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Non-Stationary Helical Flows for Incompressible Couple Stress Fluid

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Abstract: We explored here the case of three-dimensional non-stationary flows of *helical type* for the incompressible couple stress fluid with given *Bernoulli*-function in the whole space (the Cauchy problem). In our presentation, the case of non-stationary *helical* flows with constant coefficient of proportionality α between velocity and the curl field of flow is investigated. In the given analysis for this given type of couple stress fluid flows, an absolutely novel class of exact solutions in theoretical hydrodynamics is illuminated. Conditions for the existence of the exact solution for the aforementioned type of flows were obtained, for which non-stationary *helical* flow with invariant *Bernoulli*-function satisfying to the Laplace equation was considered. The spatial and time-dependent parts of the pressure field of the fluid flow should be determined via *Bernoulli*-function if components of the velocity of the flow are already obtained. Analytical and numerical findings are outlined, including outstanding graphical presentations of various types of constructed solutions, in order to elucidate dynamic snapshots that show the timely development of the topological behavior of said solutions.

Keywords: couple stress fluid; micropolar fluid; bipolar vector Laplacian; non-stationary helical flow; *Beltrami* flow

MSC: 35Q35; 76D17

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1. Introduction: System of Equations (Incompressible Couple Stress Fluid)

The description of flows of viscous incompressible fluids [1–53] is based mainly on the integration of the classical Navier–Stokes equations for Newtonian fluid and the continuity equation. Among the literature, [1,12,30,39] present seminal books and articles both on classical Newtonian and couple stress fluid flows where a lot of examples can be found regarding analytical aspects to their study, and [2–4,13] present works where exact, analytical solutions along with Stokes' problems for couple stress fluid flows have been investigated. The works [5–11] and also [16–29,49–53] introduce in detail the description of the effective use of helical flow conception in application to the investigation of non-linear aspects of fluid flow dynamics both in classical Newtonian and couple stress fluid flows. Let us also mention the works [31–38,40–48], where these references are within the framework of applying an analytical approach to the study of mathematical models in applications to various non-linear problems in hydrodynamics, fluid dynamics, and magnetohydrodynamics that are important in the context of the current research. It is worth paying particular attention to the work [40], where stability of fluid motions under various perturbing effects was investigated. Conversely, the alley of research [14,15] (with references therein and citing them) presents a partial class of solution with given constant Bernoulli-function in the whole space for fluid flows.

In the case of the classical Navier–Stokes equations, viscous internal forces are described by the symmetric Cauchy tensor [32,33]. In other words, for a representative volume of the medium, only translational degrees of freedom of movement are taken into account. For incompressible fluids, the symmetry of the Cauchy stress tensor postulates the proportionality of the force of internal viscous friction to the Laplace operator. The proportionality coefficient is the coefficient of kinematic or dynamic viscosity.

It is known that the study of the properties of solutions to the Navier–Stokes equations is still far from complete [12]. Various hypotheses about the structure of flows are used to prove various theorems and construct classes of exact solutions [32,33]. One of the approaches to studying the properties of solutions for the hydrodynamical equations is the use of a regularizing perturbing force proportional to the linear biharmonic Laplace operator [1]; in this case, they describe dynamics of flows of couple stress fluid. Such a mathematical way of integrating the hydrodynamical equations has a clear physical interpretation, namely, if it is for a representative volume, we take into account not only translational degrees of freedom, but also rotational degrees of freedom, with further clear defining relations for describing fluid flows that contain the biharmonic Laplace operator [1–4].

For the analytical integration of the Navier–Stokes equations, the class of Beltrami flows is known, for which a significant supply of exact solutions has been constructed [5–10]. These exact solutions are important for understanding the mechanisms of interaction of convective mixing of a liquid with internal friction forces [5–10]. The exact solutions obtained can illustrate even arising chaos in dynamical systems.

The article [13] presents several classes of exact solutions for the Navier–Stokes equations with paired couple stresses. The results of this article generalize the pioneering exact solutions [1–4] and make it possible to describe spatially inhomogeneous flows of non-classical fluids.

Given the importance of finding classes of exact solutions, this paper studies Beltrami-type flows of non-classical fluids. It is worth noting that helical (Beltrami) flows are known to be used in rotor turbine design in modelling the flows in technological processes such as Gorlov's water helical turbine (or a kind of Stampa's helical turbine used to generate electric power from wind energy), modelling the coaxial propellers or air-screws for aircrafts or helicopters. These solutions will be useful for studying the influence of competing dissipative mechanisms on the structure of the velocity field. In accordance with [1–4], a system of equations for incompressible flow of micropolar fluid (or incompressible couple stress fluid flow) with conservative body forces should be presented in the Cartesian coordinates as below, under the no-slip conditions over rigid surface:

$$\nabla \cdot \vec{u} = 0, \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \vec{F} + \nu \nabla^2 \vec{u} - \eta \nabla^4 \vec{u}, \tag{2}$$

where $\boldsymbol{u}$ is the flow velocity, a vector field; ρ is the fluid density; p is the pressure; ν is the kinematic viscosity; η is the parameter due to couple stress; $\boldsymbol{F}$ represents external force (*per unit of mass in a volume*) acting on the fluid; and notation $\boldsymbol{u}$ or $\vec{u}$ means a common used notations of vector field. Moreover, we assume here external force $\boldsymbol{F}$ above to be the force, which has a potential ϕ represented by $\boldsymbol{F} = -\nabla \phi$. As for the domain in which the flow occurs and the boundary conditions, let us consider only the Cauchy problem in the whole space.

Let us search for solutions of the system (1)–(2) in a form of *helical* or *Beltrami* flow [5–11] as below:

$$\vec{\Omega}\,(x,y,z,t) = \alpha(x,y,z)\vec{u}\,(x,y,z,t) \tag{3}$$

Here, we denote the curl field $\boldsymbol{\Omega}$ = ($\nabla \times \boldsymbol{u}$), a pseudovector *time-dependent* field (which means the vorticity of the fluid flow); α is variable parameter as in [6] (which differs from the case α = const, considered in [5]).

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$, we could present momentum Equation (2) for incompressible couple stress fluid flow $\boldsymbol{u} = \{u_1, u_2, u_3\}$ as below [6]:

$$\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{\Omega} + \nu \nabla^2 \vec{u} - \eta \nabla^4 \vec{u} - \left(\frac{1}{2}\nabla(\vec{u}^{\,2}) + \frac{\nabla p}{\rho} + \nabla \varphi\right) \tag{4}$$

where we chose ρ = 1 for simplicity (thus, the class of fluids is restricted). It is worth noting that continuity Equation (1) should be satisfied automatically if we use the presentation of solution in a form (3) for the case α = const. If $\alpha \neq$ const as formulated in more general case (3), the continuity Equation (1) yields demand as below, using (3):

$$\nabla \cdot \vec{u} = 0 \;\Rightarrow\; \left(\frac{1}{\alpha}\right)\nabla \cdot \vec{\Omega} - \left(\frac{\nabla \alpha}{\alpha^2} \cdot (\alpha \vec{u})\right) = 0 \;\Rightarrow\; \left(\nabla \alpha \cdot \vec{u}\right) = 0 \tag{5}$$

$$\frac{\partial \alpha}{\partial x} \cdot u_1 + \frac{\partial \alpha}{\partial y} \cdot u_2 + \frac{\partial \alpha}{\partial z} \cdot u_3 = 0 \tag{6}$$

Let us note that the simple choice $\nabla \alpha = \vec{0}$ in (5) and (6) corresponds to the obvious case α = const; we investigate such a case here in the current research.

