

Article

Oscillatory Properties of Fourth-Order Advanced Differential Equations

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Abstract: This paper presents a study on the oscillatory behavior of solutions to fourth-order advanced differential equations involving p -Laplacian-like operator. We obtain oscillation criteria using techniques from first and second-order delay differential equations. The results of this work contribute to a deeper understanding of fourth-order differential equations and their connections to various branches of mathematics and practical sciences. The findings emphasize the importance of continued research in this area.

Keywords: oscillatory behavior; p -Laplace operator; fourth-order equation; advanced differential equations

MSC: 34C10; 34K11



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1. Introduction

We take into consideration in our work the oscillatory criteria of the following fourth-order advanced differential equation

$$\begin{cases} \left(a(t)|u'''(t)|^{p-2}u'''(t) \right)' + q(t)g(u(\delta(t))) = 0, \\ a(t) > 0, a'(t) \geq 0, q(t) \geq 0, \delta(t) \geq t, \end{cases} \quad (1)$$

where the p -Laplace operator is represented by the first term with ($1 < p < \infty$), and with coefficient function $a \in C^1([t_0, \infty), \mathbb{R})$, $t \geq t_0 > 0$. Furthermore, $q, \delta \in C([t_0, \infty), \mathbb{R})$ and $g \in C(\mathbb{R}, \mathbb{R})$.

The oscillatory behavior of solutions for various classes of functional differential equations was a largely investigated field of research in recent decades. Here we recall the pioneering papers by Nehari [1] and Philos [2], and the comprehensive book of Agarwal–Grace–O’Regan [3]. We mention the works of Grace–Lalli [4], Zhang–Agarwal–Bohner–Li [5], Zhang–Li–Sun [6] (higher order equations), and Bartusek–Cecchi–Dosla–Marini [7], Ali and Bazighifan [8–14], and Agarwal–Shieh–Yeh [15] (second order equations). More related studies have been done recently, see [16–18].

Advanced differential equations include several applications in optimization, dynamical systems, and simulation techniques of engineering problems, including power systems,

control mechanisms, networking, and nanomaterials (see Hale’s book [19]). The importance of p -Laplace equations appears in several applications in the theory of elasticity as well as in the theory of continuum mechanics, see [20–22]. We recommend the publications of Li–Baculikova–Dzurina–Zhang [23] for some findings on the oscillatory behavior of equations resulting from a p -Laplace differential operator as well as the papers of Liu–Zhang–Yu [24], and Zhang–Agarwal–Li [25].

Therefore, the purpose of this study is to supplement previous work by focusing on the results in [4,26,27]. Using the integral averaging technique (see, for example, Xu–Xia [28]), together with the Riccati transformation technique (see, for example, Zhang–Li–Saker [29]) and comparison method with second-order differential equations, we obtain new criteria for the oscillation of Equation (1). We point out that when all the solutions of (1) are oscillatory, then the equation itself is called oscillatory; otherwise (1) is said non-oscillatory.

2. Auxiliary Results—Hypotheses

In this section, we summarize significant information and supplementary results from the literature that will be useful for the remainder of the paper. Additionally, we establish the notation used.

The definition below pertains to the non-oscillatory behavior of a second-order differential equation. We will utilize this definition in the technique of comparing with second-order differential equations to prove our second theorem.

Now, we take into account the following sets:

$$D = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \text{ and } D_0 = \{(t, s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

Definition 1 ([24]). *A couple of functions $(H_1, H_2) \in C(D, \mathbb{R}) \times C(D, \mathbb{R})$ is said to be of class \mathfrak{S} , if the following conditions hold:*

- (i) $H_i(t, t) = 0$ for $t \geq t_0$, and $H_i(t, s) > 0$ for $(t, s) \in D_0$, with $i = 1, 2$;
- (ii) there exist $\eta, \vartheta \in C^1([t_0, \infty), (0, \infty))$ and $(h_1, h_2) \in C(D, \mathbb{R}) \times C(D, \mathbb{R})$ such that:

$$\frac{\partial}{\partial s} H_1(t, s) + \frac{\eta'(s)}{\eta(s)} H_1(t, s) = h_1(t, s) H_1^{(p-1)/p}(t, s), \tag{2}$$

$$\frac{\partial}{\partial s} H_2(t, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(t, s) = h_2(t, s) \sqrt{H_2(t, s)}, \tag{3}$$

whenever the partial derivatives $\frac{\partial H_i}{\partial s}, i = 1, 2$, are continuous and nonpositive on D_0 .

