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Existence Results for Systems of Nonlinear Second-Order and Impulsive Differential Equations with Periodic Boundary

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Abstract: A class for systems of nonlinear second-order differential equations with periodic impulse action are considered. An urgent problem for this class of differential equations is the problem of the quantitative study (existence) in the case when the phase space of the equation is, in the general case, some Banach space. In this work, sufficient conditions for the existence of solutions for a system with parameters are obtained. The results are obtained by using fixed point theorems for operators on a cone. Our approach is based on Schaefer's fixed point theorem more precisely. In addition, the existence of positive solutions is also investigated.

Keywords: iterative methods; periodic solutions; impulses; matrix convergent to 0; generalized banach space; Schaefer's fixed point theorem; differential equations

MSC: 39A12; 34A37; 34K45; 54H25

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1. Introduction and Some Historical Review

Differential equations with impulse action are an important subclass of hybrid systems, i.e., systems of differential equations that merge continuous and discrete time dynamics. They are a new direction in differential equations theory, which has many applications to models of mathematical problems in mechanics, biology, and engineering [1,2]. The very important problem for this type of system is the question of the qualitative properties of solutions. The fundamentals of the theory of the stability of solutions to problems of differential equations with impulse action are presented in [3], where a direct Lyapunov method for this type of problem was also developed. Certain of the relevant results are generalized in monograph [4] using piecewise differentiable auxiliary functions. The works [5,6] show the universality of the direct Lyapunov method in this class of auxiliary functions. In [7], stability conditions for solutions of a nonlinear system with impulse action were obtained based on two auxiliary functions. It is shown that the obtained stability conditions generalize theorems from the monograph [3]. Relevant and important from a practical point of view is the question of the stability for systems of differential equations with impulse action in critical cases. In [8], the problem of generalizing the reduction principle for certain classes of systems of differential equations with impulse action is considered. This principle is a very important tool for study of critical cases; actually, it is reduced to the study of the qualitative properties for a system on the central manifold. Studying the properties of a system of differential equations on a central manifold requires a certain skill of the researcher, since there are no general research methods.

We consider a system of nonlinear second-order and impulsive differential equations with the periodic boundary

$$\begin{cases} u'' - \lambda^2 u = -f_1(t,u,v,\theta), & t \in \mathcal{J} = [0,2\pi], t \neq t_k, k \in [1,m]_{\mathbb{N}}, \\ v'' - \lambda^2 v = -f_2(t,u,v,\theta) & t \in \mathcal{J}, t \neq t_k, k \in [1,m]_{\mathbb{N}}, \\ u(t_k^+) - u(t_k^-) = I_k^1(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ v(t_k^+) - v(t_k^-) = I_k^2(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ u'(t_k^+) - u'(t_k^-) = \bar{I}_k^1(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ v'(t_k^+) - v'(t_k^-) = \bar{I}_k^2(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ u(t=0) = u(t=2\pi), \quad u'(t=0) = u'(t=2\pi), & \\ v(t=0) = v(t=2\pi), \quad v'(t=0) = v'(2\pi), & \end{cases} \tag{1}$$

where $[n,m]_{\mathbb{N}} = \{n, n+1, ..., m\}$, for all $n,m \in \mathbb{N}$, $\lambda \in \mathbb{R}^*$, $u = u(t), v = v(t)$, and θ is a real parameter, $f_1, f_2 \in C^0(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are given functions, I_k^i, $\bar{I}_k^i \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $t_k \in \mathcal{J}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 2\pi$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^+} u(t_k - h)$ represent the right and left limits of $u(t)$ at $t = t_k$. It is well known that much research has been conducted on the question of the existence of solutions for a system of impulsive differential equations; see, for example, [9–13]. In [9,14], the existence of solutions for systems with a nonlocal coupled nonlinear initial condition is studied owing to the different fixed point principles. In this work, in particular, the problem of the existence of positive solutions for the original system of differential equations in second order with impulse action is reduced to the study of a system of nonlinear differential equations with impulse action with equidistant moments of impulse action. The purpose of this work is to develop some existing ideas for differential equations with impulse action. At the same time, along with the ideas in the work [15], new ideas are proposed related to the second-order differential equations with impulse action, see Refs. [16–18].