2. The Solving Procedure for Time-Dependent Solution: α = Const

Using (3) and (4), we should note that each equation of the *vector* equation in system (4) could be transformed as below in case α = const:

$$\begin{aligned}
&\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{\Omega} + \nu \nabla^2 \vec{u} - \eta \nabla^4 \vec{u} - \left(\tfrac{1}{2}\nabla(\vec{u}^{\,2}) + \nabla p + \nabla \varphi\right) \;\Rightarrow \\
&\frac{\partial \vec{u}}{\partial t} = \vec{u} \times (\alpha \vec{u}) - \nu \nabla \times (\alpha \vec{u}) - \eta \nabla \times \nabla \times \nabla \times (\alpha \vec{u}) - \nabla B \;\Rightarrow \\
&\frac{\partial \vec{u}}{\partial t} = (-\nu \alpha^2 - \eta \alpha^4)\, \vec{u} - \nabla B
\end{aligned} \tag{7}$$

where Bernoulli-function B is given by expression below:

$$B = \frac{1}{2}(\vec{u}^{\,2}) + p + \varphi. \tag{8}$$

So, first we obtain from (7), using (1):

$$\Delta B = 0 \tag{9}$$

i.e., Bernoulli-function B is a harmonic function. The second, (7) yields

$$\begin{cases}
\frac{\partial u_1}{\partial t} = (-\nu \alpha^2 - \eta \alpha^4) u_1 - \frac{\partial B}{\partial x}, \\
\frac{\partial u_2}{\partial t} = (-\nu \alpha^2 - \eta \alpha^4) u_2 - \frac{\partial B}{\partial y}, \\
\frac{\partial u_3}{\partial t} = (-\nu \alpha^2 - \eta \alpha^4) u_3 - \frac{\partial B}{\partial z}.
\end{cases} \tag{10}$$

It is a well-known fact that 3D Laplace Equation (9) has a fundamental solution (except simple case B = const):

$$B = B_0\sqrt{x_0^{\,2} + y_0^{\,2} + z_0^{\,2}}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{11}$$

where expressions $\sqrt{x_0^{\,2} + y_0^{\,2} + z_0^{\,2}}$, $B_0 = const$ correspond to the initial values of the problem as formulated in (1) and (2).

So, we can adjust or freely choose pressure p in (8) in such a way that the profiles of velocity obtained with the help of solving (10) should satisfy in (8), where the expression of Bernoulli-function B is given by equality (11). In this case, we obtain a kind of exact solution of a type (3) for Equations (1) and (2); moreover, each equation in (9) will turn out to be an ordinary differential equation that has a fundamental solution as below:

$$\begin{cases} u_1 &= \exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right)\left[\left(-\frac{\partial B}{\partial x}\right)\int\left(\exp(\nu\alpha^2 + \eta\alpha^4)t\right)dt + C_1(x,y,z)\right] = \\ &= \frac{\left(-\frac{\partial B}{\partial x}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_1(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \\ u_2 &= \frac{\left(-\frac{\partial B}{\partial y}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_2(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \\ u_3 &= \frac{\left(-\frac{\partial B}{\partial z}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_3(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \end{cases} \tag{12}$$

where $\{C_1, C_2, C_3\}$ are three functions that should be chosen according to initial values of the problem as formulated in (1) and (2).

Three aforementioned equalities of the system (12) should determine three time-dependent functions $\{u_1, u_2, u_3\}$ in regard to the time t, with expression for Bernoulli-function B given in (11). Expression for pressure field p should be obtained or expressed via equality (8).

3. Final Presentation of the Solution (the Helical Flows for Incompressible Couple Stress Fluid)

Let us present the non-stationary solution $\{p, \boldsymbol{u}\}$ ($\boldsymbol{u} = \{u_1, u_2, u_3\}$) of helical flow of type (3) at α = const for the flows of incompressible couple stress fluid (1)–(2) in its final form:

$$\begin{aligned} p &= B - \left(\tfrac{1}{2}(\vec{u}^{\,2}) + \varphi\right), \\ B &= B_0\sqrt{x_0^2 + y_0^2 + z_0^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right),\ B_0 = const, \\ u_1 &= \frac{\left(-\frac{\partial B}{\partial x}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_1(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \\ u_2 &= \frac{\left(-\frac{\partial B}{\partial y}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_2(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \\ u_3 &= \frac{\left(-\frac{\partial B}{\partial z}\right)}{(\nu\alpha^2 + \eta\alpha^4)} + C_3(x,y,z)\exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right), \end{aligned} \tag{13}$$

where φ is the potential of external force, acting on a fluid; expressions $\sqrt{x_0^{\,2} + y_0^{\,2} + z_0^{\,2}}$ and $\{C_1, C_2, C_3\}$ correspond to initial values of the problem as formulated in (1)–(2).

We should note if we assume $B_0 = B_0(t)$ in (11), then (12) (and thus, (13)) can be presented in a more general form

$$\begin{cases} u_1 = \exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right)\left[-\int\left(\left(\frac{\partial B}{\partial x}\right)\exp(\nu\alpha^2 + \eta\alpha^4)t\right)dt + C_1(x,y,z)\right], \\ u_2 = \exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right)\left[-\int\left(\left(\frac{\partial B}{\partial y}\right)\exp(\nu\alpha^2 + \eta\alpha^4)t\right)dt + C_2(x,y,z)\right], \\ u_3 = \exp\left(-(\nu\alpha^2 + \eta\alpha^4)t\right)\left[-\int\left(\left(\frac{\partial B}{\partial z}\right)\exp(\nu\alpha^2 + \eta\alpha^4)t\right)dt + C_3(x,y,z)\right]. \end{cases} \tag{14}$$

4. Discussion

The system of equations of motion for incompressible couple stress fluid has already been investigated in numerous studies, including their numerical and analytical findings [1–4], even for the 3D case of *non-stationary* flows of incompressible fluid [13]. However, essential deficiency exists as ever in the studies of non-stationary solutions of this type of hydrodynamical equation.

The results presented in the current research are novel with respect to the investigation of the case of incompressible couple stress fluid, to the best of our knowledge, since the case of non-stationary flows of *helical type* for the incompressible couple stress fluid with given Bernoulli-function (11) in the whole space (the Cauchy problem) was investigated for the first time. In this respect, we should refer also to the research [14,15] (with examples including the Bernoulli-invariant as a key point in solving procedure).