For further convenience, we denote:

$$\begin{aligned} \zeta(t) &= \int_t^\infty \frac{1}{a^{1/(p-1)}(s)} ds, \\ \pi(s) &= \frac{h_1^p(t, s) H_1^{p-1}(t, s)}{p^p} \frac{2^{p-1} \eta(s) a(s)}{(\theta s^2)^{p-1}}, \quad \theta \in (0, 1), \\ \omega(v) &= \int_t^\infty \left(\frac{k}{a(\zeta)} \int_\zeta^\infty q(s) ds \right)^{1/(p-1)} d\zeta. \end{aligned}$$

Our goal here, as stated in the introduction, is to supplement findings in [4,26,27]. Therefore, we discuss in detail all these findings.

We point out that Li–Baculikova–Dzurina–Zhang [23], applied the Riccati transformation along with the integral averaging method, and concentrated on the way how the following equations oscillate

$$\left(a(t) |z'''(t)|^{p-2} z'''(t) \right)' + \sum_{i=1}^j q_i(t) |u(\delta_i(t))|^{p-2} u(\delta_i(t)) = 0, \quad 1 < p < \infty.$$

In Park–Moazz–Bazighifan [30], the Riccati method leads to important and sufficient conditions for the oscillation of

$$\begin{cases} \left(a(t) \left| u^{(\omega-1)}(t) \right|^{p-2} u^{(\omega-1)}(t) \right)' + q(t)g(u(\delta(t))) = 0, & \omega \text{ even,} \\ \int_t^\infty \frac{ds}{a^{1/(p-1)}(s)} = \infty. \end{cases}$$

In Zhang–Agarwal–Bohner–Li [5] and C. Zhang–Li–Sun–Thandapani [6], the method of comparison was applied along with first order equations in order to establish that all the solutions u of

$$\begin{cases} \left(\left(u^{(\omega-1)}(t) \right)^\beta \right)' + q(t)u^\alpha(\delta(t)) = 0, \\ \int_t^\infty \frac{ds}{a^{1/(p-1)}(s)} < \infty, \end{cases} \tag{4}$$

are oscillating or that $\lim_{t \rightarrow \infty} u(t) = 0$ holds whenever $\delta(t) \leq t, \alpha \leq \beta$ (with α, β being ratios of odd positive integers), and ω is even.

For the special case when $\beta = \alpha$, Zhang–Li–Saker [29] obtained several results illustrating the findings on the asymptotic behavior of (4), with $\omega = 4$. Agarwal–Grace [26] and Agarwal–Grace–O’Regan [27] considered the canonical even-order nonlinear advanced differential equation

$$\left(\left(u^{(\omega-1)}(t) \right)^\beta \right)' + q(t)u^\beta(\delta(t)) = 0, \tag{5}$$

using Riccati transformation method, where they provided several oscillatory results for (5) where $\delta(t) \geq t, \omega$ is even and β is the ratio of odd non-negative integers. For $\beta = 1$, Equation (5) becomes

$$u^{(\omega)}(t) + q(t)u(\delta(t)) = 0. \tag{6}$$

Now, Grace–Lalli [4] proved oscillatory theorems for (6) in the case where ω is even and under the condition

$$\int_{t_0}^\infty q(s)\delta^{\omega-1}(s)ds = \infty.$$

We point out that applying the above-mentioned theorems to

$$u^{(\omega)}(t) + \frac{q_0}{t^\omega}u(\rho t) = 0, \quad t \geq 1, \tag{7}$$

in the case where $\omega = 4$ and $\rho = 2$, then the hypotheses in [4,26,27] on (7) lead to show that the results in [27] improve the corresponding ones in [4]. Furthermore, the results in [26] refine the results in [4,27].

Finally, we mention a few tools that will come in handy as the paper progresses.