The paper proceeds as follows. After the introduction and position of problem, in Section 2, we recall some related definitions and facts, which will be useful in our analysis. In Section 3, we use the Perov and Schaefer's type to obtain additional existence results. In Section 4, some existence results based on the Krasnosel'skii-type Theorem in generalized Banach spaces is obtained. Our study concludes with a discussion.

2. Statement of the Problem and Auxiliary Results

Let us recall some important results on the existence of linear/nonlinear impulsive systems, which are generalized in this paper. To this end, let us first set

$$\mathcal{J}_0 = [0, t_1], \mathcal{J}_k = (t_k, t_{k+1}], k \in [1,m]_{\mathbb{N}},$$

and let u_k be the restriction of the function u to $\mathcal{J}_k$. We consider the space $\mathcal{C}(\mathcal{J}, \mathbb{R})$ to be the Banach space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{t \in \mathcal{J}} |u(t)|.$$

$L^1(\mathcal{J}, \mathbb{R})$ denotes the Banach space of measurable functions $u \in C^0(\mathcal{J}, \mathbb{R})$, which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_{\mathcal{J}} |u(t)| dt.$$

$\mathcal{AC}^i(\mathcal{J}, \mathbb{R})$ is the space of i-times differentiable functions $u \in C^0(\mathcal{J}, \mathbb{R})$, whose *ith* derivative, $u^{(i)}$, is absolutely continuous

$$\mathcal{PC}(\mathcal{J}, \mathbb{R}) = \left\{ \begin{array}{l} u \in C^0(\mathcal{J}, \mathbb{R}) : u \text{ is continuous everywhere except for some} \\ \text{at which } u(t_k^-) \text{ and } u(t_k^+) \text{ exist, and } u(t_k^-) = u(t_k), \forall k \in [1,m]_{\mathbb{N}} \end{array} \right\}.$$

Clearly, $\mathcal{PC}$ is a Banach space with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{t\in\mathcal{J}} |u(t)|.$$

Let $\mathcal{PC}(\mathcal{J},\mathbb{R})\times\mathcal{PC}(\mathcal{J},\mathbb{R})$ be endowed with the vector norm $\|.\|$ defined by

$$\|w\| = (\|u\|_{\mathcal{PC}}, \|v\|_{\mathcal{PC}}), \forall w = (u,v),$$

where for $u \in\mathcal{PC}(\mathcal{J},\mathbb{R})$, we set $\|u\|_{\mathcal{PC}} = \sup_{t\in\mathcal{J}} |u(t)|$; it is obvious that $(\mathcal{PC}\times\mathcal{PC}, \|.\|)$ is a generalized Banach space.

$\forall u = (u_1, u_2, \cdots, u_n), v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$, and we note the partial order relation,

$$u \leq v \equiv u_i \leq v_i, \forall i \in [1,n]_{\mathbb{N}}.$$

Let $(X, d_i)_{i\in[1,n]_{\mathbb{N}}}$ be a finite sequence of metric. Let the map $d \in C^0(X\times X, \mathbb{R}^n)$ be given by

$$d(u,v) = (d_1(u,v), ..., d_n(u,v)),$$

and the pair (X,d) is called a generalized metric space.

Let $\|.\|_i$ be an end sequence of norms on u, let the map $\|u\| \in C^0(X,\mathbb{R}^n)$ be given by

$$\|u\| = (\|u\|_1, ..., \|u\|_n),$$

and the pair $(X, \|.\|)$ is called a generalized norms space.

Remark 1. *Let $(X, \|.\|)$ be a generalized norms space, abd we pose*

$$d(u,v) = \|u - v\|, \forall u,v \in X;$$

then, $(X, \|.\|)$ is a generalized norms space.