As for the relevance of this new solution, let us discuss the essential details about the possible physical properties of the aforementioned solution (13). Firstly, we should note that all Equations (3)–(12) can be easily updated to the case $\alpha = \alpha\,(t)$, while of course formulae (14) should be updated accordingly (and specifically) to this case. As for investigating the case of spatial possible dependence of parameter $\alpha = \alpha\,(x, y, z, t)$, we do not consider such case here due to complicated vector algebra calculations in (4) with respect to taking into account vector $\nabla\,\alpha$ when calculating appropriate expressions of vector algebra for term $\nabla\,^4\vec{u}$ while condition (6) would be still valid in the aforementioned case. Secondly, it is worth noting that we considered here only the Cauchy problem in the whole space (without possible reflection of flows from solid boundaries inside the flow field), besides we considered in (3) that $\boldsymbol{u}$ is a vector field, whereas $\boldsymbol{\Omega}$ is *a pseudovector* field.

We schematically imagined a time-dependence of solution (13) for the components of velocity field in Figure 1.

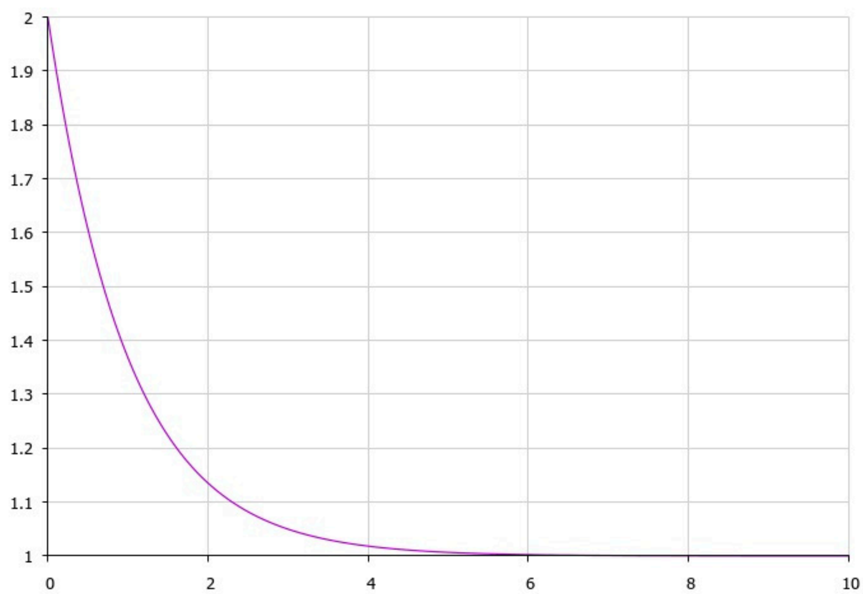

Figure 1. Schematically presented solution of a type (13) for the components of velocity field (here, we designated t on the abscissa axis, function $(1 + \exp\,(-t))$ on the ordinate axis).

As a result, let us proclaim that the case of three-dimensional non-stationary flows of *helical type* for the incompressible couple stress fluid with Bernoulli-function satisfying to the Laplace equation was investigated here for *helical* flows taking place in the whole space (the Cauchy problem) with constant coefficient of proportionality α between velocity and the curl field of flow in the current study.

As for conditions for the existence of the exact solution for the aforementioned type of flows, for which non-stationary *helical* flow is determined by the given Bernoulli-function (here, a fundamental solution of the Laplace equation), we can conclude the following:

(1) According strictly to form (3) of *helical* flow, solutions of a type (13) exist if only we choose $B_0 = 0$ or if we choose simplifying condition $B = \text{const}$ in (9) from the very beginning for the process of constructing the exact solutions; in this case, we had from (3)

$$\begin{cases} \alpha C_1(x,y,z) = \frac{\partial}{\partial y}(C_3(x,y,z)) - \frac{\partial}{\partial z}(C_2(x,y,z)), \\ \alpha C_2(x,y,z) = \frac{\partial}{\partial z}(C_1(x,y,z)) - \frac{\partial}{\partial x}(C_3(x,y,z)), \\ \alpha C_3(x,y,z) = \frac{\partial}{\partial x}(C_2(x,y,z)) - \frac{\partial}{\partial y}(C_1(x,y,z)), \end{cases} \quad (15)$$

so conditions (15) restrict the nine degrees of freedom in choosing form of $\{C_1, C_2, C_3\}$ up to six degrees of freedom. Meanwhile, condition $B_0 = 0$ means that we should take into account the potential of external force φ (which is for gravity central force known to have a sufficiently large negative value). If we, nevertheless, assume the consideration of the case of absence of any external force, this would mean that *helical* flow takes place at negative pressure. As for the negative value of pressure p in (13), we know physically reasonable cases of flows when pressure is transformed to be negative (these are very special conditions for fluids flow, see [38,39]).

(2) But nevertheless, it is worth noting that form (3) of *helical* flow has *already* been taken into account in a derived system of Equation (10) (stemming from momentum Equation (2)), from which we obtain as a result solutions (14) in the most general form. So, using continuity Equation (1), we conclude that solutions of a type (14) can exist if restriction to the form of solutions is valid as below:

$$\frac{\partial}{\partial x}(C_1(x,y,z)) + \frac{\partial}{\partial y}(C_2(x,y,z)) + \frac{\partial}{\partial z}(C_3(x,y,z)) = 0 \quad (16)$$

It is also important to note that (16) stems from (15): if we differentiate the first equation of system (15) with respect to x, second with respect to y, and third with respect to z, then sum them all together, we should obtain (16) as a result. This means that restrictions (15) are stricter and can be considered as excessive with respect to the sufficient conditions (16) for the existence of the exact solution for the aforementioned type of flows.