Lemma 1 ([6]). *Let $u \in C^\omega([t_0, \infty), (0, \infty))$ be such that $u^{(\omega)}$ does not change sign on $[t_0, \infty)$ and $u^{(\omega)} \not\equiv 0$. Assume there is $t_1 \geq t_0$ with*

$$u^{(\omega-1)}(t)u^{(\omega)}(t) \leq 0, \quad \text{for all } t \geq t_1.$$

If $\lim_{t \rightarrow \infty} u(t) \neq 0$, then we can find $t_\theta \geq t_1$ with

$$u(t) \geq \frac{\theta}{(\omega - 1)!} t^{\omega-1} \left| u^{(\omega-1)}(t) \right|, \quad \text{for all } \theta \in (0, 1), t \geq t_\theta.$$

Lemma 2 ([31]). *For $i = 0, 1, \dots, \omega$, let $u^{(i)}(t) > 0$ and $u^{(\omega+1)}(t) < 0$. Thus, we have*

$$u'(t) \frac{t^\omega}{\omega!} \leq u(t) \frac{t^{\omega-1}}{(\omega - 1)!}.$$

Remark 1 ([32]). Fixing $c_0 > 0$ and $c_1 \geq 0$, we have that

$$c_1x - c_0x^{p/(p-1)} \leq \frac{(p-1)^{p-1} c_1^p}{p^p c_0^{p-1}}, \quad \text{for all } x \in [0, +\infty).$$

By a solution of (1) we mean a function $u \in C^3([t_u, \infty), \mathbb{R})$, $t_u \geq t_0$, which has the property $a(t)|u'''(t)|^{p-2}u'''(t) \in C^1[t_u, \infty)$, and satisfies (1) on $[t_u, \infty)$. We consider only those solutions u of (1) such that $\sup\{|u(t)| : t \geq t_u\} > 0$.

The following lemma encapsulates the scenarios to be examined in the demonstrations of our results.

Lemma 3 ([29]). Let $u \in C^3([t_u, \infty), \mathbb{R})$ be an (eventually) non-negative and non-zero solution of Equation (1). Thus, one could get the below cases:

(S₁) $u(t) > 0, u'(t) > 0, u''(t) > 0, u'''(t) > 0, u^{(4)}(t) < 0,$

(S₂) $u(t) > 0, u'(t) > 0, u''(t) < 0, u'''(t) > 0, u^{(4)}(t) < 0,$

(S₃) $u(t) > 0, u''(t) > 0, u'''(t) < 0,$

for $t \geq t_1$, where $t_1 \geq t_0$ is large enough.

We are now able to present the specific hypotheses based on the facts of (1):

(H1) $\zeta(t) < \infty,$

(H2) There exist a constant k such that $\frac{g(x)}{|x|^{p-2}x} \geq k > 0,$ for $x \neq 0,$

(H3) There exist $(H_1, H_2) \in \mathfrak{S}$ and $\theta \in (0, 1)$ such that we have:

$$\limsup_{t \rightarrow \infty} \frac{1}{H_1(t, t_1)} \int_{t_1}^t (H_1(t, s)k\eta(s)q(s) - \pi(s))ds = \infty, \tag{8}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H_2(t, t_1)} \int_{t_1}^t \left(H_2(t, s)\vartheta(s)\omega(s) - \frac{\vartheta(s)h_2^2(t, s)}{4} \right) ds = \infty, \tag{9}$$

(H4) Let $\tilde{\pi}(s) := \frac{(p-1)^p}{p^p} \frac{1}{a^{1/(p-1)}(s)\zeta(s)}$. There exists $\theta_1 \in (0, 1)$ such that we have

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left(kq(s) \left(\frac{\theta_1 \delta^2(s)}{2} \right)^{p-1} \zeta^{p-1}(\delta(s)) - \tilde{\pi}(s) \right) ds = \infty. \tag{10}$$

3. Main Results

The first result of the paper introduces a theorem that employs the integral averaging technique to apply Philos-type oscillation criteria to Equation (1).

Theorem 1. If (H1)–(H4) hold, then every solution $u \in C^3([t_u, \infty), \mathbb{R})$ of (1) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} u(t) = 0$.

Proof. Arguing by contradiction, we suppose that $u \in C^3([t_u, \infty), \mathbb{R})$ is a positive solution of (1). So, we assume that $u(t)$ and $u(\delta(t))$ are positive for all $t \geq t_1$ large enough.

