,2} = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}}, \tag{10}$$

where

$$\begin{aligned} \mathcal{A} &= k_1(c^2 - b^2) + k_2(a^2 - c^2) + k_3(b^2 - a^2) + k_1k_2(b - a) + k_2k_3(c - b) + k_1k_3(a - c), \\ \mathcal{B} &= k_1(c^3 - b^3) + k_2(a^3 - c^3) + k_3(b^3 - a^3) + (k_1k_2 + 2k_3\overline{x})(b^2 - a^2) \\ &\quad + (k_1k_3 + 2k_2\overline{x})(a^2 - c^2) + (k_2k_3 + 2k_1\overline{x})(c^2 - b^2) + (k_1k_2\overline{x} - k_3\overline{y})(b - a) \\ &\quad + (k_2k_3\overline{x} - k_1\overline{y})(c - b) + (k_1k_3\overline{x} - k_2\overline{y})(a - c), \\ \mathcal{C} &= (p - \overline{xy})[k_1(c - b) + k_2(a - c) + k_3(b - a)] + \overline{x}\left[k_1(c^3 - b^3) + k_2(a^3 - c^3) + k_3(b^3 - a^3)\right] \\ &\quad + (\overline{x}^2 - \overline{y})\left[k_1(c^2 - b^2) + k_2(a^2 - c^2) + k_3(b^2 - a^2)\right]. \end{aligned}$$

The numbers $d_{1,2}$ are real and different, real and identical, or a pair of complex conjugate numbers depending on the value of $\mathcal{B}^2 - 4\mathcal{AC}$. $\square$

The values $d_{1,2}$ determine the points $G_{1,2}$, the common points of the lines AQ_1, BQ_2, and CQ_3. The points G_1 and G_2 can be real and different, complex conjugate, or coinciding depending on the value of $\mathcal{B}^2 - 4\mathcal{AC}$. They are called generalized Gergonne points for the triangle ABC and point I and lines q_1, q_2, q_3, through it.

Remark: By eliminating the parameter d from the first two equations of (9), we obtain the equation

$$\frac{(\overline{y}-a)(x-a)-(\overline{x}-a)(y-a^2)}{y-a^2-k_1(x-a)} = \frac{(\overline{y}-b)(x-b)-(\overline{x}-b)(y-b^2)}{y-b^2-k_2(x-b)}. \tag{11}$$

It represents the locus m_{12} of points M_{12} when d runs through $\mathbb{R}$. The curve m_{12} is obviously a conic. In the same manner, we conclude that the loci of M_{13} and M_{23} are conics as well. According to Theorem 2, three loci m_{12}, m_{13} and m_{23} share two further common points $G_{1,2}$ except the fixed point I, see Figure 2.

Note that, if directions k_1, k_2, k_3 are given, $\mathcal{B}^2 - 4\mathcal{AC}$ from (10) is a quadratic function of $I(\overline{x}, \overline{y})$. This means that there will be two, one, or none real points $G_{1,2}$ depending on whether the point I is located outside, on or inside the conic i with the equation $\mathcal{B}^2 - 4\mathcal{AC} = 0$, see Figure 3.