The spatial and time-dependent parts of the pressure field of the fluid flow should be determined via Bernoulli-function if components of the velocity of the flow are already obtained. Analytical and numerical findings were outlined, including outstanding graphical presentations of various types of constructed solution in illuminating dynamical snap-shots that demonstrate developing in time the structural behaviour of topology of the aforepresented solutions. For example, we can choose in (13) as follows (here, below, parameters $B_0 = 0$, $\tau = t$ were chosen just for the simplicity of the presentation of velocity's components on Figures 2–5):

$$(\nu\alpha^2 + \eta\alpha^4) = 1[\mathrm{s}^{-1}],\ \sqrt{x_0^2 + y_0^2 + z_0^2} = 1[\mathrm{m}^2],\ B_0 = 0,$$

$$C_1 = \cos((1/\exp(\cos(\tau))) \cdot 5\tau),\ C_2 = \sin((1/\exp(\cos(\tau))) \cdot 5\tau),\ \tau = t,\ C_3 = 0,$$

$$\Rightarrow \begin{cases} u_1 = C_1 \exp(-(\nu\alpha^2 + \eta\alpha^4)t),\ C_1(x(\tau), y(\tau), z(\tau)) = C_1(\tau) \\ u_2 = C_2 \exp(-(\nu\alpha^2 + \eta\alpha^4)t),\ C_2(x(\tau), y(\tau), z(\tau)) = C_2(\tau) \\ u_3 = 0 \end{cases} \quad (17)$$

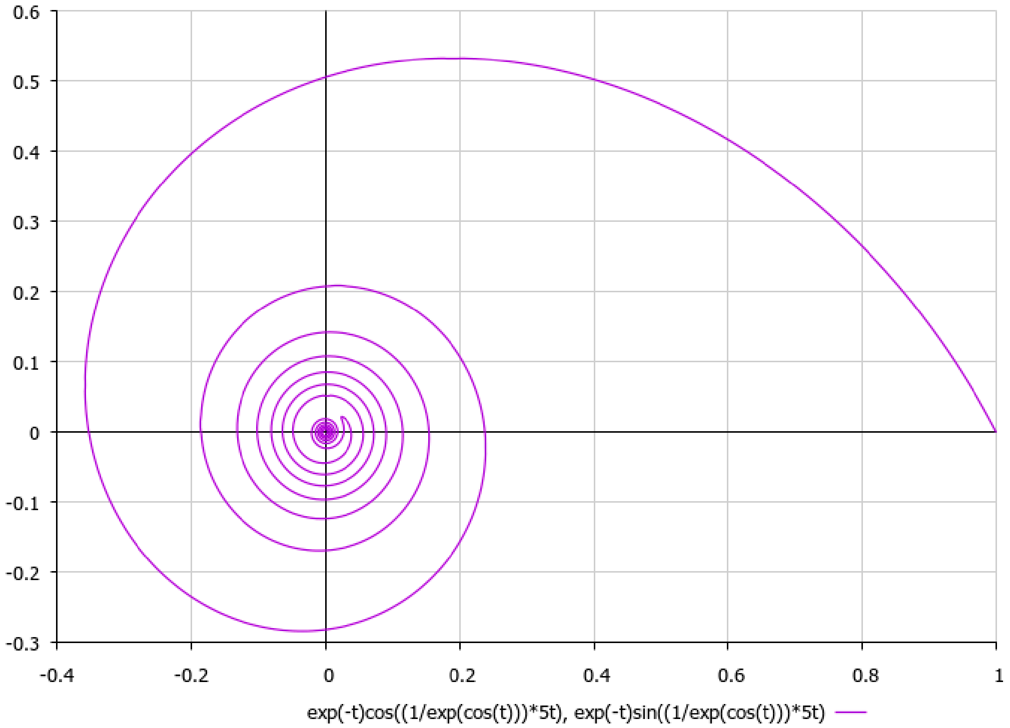

Figure 2. Schematically presented solution of a type (17) for components of the velocity field.

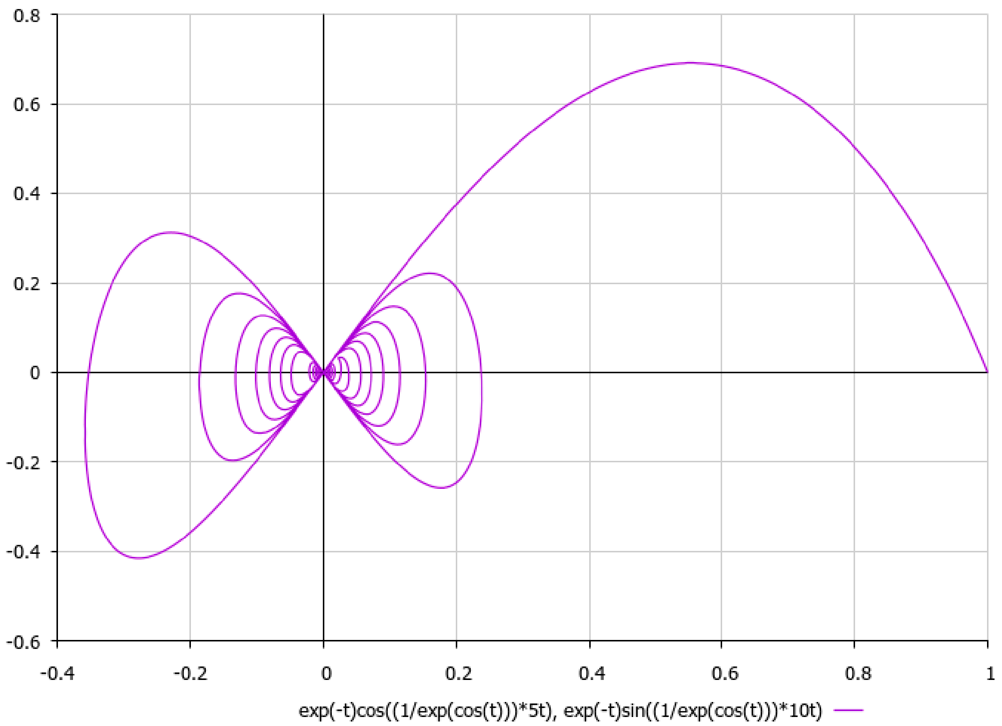

Figure 3. Schematically presented solution of a type (17) for the components of the velocity field if we choose in (15): $C_1 = \cos((1/\exp(\cos(\tau))) \cdot 5\tau)$, $C_2 = \sin((1/\exp(\cos(\tau))) \cdot 10\tau)$.