Now, we distinguish the following three cases (see Lemma 3):

Case 1. If (S₁) holds, then by Lemma 1, we have

$$u'(t) \geq \frac{\theta}{2} t^2 u'''(t), \quad \text{for all } \theta \in (0, 1), t \text{ large enough.} \tag{11}$$

Putting

$$\varphi(t) := \eta(t) \left(\frac{a(t)(u'''(t))^{p-1}}{u^{p-1}(t)} \right) \quad (\text{with } \eta \text{ given as in (H3)}), \tag{12}$$

we note that $\varphi(t) > 0$ for $t \geq t_1$ and we have

$$\begin{aligned} \varphi'(t) &= \eta'(t) \frac{a(t)(u'''(t))^{p-1}}{u^{p-1}(t)} + \eta(t) \frac{(a(t)(u'''(t))^{p-1})'}{u^{p-1}(t)} \\ &\quad - (p-1)\eta(t) \frac{u^{p-2}(t)u'(t)a(t)(u'''(t))^{p-1}}{u^{2(p-1)}(t)}. \end{aligned}$$

By (11) and (12), we deduce that

$$\begin{aligned} \varphi'(t) &\leq \frac{\eta'(t)}{\eta(t)} \varphi(t) + \eta(t) \frac{(a(t)(u'''(t))^{p-1})'}{u^{p-1}(t)} - (p-1)\eta(t) \frac{\theta t^2 a(t)(u'''(t))^p}{2 u^p(t)} \\ &\leq \frac{\eta'(t)}{\eta(t)} \varphi(t) + \eta(t) \frac{(a(t)(u'''(t))^{p-1})'}{u^{p-1}(t)} - \frac{(p-1)\theta t^2}{2[\eta(t)a(t)]^{\frac{1}{p-1}}} \varphi(t)^{\frac{p}{p-1}}. \end{aligned} \tag{13}$$

The equation in (1), and (13) give

$$\varphi'(t) \leq \frac{\eta'(t)}{\eta(t)} \varphi(t) - k\eta(t) \frac{q(t)u^{p-1}(\delta(t))}{u^{p-1}(t)} - \frac{(p-1)\theta t^2}{2[\eta(t)a(t)]^{\frac{1}{p-1}}} \varphi(t)^{\frac{p}{p-1}}.$$

Since $u'(t) > 0$ and $\delta(t) \geq t$, we get

$$\varphi'(t) \leq \frac{\eta'(t)}{\eta(t)} \varphi(t) - k\eta(t)q(t) - \frac{(p-1)\theta t^2}{2[\eta(t)a(t)]^{\frac{1}{p-1}}} \varphi(t)^{\frac{p}{p-1}}. \tag{14}$$

Next, we multiply both sides of (14) by $H_1(t, s)$, then we integrate each side over the interval $[t_1, t]$. So, we have

$$\begin{aligned} &\int_{t_1}^t H_1(t, s)k\eta(s)q(s)ds \\ &\leq \varphi(t_1)H_1(t, t_1) + \int_{t_1}^t \left(\frac{\partial}{\partial s} H_1(t, s) + \frac{\eta'(s)}{\eta(s)} H_1(t, s) \right) \varphi(s)ds \\ &\quad - \int_{t_1}^t \frac{(p-1)\theta s^2}{2[\eta(s)a(s)]^{\frac{1}{p-1}}} \varphi(s)^{\frac{p}{p-1}} H_1(t, s)ds. \end{aligned}$$

Using (2) (that is, the first equation of Definition 1 (ii)), we deduce that

$$\begin{aligned} &\int_{t_1}^t H_1(t, s)k\eta(s)q(s)ds \\ &\leq \varphi(t_1)H_1(t, t_1) + \int_{t_1}^t h_1(t, s)H_1^{(p-1)/p}(t, s)\varphi(s)ds \\ &\quad - \int_{t_1}^t \frac{(p-1)\theta s^2}{2[\eta(s)a(s)]^{\frac{1}{p-1}}} \varphi(s)^{\frac{p}{p-1}} H_1(t, s)ds. \end{aligned} \tag{15}$$