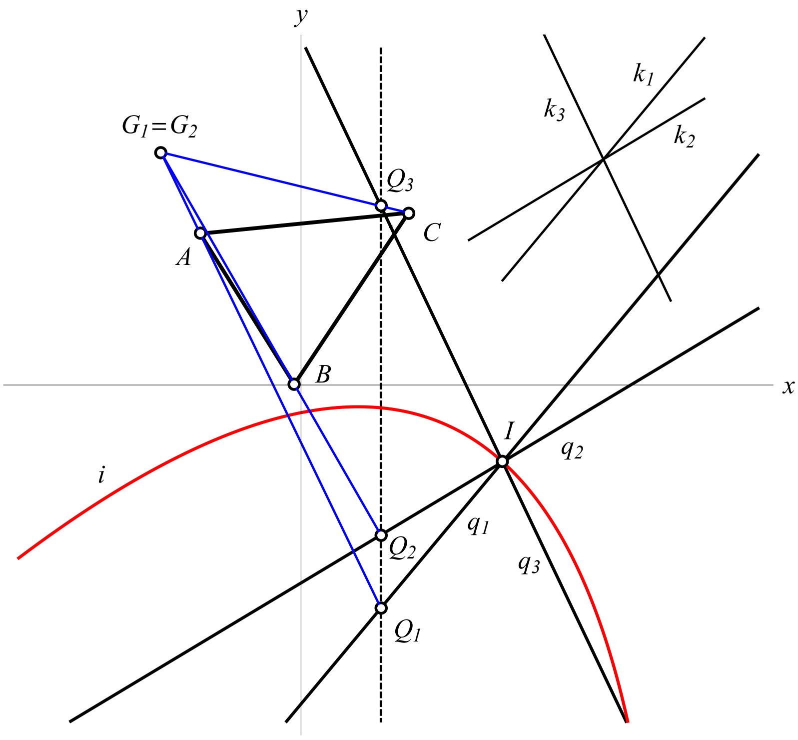

Figure 3. The locus i of all points I for which two generalized Gergonne points G_1, G_2 of the triangle ABC in directions k_1, k_2, k_3 coincide.

Now, we can also state:

Theorem 3. *Let ABC be the standard triangle and k_1, k_2, k_3 three directions. All points I for which there is a unique value $d \in \mathbb{R}\backslash\{0\}$ such that for points Q_1, Q_2, Q_3 on lines q_1, q_2, q_3 in directions k_1, k_2, k_3 with $d(I, Q_1) = d(I, Q_2) = d(I, Q_3) = d$ the lines AQ_1, BQ_2, and CQ_3 are concurrent lie on a parabola.*

Proof of Theorem 3. It is left to prove that the conic i with the equation $\mathcal{B}^2 - 4\mathcal{AC} = 0$ is a parabola. After replacing $\overline{x}, \overline{y}$ with x, y and introducing notations

$$\begin{aligned} \mathcal{D} &= k_1(c^2 - b^2) + k_2(a^2 - c^2) + k_3(b^2 - a^2), \\ \mathcal{F} &= k_1(c - b) + k_2(a - c) + k_3(b - a), \end{aligned}$$

the terms of the highest degree in the equation of i are

$$[(\mathcal{A} - \mathcal{D})x - \mathcal{F}y]^2.$$

Thus, the conic i touches the absolute line in one point, the isotropic point of the line $y = \dfrac{\mathcal{A} - \mathcal{D}}{\mathcal{F}}x$. □

4. Discussion and Conclusions

This study gives a contribution to the very rich base of triangle properties in the isotropic plane. We have proved that for a triangle ABC and its contact triangle $A_iB_iC_i$, there is a pencil of circles K_m such that for each circle k_m from the pencil the lines AA_m, BB_m, CC_m are concurrent at a point G_m, where A_m, B_m, C_m are points on k_m parallel to A_i, B_i, C_i, respectively. When k_m runs through $K(m)$, the generalized Gergonne points G_m form a special hyperbola.

Further on, to each triangle ABC, a point I and three lines q_1, q_2, q_3 through I we have associated three conics intersecting at I and two generalized Gergonne points G_1 and G_2. The existence of G_1 and G_2 follows from the existence of two values d such that for points Q_1, Q_2, Q_3 on q_1, q_2, q_3 with $d(I, Q_1) = d(I, Q_2) = d(I, Q_3) = d$ the lines AQ_1, BQ_2 and CQ_3 are concurrent. For arbitrary directions, the points I, such that G_1, G_2 coincide, lie on a parabola.

In the papers [7,8], the authors studied some further curves related to Gergonne points; they studied the loci of Gergonne points in different pencils of triangles in the isotropic plane. Hence, this paper completes the investigations given there.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of Open-Access Journals

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