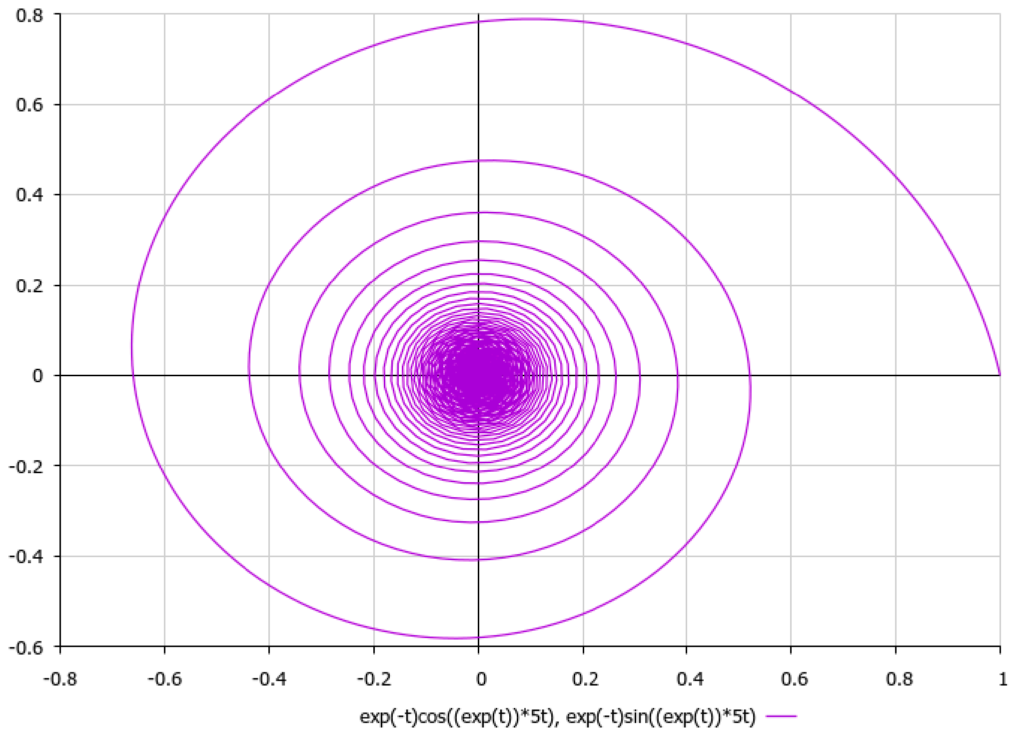

Figure 4. Schematically presented solution of a type (17) for the components of the velocity field if we choose in (15): $C_1 = \cos(\exp(\tau) \cdot 5\tau)$, $C_2 = \sin(\exp(\tau) \cdot 5\tau)$.

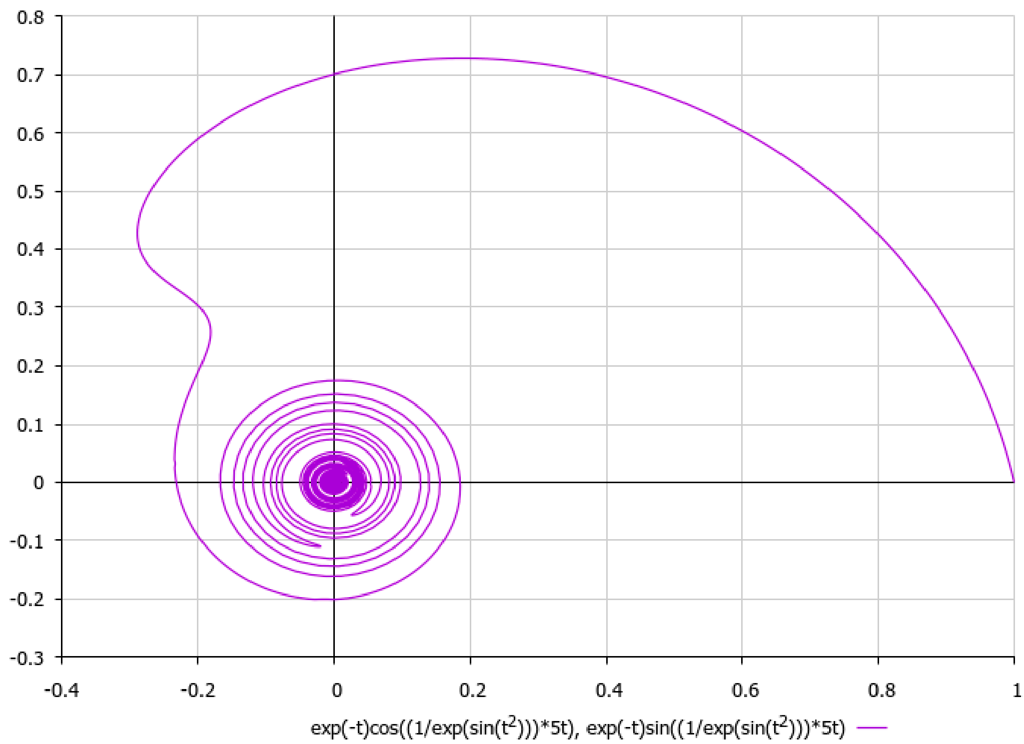

Figure 5. Schematically presented solution of a type (17) for the components of the velocity field if we choose in (15): $C_1 = \cos((1/\exp(\sin(\tau^2))) \cdot 5\tau)$, $C_2 = \sin((1/\exp(\sin(\tau^2))) \cdot 5\tau)$.

Thus, we obtained the following Figures 2–5 (we designate u_1 on the abscissa axis, u_2 on the ordinate axis in Figures 2–5).

We should note that in (17), additional restriction should be valid for choosing functions $\{C_1, C_2\}$:

$$\frac{\partial}{\partial x}(C_2(x,y,z)) - \frac{\partial}{\partial y}(C_1(x,y,z)) = 0.$$

Let us demonstrate also on Figures 6–9 the interesting cases of three-dimensional dynamics of solutions that generalize the solutions presented earlier on Figures 2–5, but with a non-zero third component of the velocity field as below:

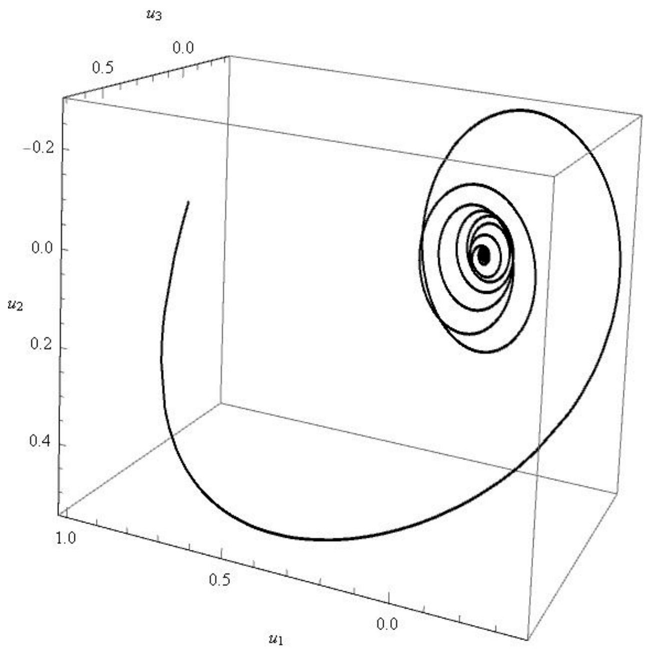

Figure 6. Schematically presented solution of a type (13) for the components of the velocity field if we choose in (13) (with parameters given in (17)): $C_1 = \cos((1/\exp(\cos(\tau))) \cdot 5\tau)$, $C_2 = \sin((1/\exp(\cos(\tau))) \cdot 5\tau)$, $C_3 = \sin(4\tau)$, $\tau = t$.