Theorem 1 ([19])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, and the next claims are equivalent:*

(a) *M is convergent towards 0,*

(b) *$M^k \to 0$ as $k \to \infty$,*

(c) *The matrix $(Id - M)$ is nonsingular, and $(Id - M)^{-1} = \sum_{k=0}^{k=\infty} I^k$,*

(d) *The matrix $(Id - M)$ is nonsingular, and $(Id - M)^{-1}$ has nonnegative elements.*

Let (X,d) be a generalized metric space. An operator $N \in C^0(X,X)$ is said to be contractive associated with d on X, if there exists a convergent to 0 matrix M, such that

$$d(T(u), T(v)) \leq Md(u,v), \forall u,v \in X.$$

The function $f \in C^0([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is an L^1-Carathéodory function, if

1. $t \to f(t,u,v)$ is measurable for any $(u,v) \in \mathbb{R}^2$,
2. $(u,v) \to f(t,.,.)$ is a continuous almost everywhere $0 \leq t \leq 1$,
3. For r_1, $r_2 > 0$, there exists $\phi_{r_1,r_2} \in L^1([0,+\infty[)$, so that $|f(t,u,v)| \leq \phi_{r_1,r_2}(t), \forall(u,v) \in \mathbb{R}^2$, with $|u| \leq r_1, |v| \leq r_2$ and almost everywhere $t \in [0,1]$.

Theorem 2 ([15], page 149)**.** *Let X be a generalized Banach space, and let $T : X \to X$ be a completely continuous map. If the set*

$$\Phi = \{u \in X : u = \lambda_1 Tu, \textit{for some } \lambda_1 \in (0,1)\}$$

is bounded, then T has a fixed point.

Then, we recall the vectorial version of the fixed point Theorem, see [20].

Theorem 3 (Perov fixed point Theorem)**.** *Let (X, d) be a complete generalized metric space and $T \in C^0(X, X)$ be a contractive operator with Lipschitz matrix M. Then, T has a unique fixed point u, and for each $u_0 \in X$,*

$$d(T^k(u_0, u) \leq M^k(Id - M)^{-1} d(u_0, T(u_0)), \text{ where } k \in \mathbb{N}.$$

3. Existence of Solutions

A sufficient condition is given to prove the existence of solutions to (1) owing to Perov's fixed point Theorem. We state without proof the next Lemma 1, which is useful to transform problem (1) into a fixed point problem. Its proof is not difficult.

Lemma 1. *Let $u, v \in \mathcal{PC}(\mathcal{J}, \mathbb{R}) \cap C^1((t_k, t_{k+1}),\, k \in [0, m]_{\mathbb{N}}$ be a solution of (1), if and only if $u, v \in \mathcal{PC}(\mathcal{J}, \mathbb{R})$ is a solution of the following impulsive integral equation*

$$u(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_1(t, u(s), v(s), \theta) ds \\ - \sum\limits_{k=1}^{m} \left[H(t, t_k) I_k^1(u(t_k), v(t_k)) + L(t, t_k) \bar{I}_k^1(u(t_k), v(t_k)) \right], \quad t \in \mathcal{J}, \end{cases}$$

and

$$v(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_2(t, u(s), v(s), \theta) ds \\ - \sum\limits_{k=1}^{m} \left[H(t, t_k) I_k^2(u(t_k), v(t_k)) + L(t, t_k) \bar{I}_k^2(u(t_k), v(t_k)) \right], \quad t \in \mathcal{J}, \end{cases}$$

where

$$H(t,s) = \frac{1}{2\lambda(e^{2\lambda\pi} - 1)} \begin{cases} e^{\lambda(t-s)} + e^{\lambda(2\pi - t + s)}, & 0 \leq s < t \leq 2\pi, \\ e^{\lambda(s-t)} + e^{\lambda(2\pi - s + t)}, & 0 \leq t < s \leq 2\pi, \end{cases}$$

and

$$\begin{aligned} L(t,s) &= \frac{\partial}{\partial t} H(t,s) \\ &= \frac{-1}{2(e^{2\lambda\pi} - 1)} \begin{cases} e^{\lambda(2\pi - t + s)} - e^{\lambda(t-s)}, & 0 \leq s < t \leq 2\pi, \\ e^{\lambda(s-t)} - e^{\lambda(2\pi - s + t)}, & 0 \leq t < s \leq 2\pi. \end{cases} \end{aligned}$$

We assume that

(H_1) There exist constants a_{f_i}, $b_{f_i} \in \mathbb{R}^+$ for each $i = 1, 2$, such that

$$|f_i(t, u, v, \theta) - f_i(t, \overline{u}, \overline{v}, \theta)| \leq a_{f_i} |u - \overline{u}| + b_{f_i} |v - \overline{v}|,$$

for each $t \in \mathcal{J}$ and all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$.