If we apply the inequality given in Remark 1 for

$$\begin{aligned} c_0 &= \frac{(p-1)\theta s^2}{2[\eta(s)a(s)]^{\frac{1}{p-1}} H_1(t, s)}, \\ c_1 &= h_1(t, s)H_1^{(p-1)/p}(t, s), \\ x &= \varphi(s), \end{aligned}$$

we obtain that

$$\begin{aligned} & h_1(t,s)H_1^{(p-1)/p}(t,s)\varphi(s) - \frac{(p-1)\theta s^2}{2[\eta(s)a(s)]^{\frac{1}{p-1}}}\varphi(s)^{\frac{p}{p-1}}H_1(t,s) \\ & \leq \frac{h_1^p(t,s)H_1^{p-1}(t,s)}{p^p} \frac{2^{p-1}\eta(s)a(s)}{(\theta s^2)^{p-1}}, \\ \Rightarrow & \frac{1}{H_1(t,t_1)} \int_{t_1}^t (H_1(t,s)k\eta(s)q(s) - \pi(s))ds \leq \varphi(t_1), \end{aligned}$$

a contradiction to (8).

Case 2. If (S₂) holds, for $t \geq t_1$, we have

$$\begin{aligned} \psi(t) & := \vartheta(t) \frac{u'(t)}{u(t)} > 0 \quad (\text{with } \vartheta \text{ given as in (H3)}), \\ \Rightarrow \psi'(t) & = \frac{\vartheta'(t)}{\vartheta(t)}\psi(t) + \vartheta(t) \frac{u''(t)}{u(t)} - \frac{1}{\vartheta(t)}\psi(t)^2 \text{ (by differentiation)}. \end{aligned} \tag{16}$$

If we integrate the equation in (1) over the interval $[t, m]$ and use $u'(t) > 0$, then

$$a(m)(u'''(m))^{p-1} - a(t)(u'''(t))^{p-1} = - \int_t^m q(s)g(u(\delta(s)))ds.$$

Since $u'(t) > 0$ and $\delta(t) \geq t$, we obtain that

$$\begin{aligned} & a(m)(u'''(m))^{p-1} - a(t)(u'''(t))^{p-1} \leq -ku^{p-1}(t) \int_t^m q(s)ds \quad (\text{by (H2)}) \\ \Rightarrow & a(t)(u'''(t))^{p-1} \geq ku^{p-1}(t) \int_t^\infty q(s)ds \quad (\text{passing to the limit as } m \rightarrow \infty), \\ \Rightarrow & u'''(t) \geq u(t) \left(\frac{k}{a(t)} \int_t^\infty q(s)ds \right)^{1/(p-1)}. \end{aligned}$$

Now, we integrate over the interval $[t, \infty)$, so that we have

$$\begin{aligned} & u''(t) + u(t) \int_t^\infty \left(\frac{k}{a(\zeta)} \int_\zeta^\infty q(s)ds \right)^{1/(p-1)} d\zeta \leq 0, \\ & \quad (\text{recall that } u''(t) < 0 \text{ and } u'''(t) > 0), \\ \Rightarrow \psi'(t) & \leq \frac{\vartheta'(t)}{\vartheta(t)}\psi(t) - \vartheta(t)\omega(v) - \frac{1}{\vartheta(t)}\psi(t)^2. \end{aligned} \tag{17}$$

Next, we multiply both sides of (17) (with $v = s$) by $H_2(t, s)$, then we integrate each side over the interval $[t_1, t]$. So, we have

$$\begin{aligned} & \int_{t_1}^t H_2(t,s)\vartheta(s)\omega(s)ds \\ & \leq \psi(t_1)H_2(t,t_1) + \int_{t_1}^t \left(\frac{\partial}{\partial s} H_2(t,s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(t,s) \right) \psi(s)ds \\ & \quad - \int_{t_1}^t \frac{1}{\vartheta(s)} H_2(t,s)\psi^2(s)ds. \end{aligned}$$