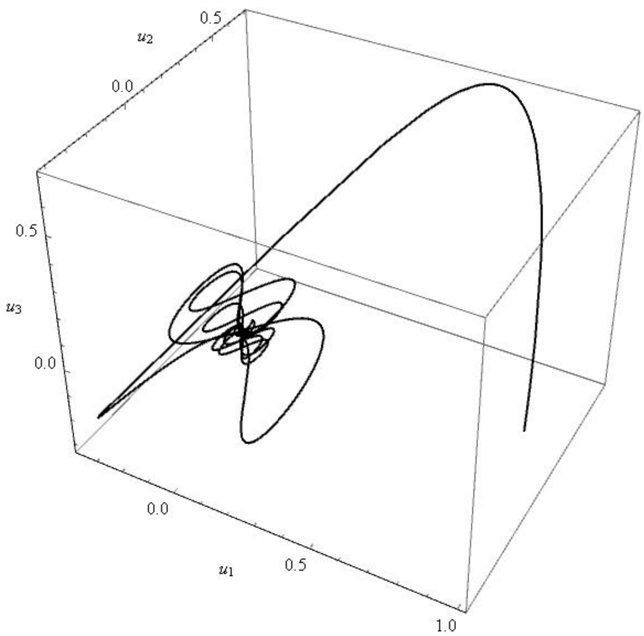

Figure 7. Schematically presented solution of a type (13) for the components of the velocity field if we choose in (13) (with parameters given in (17)): $C_1 = \cos((1/\exp(\cos(\tau))) \cdot 5\tau)$, $C_2 = \sin((1/\exp(\cos(\tau))) \cdot 10\tau)$, $C_3 = \sin(4\tau)$, $\tau = t$.

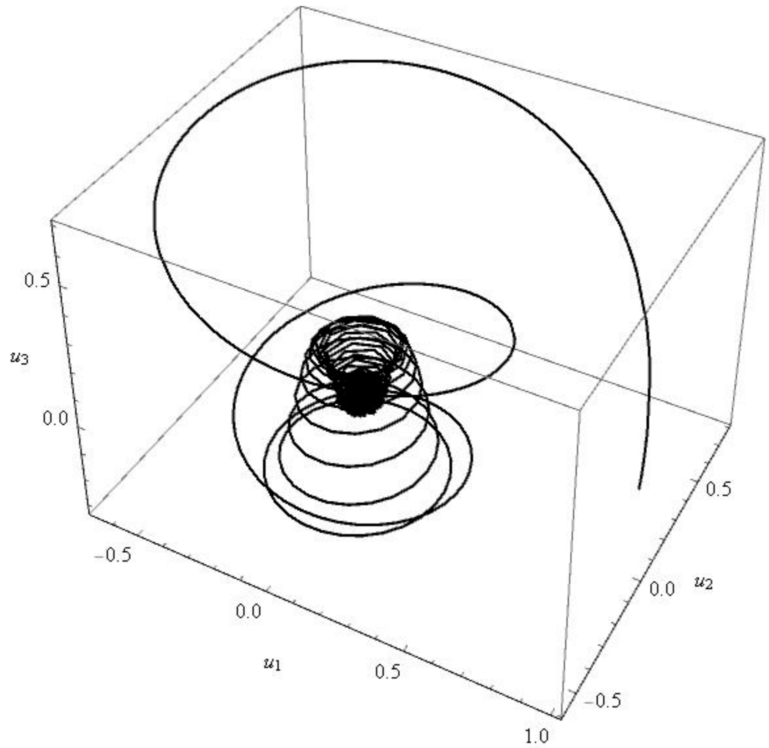

Figure 8. Schematically presented solution of a type (13) for the components of the velocity field if we choose in (13) (with parameters given in (17)): $C_1 = \cos(\exp(\tau)\cdot 5\tau)$, $C_2 = \sin(\exp(\tau)\cdot 5\tau)$, $C_3 = \sin(4\ \tau)$, $\tau = t$.

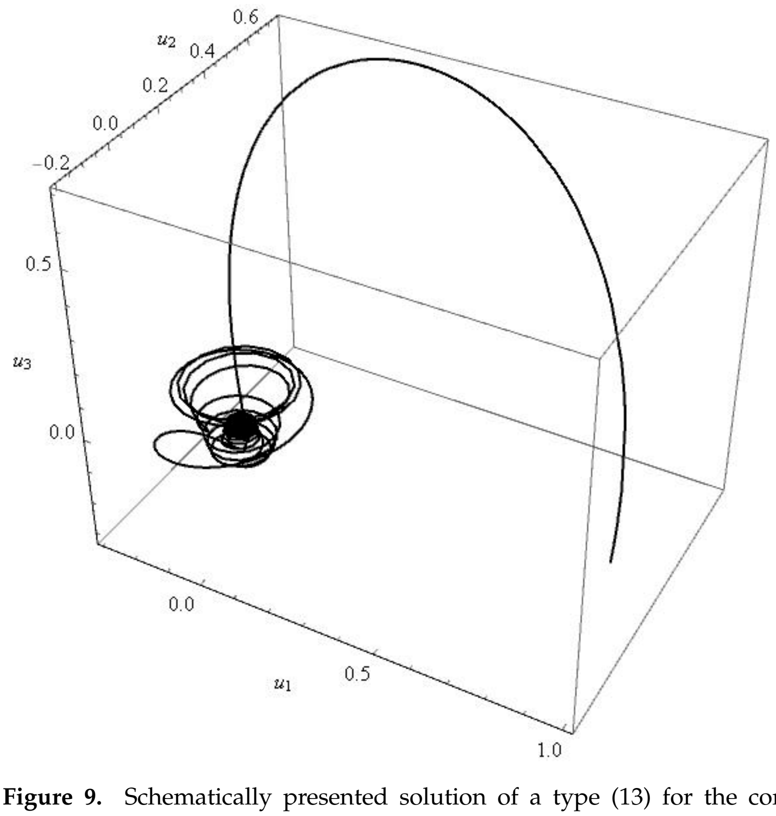

Figure 9. Schematically presented solution of a type (13) for the components of the velocity field if we choose in (13) (with parameters given in (17)): $C_1 = \cos\left((1/\exp\left(\sin(\tau^2)\right))\cdot 5\tau\right)$, $C_2 = \sin\left((1/\exp\left(\sin(\tau^2)\right))\cdot 5\tau\right)$, $C_3 = \sin(4\ \tau)$, $\tau = t$.

While such a theoretical motivation is of course realized, it is worth noting that helical flows are very important in some practical problems, for example, in rotor turbine design, fast rotating of coaxial propellers, or air-screws of various types of aircrafts, etc. It should be additionally noted that some mathematical solutions do not reflect physical phenomena, and the equation for the force, $\boldsymbol{F}$, used in the analysis is valid only for conservative forces [6]. Also, the question under what conditions is alpha in (3) suitable to be set for taking a constant value appears to be of practical significance for the flow of pipelines, water, rivers, sea currents, or for any other situations of modelling the flows in technological processes. It is additionally worth noting that according to the system of Equation (15) (which should be valid also in case $\alpha = \alpha(t)$), the correct choice of spatial part of functions $\{C_1, C_2, C_3\}$ depends on $\alpha(t)$.