(H_2) There exist $d_k^i, d_k^{i+1}, \overline{d}_k^i, \overline{d}_k^{i+1} \in \mathbb{R}^+$ for each $k \in [1, m]_{\mathbb{N}}$, $i = 1, 2$, such that

$$\begin{cases} |I_k^i(u, v) - I_k^i(\bar{u}, \bar{v})| \leq d_k^i |u - \bar{u}| + d_k^{i+1} |v - \bar{v}| \\ |\overline{I}_k^i(u, v) - \overline{I}_k^i(\bar{u}, \bar{v})| \leq \overline{d}_k^i |u - \bar{u}| + \overline{d}_k^{i+1} |v - \bar{v}|, \end{cases}$$

for each $t \in \mathcal{J}$ and all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$.

Theorem 4. *Assume that (H_1)–(H_2) are satisfied, and the matrix*

$$M_{trix} = \begin{pmatrix} q_1 & \bar{q}_1 \\ q_2 & \bar{q}_2 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+),$$

where

$$q_1 = 2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))a_{f_1} + \sum_{k=1}^{m} \left[d_k^1 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^1 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right],$$

$$\bar{q}_1 = 2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))b_{f_1} + \sum_{k=1}^{m} \left[d_k^2 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^2 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right],$$

and

$$q_2 = 2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))a_{f_2} + \sum_{k=1}^{m} \left(d_k^2 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^3 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right),$$

$$\bar{q}_2 = 2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))b_{f_2} + \sum_{k=1}^{m} \left[d_k^2 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^3 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right].$$

If M_{trix} converges to 0, then the problem (1) has a solution on $\mathcal{J}$.

Proof. Consider the operator

$$\begin{array}{rcl} N : \mathcal{PC}\times\mathcal{PC} & \to & \mathcal{PC}\times\mathcal{PC} \\ (u,v) & \to & (N_1(t,u,v), N_2(t,u,v)), \end{array}$$

$$N_1(u,v)(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_1(t,u(s),v(s),\theta)ds \\ -\sum_{k=1}^{m} \left(H(t,t_k) I_k^1(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^1(u(t_k),v(t_k)) \right), \quad t\in\mathcal{J}, \end{cases}$$

and

$$N_2(u,v)(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_2(t,u(s),v(s),\theta)ds \\ -\sum_{k=1}^{m} \left(H(t,t_k) I_k^2(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^2(u(t_k),v(t_k)) \right), \quad t\in\mathcal{J}. \end{cases}$$

We use Theorem 3 to prove that N has a fixed point. Indeed, let $(u,v),(\bar{u},\bar{v}) \in \mathcal{PC}\times\mathcal{PC}$. Then, we have, for each $t\in\mathcal{J}$,

$$\begin{aligned} & |N_1(t,u,v) - N_1(t,\bar{u},\bar{v})| \\ & \le 2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))\Big(a_{f_1}\|u-\overline{u}\|_{\mathcal{PC}} + b_{f_1}\|v-\overline{v}\|_{\mathcal{PC}} \Big) \\ & + \sum_{k=1}^{m} \left(d_k^1 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^1 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right) \|u-\bar{u}\|_{\mathcal{PC}} \\ & + \sum_{k=1}^{m} \left(d_k^2 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^2 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right) \|v-\bar{v}\|_{\mathcal{PC}} \\ & \le \left(2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))a_{f_1} + \sum_{k=1}^{m} \left[d_k^1 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^1 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right] \right) \|u-\bar{u}\|_{\mathcal{PC}} \\ & + \left(2\pi \sup_{(t,s)\in\mathcal{J}^2} (H(t,s))b_{f_1} + \sum_{k=1}^{m} \left[d_k^2 \sup_{t\in\mathcal{J}} |H(t,t_k)| + \bar{d}_k^2 \sup_{t\in\mathcal{J}} |L(t,t_k)| \right] \right) \|v-\bar{v}\|_{\mathcal{PC}}. \end{aligned}$$

Thus,

$$\|N_1(t,u,v) - N_1(t,\bar{u},\bar{v})\|_{\mathcal{PC}} \le q_1\|u-\bar{u}\|_{\mathcal{PC}} + \bar{q}_1\|v-\bar{v}\|_{\mathcal{PC}}.$$