Using (3) (that is, the second equation of Definition 1 (ii)), we deduce that

$$\begin{aligned}
 & \int_{t_1}^t H_2(t,s)\vartheta(s)\omega(s)ds \\
 & \leq \psi(t_1)H_2(t,t_1) + \int_{t_1}^t h_2(t,s)\sqrt{H_2(t,s)}\psi(s)ds \\
 & \quad - \int_{t_1}^t \frac{1}{\vartheta(s)}H_2(t,s)\psi^2(s)ds \\
 & \leq \psi(t_1)H_2(t,t_1) + \int_{t_1}^t \frac{\vartheta(s)h_2^2(t,s)}{4}ds \\
 \Rightarrow & \frac{1}{H_2(t,t_1)} \int_{t_1}^t \left(H_2(t,s)\vartheta(s)\omega(s) - \frac{\vartheta(s)h_2^2(t,s)}{4} \right) ds \leq \psi(t_1),
 \end{aligned}$$

a contradiction to (9).

Case 3. If (S₃) holds and $\lim_{t \rightarrow \infty} u(t) \neq 0$, then since $a(t)|u'''(t)|^{p-2}u'''(t)$ is nonincreasing by (1), we have

$$a^{1/(p-1)}(s)u'''(s) \leq a^{1/(p-1)}(t)u'''(t), \quad s \geq t \geq t_1.$$

Now, we multiply both sides by $[a^{1/(p-1)}(s)]^{-1}$, then we integrate each side over the interval $[t, m]$. Thus, we get

$$\begin{aligned}
 u''(u) & \leq u''(t) + a^{1/(p-1)}(t)u'''(t) \int_t^m a^{-1/(p-1)}(s)ds, \\
 \Rightarrow 0 & \leq u''(t) + a^{1/(p-1)}(t)u'''(t)\zeta(t) \quad (\text{letting } m \rightarrow \infty), \\
 \Rightarrow \frac{-a^{1/(p-1)}(t)u'''(t)\zeta(t)}{u''(t)} & \leq 1, \tag{18}
 \end{aligned}$$

$$\Rightarrow \left(\frac{u''(t)}{\zeta(t)} \right)' \geq 0. \tag{19}$$

For $t \geq t_1$, we have

$$\begin{aligned}
 \phi(t) & = \frac{a(t)|u'''(t)|^{p-2}u'''(t)}{(u''(t))^{p-1}} < 0, \tag{20} \\
 \Rightarrow \phi'(t) & = \frac{\left(a(t)|u'''(t)|^{p-2}u'''(t) \right)'}{(u''(t))^{p-1}} - \frac{(p-1)a(t)|u'''(t)|^p}{(u''(t))^p}.
 \end{aligned}$$

Using the equation in (1), then (18) gives

$$\phi'(t) \leq \frac{-kq(t)u^{p-1}(\delta(t))}{(u''(t))^{p-1}} - \frac{(p-1)(-\phi)^{p/(p-1)}(t)}{a^{1/(p-1)}(t)}.$$

Now, Lemma 1 leads to

$$u(t) \geq \frac{\theta_1}{2}t^2u''(t). \tag{21}$$

Using (19) we get $\frac{\zeta(\delta(v))}{\zeta(v)} \leq \frac{u''(\delta(v))}{u''(v)}$. Then, we obtain that

$$\begin{aligned}
 \phi'(t) & = \frac{-kq(t)u^{p-1}(\delta(t))}{(u''(\delta(t)))^{p-1}} \frac{(u''(\delta(t)))^{p-1}}{(u''(t))^{p-1}} - \frac{(p-1)(-\phi)^{p/(p-1)}(t)}{a^{1/(p-1)}(t)}, \\
 \Rightarrow \phi'(t) & \leq -kq(t) \left(\frac{\theta_1\delta^2(t)}{2} \right)^{p-1} \left(\frac{\zeta(\delta(t))}{\zeta(t)} \right)^{p-1} - \frac{(p-1)(-\phi)^{p/(p-1)}(t)}{a^{1/(p-1)}(t)}. \tag{22}
 \end{aligned}$$