The stability of the presented solution was not considered. Last but not least, we should specifically mention comprehensive research [16–24], where a lot of unknown details concerning the close-related area of helical flows are remarked upon, including methods of investigating the stability of such flows.

We should also note that the construction of exact solutions of the Navier–Stokes equations is often carried out by the method proposed in [7–9]. In [7–9], fluid flows were considered in which the velocity vector has a functional relationship with the vorticity pseudovector (the Lamb vector of the cross product of the velocity vector to vorticity pseudovector is equal to zero). In the article [8], a special case of results [7,9] was considered. In [8], the velocity vectors and the vorticity pseudovector were considered parallel, that is, they are parallel. Trkal's idea [8] was developed in articles [19,22,26,29]. The results announced in [7,9] were generalized and modified in [24] (meanwhile, article [18] is a qualified translation of work [8] into English).

5. Conclusions

We explored here the case of non-stationary flows of *helical type* for the incompressible couple stress fluid with a given Bernoulli-function (11) in the whole space (the Cauchy problem). The case of non-stationary *helical* flows with a constant coefficient of proportionality α between velocity and the curl field of flow was investigated in our study for such a type of couple stress fluid flows, illuminating a novel class of exact solutions in theoretical hydrodynamics. Conditions for the existence of the exact solution for the aforementioned type of flows were obtained, for which non-stationary *helical* flow with invariant Bernoulli-function was considered satisfying to the Laplace equation. The spatial and time-dependent parts of the pressure field of the fluid flow should be determined via the Bernoulli-function if components of the velocity of the flow are already obtained. Analytical and numerical findings were outlined, including outstanding graphical presentations of various types of constructed solutions to elucidate dynamic snapshots that show the timely development of the topological behavior of said solutions.

Let us also apply conditions similar to (15) to restrict the results of the work [6] obtained earlier (we considered there the case $\partial\alpha/\partial y = 0$, $\partial\alpha/\partial z \neq 0$):

$$\left\{\begin{array}{l}
\alpha U = \frac{\partial}{\partial y}\Big(-\frac{\left(\frac{\partial\alpha}{\partial x}\right)}{\left(\frac{\partial\alpha}{\partial z}\right)}\cdot U\Big) - \frac{\partial}{\partial z}\Big(\pm\sqrt{(U^2(t_0) + V^2(t_0) + W^2(t_0))\cdot\exp(-2\,\nu\cdot\alpha^2\cdot(t-t_0)) - \left(1 + \frac{\left(\frac{\partial\alpha}{\partial x}\right)^2}{\left(\frac{\partial\alpha}{\partial z}\right)^2}\right)\cdot U^2}\Big), \\
\alpha\left(\pm\sqrt{(U^2(t_0) + V^2(t_0) + W^2(t_0))\cdot\exp(-2\,\nu\cdot\alpha^2\cdot(t-t_0)) - \left(1 + \frac{\left(\frac{\partial\alpha}{\partial x}\right)^2}{\left(\frac{\partial\alpha}{\partial z}\right)^2}\right)\cdot U^2}\right) = \frac{\partial}{\partial z}U - \frac{\partial}{\partial x}\Big(-\frac{\left(\frac{\partial\alpha}{\partial x}\right)}{\left(\frac{\partial\alpha}{\partial z}\right)}\cdot U\Big), \\
\alpha\Big(-\frac{\left(\frac{\partial\alpha}{\partial x}\right)}{\left(\frac{\partial\alpha}{\partial z}\right)}\cdot U\Big) = \frac{\partial}{\partial x}\Big(\pm\sqrt{(U^2(t_0) + V^2(t_0) + W^2(t_0))\cdot\exp(-2\,\nu\cdot\alpha^2\cdot(t-t_0)) - \left(1 + \frac{\left(\frac{\partial\alpha}{\partial x}\right)^2}{\left(\frac{\partial\alpha}{\partial z}\right)^2}\right)\cdot U^2}\Big) - \frac{\partial}{\partial y}U,
\end{array}\right. \tag{18}$$

where (see (19)–(20) and (17) in [6])

$$U = U(t_0) \cdot \exp\left(\int \left(\frac{1}{\lambda}\right) dt\right), \quad (F_1 + F_2 \cdot \lambda) \cdot \lambda' = F_3 \cdot \lambda^3 + F_4 \cdot \lambda^2 + F_5 \cdot \lambda + F_6,$$
$$F_1 = 2h,\ F_2 = 2f \cdot h,\ F_3 = (g' \cdot h - g \cdot h'),\ F_4 = 2(g \cdot h + f' \cdot h - f \cdot h'),\ F_5 = (4f \cdot h - h'),\ F_6 = 2h, \tag{19}$$
$$f = \nu \cdot \alpha^2,\ g = \nu^2 \cdot \left(\alpha^4 + \left(\frac{\partial\alpha}{\partial z}\right)^2 + \left(\frac{\partial\alpha}{\partial x}\right)^2\right),\ h = -\nu^2 \cdot \left(\frac{\partial\alpha}{\partial z}\right)^2 \cdot \left(U^2(t_0) + V^2(t_0) + W^2(t_0)\right) \cdot \exp\left(-2\,\nu \cdot \alpha^2 \cdot (t - t_0)\right)$$

It is more than obvious that (18) should be valid for any moment of time *t* if condition (20) below is satisfied:

$$\begin{gathered}
\left(\frac{1}{\lambda}\right) = -\nu \cdot \alpha^2 \qquad \Rightarrow \\
\Rightarrow\ 0 = (2\nu \cdot \alpha^2 \cdot g \cdot h) \cdot \lambda^3 + 2(g \cdot h + 2(\nu \cdot \alpha^2)^2 \cdot h) \cdot \lambda^2 + (4\nu \cdot \alpha^2 \cdot h + 2\nu \cdot \alpha^2 h) \cdot \lambda + 2h\ \Rightarrow \\
\left\{g = \nu^2 \cdot \left(\alpha^4 + \left(\frac{\partial\alpha}{\partial z}\right)^2 + \left(\frac{\partial\alpha}{\partial x}\right)^2\right)\right\}\ \Rightarrow\ g\,\lambda^3 + \left(\frac{g + 2\,(\nu \cdot \alpha^2)^2}{\nu \cdot \alpha^2}\right) \cdot \lambda^2 + 3\lambda + \left(\frac{1}{\nu \cdot \alpha^2}\right) = 0\ \Rightarrow \\
g\left(-\frac{1}{(\nu \cdot \alpha^2)^3}\right) + \left(\frac{g + 2\,(\nu \cdot \alpha^2)^2}{(\nu \cdot \alpha^2)^3}\right) - 2\left(\frac{1}{\nu \cdot \alpha^2}\right) = 0\ \Rightarrow\ g\left(\frac{-1+1}{(\nu \cdot \alpha^2)^3}\right) + \left(\frac{2-2}{(\nu \cdot \alpha^2)}\right) = 0
\end{gathered} \tag{20}$$

where the last equality is automatically satisfied, being the result of a substitution of first of equalities (20) to (19) directly.

Thus, a system of Equation (18) should be used to diminish three degrees of freedom at choosing functions below, which depend on spatial variables ($\partial\alpha/\partial y = 0$, $\partial\alpha/\partial z \neq 0$) according to initial conditions:

$$\alpha(x, z),\ \{U(x, y, z, t_0),\ V(x, y, z, t_0),\ W(x, y, z, t_0)\}$$

For example, if we choose

$$\alpha = \frac{1}{z}\ \Rightarrow \begin{cases} U(t_0) = -z\left(\frac{\partial}{\partial z}\left(\pm\sqrt{V^2(t_0) + W^2(t_0)}\right)\right), \\ \left(\pm\sqrt{V^2(t_0) + W^2(t_0)}\right) = z\left(\frac{\partial}{\partial z}U(t_0)\right), \\ 0 = \frac{\partial}{\partial x}\left(\pm\sqrt{V^2(t_0) + W^2(t_0)}\right) - \frac{\partial}{\partial y}U(t_0), \end{cases} \Rightarrow \begin{cases} \frac{U(t_0)}{z} = -\frac{\partial}{\partial z}U(t_0) - z\frac{\partial^2}{\partial z^2}U(t_0), \\ \left(\pm\sqrt{V^2(t_0) + W^2(t_0)}\right) = z\left(\frac{\partial}{\partial z}U(t_0)\right), \\ 0 = \frac{\partial}{\partial x}\left(z\left(\frac{\partial}{\partial z}U(t_0)\right)\right) - \frac{\partial}{\partial y}U(t_0), \end{cases} \tag{21}$$

then the *partial* class of self-similar (with respect to coordinate *z*) solutions of (21) can be assumed as below in (22)–(23):

$$\begin{gathered}
\{U(t_0) = U_0(x, y) \cdot U_0(z),\ V(t_0) = V_0(x, y) \cdot V_0(z),\ W(t_0) = W_0(x, y) \cdot W_0(z)\},\ V_0(z) = W_0(z) = \pm\, z\left(\frac{d}{dz}U_0(z)\right),\ \Rightarrow \\
\Rightarrow \begin{cases} \frac{d^2}{dz^2}U_0(z) + \left(\frac{1}{z}\right)\frac{d}{dz}U_0(z) + \left(\frac{1}{z^2}\right)U_0(z) = 0, \\ \sqrt{V_0{}^2(x, y) + W_0{}^2(x, y)} = 1, \\ 0 = \frac{\partial}{\partial x}\left(z\left(\frac{\partial}{\partial z}U(t_0)\right)\right) - \frac{\partial}{\partial y}U(t_0), \end{cases} \Rightarrow \begin{cases} \frac{d^2}{dz^2}U_0(z) + \left(\frac{1}{z}\right)\frac{d}{dz}U_0(z) + \left(\frac{1}{z^2}\right)U_0(z) = 0, \\ \sqrt{V_0{}^2(x, y) + W_0{}^2(x, y)} = 1, \\ 0 = z\left(\frac{d}{dz}U_0(z)\right) \cdot \left(\frac{\partial}{\partial x}U_0(x, y)\right) - U_0(z) \cdot \left(\frac{\partial}{\partial y}U_0(x, y)\right), \end{cases}
\end{gathered} \tag{22}$$

$$U_0(x,y) = const \quad \Rightarrow \quad \begin{cases} \underline{Euler\ equation\ for\ U_0(z)}: \\ U_0(z) = A\sin(\ln|z|) + B\cos(\ln|z|),\ \{A,B\} = const, \\ \\ \{V_0(x,y) = \sin(f(x,y)),\ W_0(x,y) = \cos(f(x,y))\}. \end{cases} \tag{23}$$

Thus, we can present the aforementioned *partial* class of helical solutions (19)–(21) as below in (24) (according to initial conditions as well):

$$\begin{aligned} &\alpha(x,z) = \left(\frac{1}{z}\right),\ \vec{u} = \{U(x,y,z,t),\ V(x,y,z,t),\ W(x,y,z,t)\}, \\ &U(x,y,z,t) = C\Big(A\sin(\ln|z|) + B\cos(\ln|z|)\Big)\cdot \exp\left(-\nu\cdot\alpha^2(z)\cdot t\right),\ \{A,B,C\} = const \\ &V(x,y,z,t) = \pm C\sin(f(x,y))\cdot\ z\left(\frac{d}{dz}\Big(A\sin(\ln|z|) + B\cos(\ln|z|)\Big)\right)\cdot \exp\left(-\nu\cdot\alpha^2(z)\cdot t\right), \\ &W(x,y,z,t) = \pm C\cos(f(x,y))\cdot\ z\left(\frac{d}{dz}\Big(A\sin(\ln|z|) + B\cos(\ln|z|)\Big)\right)\cdot \exp\left(-\nu\cdot\alpha^2(z)\cdot t\right), \end{aligned} \tag{24}$$

where $\{A, B\}$ are dimensionless constants, C has the dimension of velocity, and $f(x, y)$ is an arbitrary *dimensionless* function satisfying initial conditions. For example, we imagined a solution with $f(x,y) = \frac{1}{\sqrt{x^2 + y^2 + D^2}}, D = const$ (see Figure 10 below).

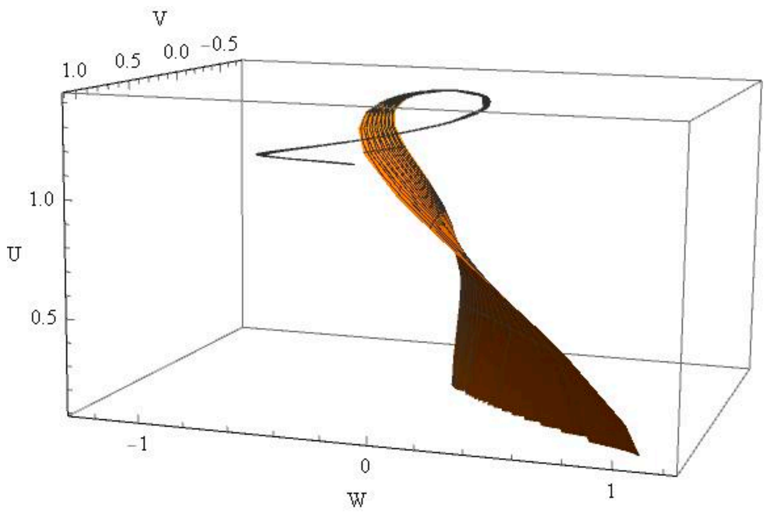

Figure 10. Schematically presented solution of a type (24) for the components of the velocity field if we choose $f(x,y) = \frac{1}{\sqrt{x^2+y^2+D^2}}, D = const.$

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