From (18) and (20), we deduce that

$$-\phi(t)\zeta^{p-1}(t) \leq 1. \tag{23}$$

Now, we multiply both sides of (22) (with $v = s$) by $\zeta^{p-1}(s)$, then we integrate over the interval $[t_1, t]$. So, we obtain that

$$\begin{aligned} & \zeta^{p-1}(t)\phi(t) - \zeta^{p-1}(t_1)\phi(t_1) + (p-1) \int_{t_1}^t a^{-1/(p-1)}(s)\zeta^{p-2}(s)\phi(s)ds \\ & \leq - \int_{t_1}^t kq(s) \left(\frac{\theta_1\delta^2(s)}{2}\right)^{p-1} \zeta^{p-1}(\delta(s))ds \\ & \quad - (p-1) \int_{t_1}^t \frac{(-\phi)^{p/(p-1)}(s)}{a^{1/(p-1)}(s)} \zeta^{p-1}(s)ds. \end{aligned}$$

It follows that

$$\begin{aligned} & \int_{t_1}^t kq(s) \left(\frac{\theta_1\delta^2(s)}{2}\right)^{p-1} \zeta^{p-1}(\delta(s))ds \\ & \leq \zeta^{p-1}(t_1)\phi(t_1) - \zeta^{p-1}(t)\phi(t) + \int_{t_1}^t (p-1)a^{-1/(p-1)}(s)\zeta^{p-2}(s)(-\phi)(s)ds \\ & \quad - (p-1) \int_{t_1}^t \frac{(-\phi)^{p/(p-1)}(s)}{a^{1/(p-1)}(s)} \zeta^{p-1}(s)ds. \end{aligned}$$

If we apply the inequality given in Remark 1 for

$$\begin{aligned} c_0 &= \frac{\zeta^{p-1}(s)}{a^{1/(p-1)}(s)}, \\ c_1 &= a^{-1/(p-1)}(s)\zeta^{p-2}(s), \\ x &= -\phi(s), \end{aligned}$$

we obtain that

$$\begin{aligned} & (p-1)a^{-1/(p-1)}(s)\zeta^{p-2}(s)(-\phi)(s) - \frac{(-\phi)^{p/(p-1)}(s)}{a^{1/(p-1)}(s)}\zeta^{p-1}(s) \\ & \leq \frac{(p-1)^p}{p^p} \frac{1}{a^{1/(p-1)}(s)\zeta(s)}, \end{aligned}$$

which leads to

$$\begin{aligned} & \int_{t_1}^t \left(kq(s) \left(\frac{\theta_1\delta^2(s)}{2}\right)^{p-1} \zeta^{p-1}(\delta(s)) - \tilde{\pi}(s) \right) ds \\ & \leq \zeta^{p-1}(t_1)\phi(t_1) + 1, \quad \text{(by (23)).} \end{aligned}$$

So, we have a contradiction to (10).

Therefore, we conclude that $u \in C^3([t_u, \infty), \mathbb{R})$ can not be a positive solution. So, every solution $u \in C^3([t_u, \infty), \mathbb{R})$ of (1) is oscillatory or $\lim_{t \rightarrow \infty} u(t) = 0$ is satisfied. \square

The next finding of the work is a theorem that establishes oscillation criteria to Equation (1). For this purpose, we apply the technique of comparison with second-order differential equations.

The new hypothesis is as follows:

(H5) For every $\theta, \theta_1 \in (0, 1)$, the equations:

$$\left(\frac{2^{p-1}a(t)}{(\theta t^2)^{p-1}} |u'(t)|^{p-2} u'(t) \right)' + kq(t)u^{p-1}(t) = 0, \tag{24}$$

$$u''(t) + u(t) \int_t^\infty \left(\frac{1}{a(\zeta)} \int_\zeta^\infty q(s)ds \right)^{1/(p-1)} d\zeta = 0, \tag{25}$$

$$\left(a(t)|u'(t)|^{p-2} u'(t) \right)' + u^{p-1}(t)kq(t) \left(\frac{\zeta(\delta(t))}{\zeta(t)} \right)^{p-1} \left(\frac{\theta_1}{2} \delta^2(t) \right)^{p-1} = 0 \tag{26}$$

are oscillatory.

Theorem 2. *If (H1), (H2) and (H5) hold, then every solution $u \in C^3([t_u, \infty), \mathbb{R})$ of (1) is oscillatory or $\lim_{t \rightarrow \infty} u(t) = 0$ is satisfied.*

Proof. Arguing by contradiction, we suppose that $u \in C^3([t_u, \infty), \mathbb{R})$ is a positive solution of (1). So, we assume that $u(t)$ and $u(\delta(t))$ are positive for all $t \geq t_1$ large enough.

Now, we distinguish the following three cases (see Lemma 3):

Case 1. If (S_1) holds, then using the same arguments as in the proof of Theorem 1 (Case 1), we get that the inequality (14) is true. Putting $\eta(t) = k = 1$, from (14) we deduce that

$$\varphi'(t) + \frac{(p-1)\theta t^2}{2a^{1/(p-1)}(t)} \varphi(t)^{\frac{p}{p-1}} + q(t) \leq 0, \quad \text{for all } \theta \in (0, 1),$$

\Rightarrow (24) is non-oscillatory,

a contradiction to hypothesis (H5).

Case 2. If (S_2) holds, then proceeding with a similar statement as in the proof of Theorem 1 (Case 2), we get that the inequality (17) is true. From (17), with $\vartheta(t) = k = 1$, we deduce that

$$\psi'(t) + \psi^2(t) + \omega(s)\zeta \leq 0$$

$$\Rightarrow \text{(25) is non-oscillatory,}$$

a contradiction to hypothesis (H5).

Case 3. If (S_3) holds and $\lim_{t \rightarrow \infty} u(t) \neq 0$, then proceeding with a similar statement as in the proof of Theorem 1 (Case 3), we get that the inequality (22) is true. So, we have

$$\phi'(t) + \frac{\beta(-\phi)^{p/(p-1)}(t)}{a^{1/(p-1)}(t)} + kq(t) \left(\frac{\theta_1 \delta^2(t)}{2} \right)^{p-1} \left(\frac{\zeta(\delta(t))}{\zeta(t)} \right)^{p-1} \leq 0$$

\Rightarrow (26) is non-oscillatory,

a contradiction to hypothesis (H5). We conclude that $u \in C^3([t_u, \infty), \mathbb{R})$ can not be a positive solution. It follows that every solution $u \in C^3([t_u, \infty), \mathbb{R})$ of (1) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} u(t) = 0$. \square

It is worth mentioning that the existence and regularity of the solution in Theorem 1 and Theorem 2 have been proven by Philos in [2].

Next, we provide a simple illustrative example.

Example 1. Consider the fourth order equation given as

$$(e^t u'''(t))' + \frac{1}{16} e^{t+\frac{1}{2}} u(t+1) = 0, \quad t \geq 1, \tag{27}$$

that is, we put $p = 2$, $a(t) = e^t$, $q(t) = e^{t+\frac{1}{2}}/16$ and $\delta(t) = t + 1$ in (1). Furthermore, we choose $\eta(s) = \vartheta(s) = 1$ for all $s \in [t_0, \infty)$, and $(H_1, H_2) \in \mathfrak{S}$ with

$$H_1(v, s) = H_2(v, s) = (v - s)^2.$$

The above equation satisfies all the hypotheses of Theorem 1. So, we conclude that every solution $u \in C^3([t_0, \infty), \mathbb{R})$ of (27) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} u(t) = 0$.

Since $\delta(t) = t + 1 > t$, we observe that theorems in [5,6] do not work for Equation (27).

Finally, we note that if we continue along this path, we can obtain oscillatory results for a fourth order equation of the type:

$$\begin{cases} \left(a(t)|u'''(t)|^{p_1-2}u'''(t) \right)' + \sum_{i=1}^j q_i(t)|u(\delta_i(t))|^{p_2-2}u(\delta_i(t)) = 0, & j \geq 1, \\ t \geq t_0, \delta_i(t) \leq t, 1 < p_2 \leq p_1 < \infty. \end{cases}$$

Remark 2. There is an interesting open problem concerning the above equation:

- Is it possible to have similar results in the case $p_2 > p_1$?

4. Conclusions

In conclusion, this study aimed at investigating the oscillatory properties of solutions to fourth-order differential equations with a p -Laplacian. The findings of this paper contribute to the understanding of the asymptotic and oscillatory behavior of such equations and provide new oscillation criteria through the use of comparison methods with first and second-order differential equations. This work highlights the relevance of the theory of fourth-order differential equations to various fields of mathematics and practical sciences, emphasizing the importance of continued research in this area.

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