Similarly, we have

$$\|N_2(t,u,v) - N_2(t,\bar{u},\bar{v})\|_{\mathcal{PC}} \le q_2\|u-\bar{u}\|_{\mathcal{PC}} + \bar{q}_2\|v-\bar{v}\|_{\mathcal{PC}}.$$

Hence,

$$\begin{aligned}\|N(u,v)-N(\overline{u},\overline{v})\|_{\mathcal{PC}} &= \begin{pmatrix}\|N_1((u,v)-N_1(\overline{u},\overline{v})\|_{\mathcal{PC}}\\ \|N_2(u,v)-N_2(\overline{u},\overline{v})\|_{\mathcal{PC}}\end{pmatrix}\\ &\leq \begin{pmatrix}q_1 & \bar{q}_1\\ q_2 & \bar{q}_2\end{pmatrix}\begin{pmatrix}\|u-\overline{u}\|_{\mathcal{PC}}\\ \|v-\overline{v}\|_{\mathcal{PC}}\end{pmatrix}.\end{aligned}$$

This implies that

$$\|N(u,v)-N(\overline{u},\overline{v})\|_{\mathcal{PC}} \leq M_{trix}\begin{pmatrix}\|u-\overline{u}\|_{\mathcal{PC}}\\ \|v-\overline{v}\|_{\mathcal{PC}}.\end{pmatrix}\forall(u,v),(\overline{u},\overline{v})\in\mathcal{PC}\times\mathcal{PC}.$$

From the Perov fixed point Theorem, the mapping N has a unique fixed $(u,v)\in\mathcal{PC}\times\mathcal{PC}$, which is the unique solution of problem (1). This completes the proof. □

Existence Results

In this section, we state our main existence results for problem (1). To this end, we assume

(H_3) There exist a function $p_i\in L^1(\mathcal{J},\mathbb{R}^+)$ and constants $0\leq\alpha_i,\beta_i<1$, such that

$$|f_i(t,u,v,\theta)|\leq p_{f_i}(t)|u|^{\alpha_i}+\bar{p}_{f_i}(t)|v|^{\beta_i},$$

$\forall t\in\mathcal{J}$ and $u,v\in\mathbb{R}$, $i=1,2$.

(H_4) There exist constants $d_k^i, d_k^{i+1}, \overline{d}_k^i, \overline{d}_k^{i+1}\in\mathbb{R}^+$ for each $k\in[1,m]_{\mathbb{N}}$, $i=1,2$ and a constants $0\leq\alpha_i,\beta_i<1$ such that

$$\begin{cases}|I_k^i(u,v)|\leq d_k^i|u|^{\alpha_i}+d_k^{i+1}|v|^{\beta_i}\\ |\overline{I}_k^i(u,v)|\leq \overline{d}_k^i|u|^{\alpha_i}+\overline{d}_k^{i+1}|v|^{\beta_i},\end{cases}$$

$\forall t\in\mathcal{J}$ and all $u,v\in\mathbb{R}, i=1,2$.

(H_5) $f_i\in C^0(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a Carath*é*odory function, and I_k^i, $\overline{I}_k^i\in\mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$.

Theorem 5. *Assume that (H_3)–(H_5) hold. Then, (1) has at least one solution on $\mathcal{J}$. Moreover, the solution set*

$$S=\{(u,v)\in\mathcal{PC}\times\mathcal{PC} : (u,v) \text{ is the solution of } (1)\},$$

and it is compact.

Proof. Clearly, the fixed points of N are solutions to (1), where N is defined in Theorem 4. In order to apply Theorem 2, we first show that N is completely continuous. The proof is given in several steps.

Step 1: $N=(N_1,N_2)$ is continuous.

Let (u_n,v_n) be a sequence, such that $(u_n,v_n)\to(u,v)\in\mathcal{PC}\times\mathcal{PC}$, as $n\to\infty$. Since f_1,f_2 is a Carath*é*odory function, by the Lebesgue dominated convergence Theorem, we obtain

$$\begin{aligned}&|N_1(u_n,v_n)(t)-N_1(u,v)(t)|\\ \leq\ & \sup_{(t,s)\in\mathcal{J}^2}|H(t,s)|\int_0^{2\pi}|f_1(t,u_n(s),v_n(s),\theta)-f_1(t,u(s),v(s),\theta)|ds\\ &+\sum_{k=1}^m\sup_{t\in\mathcal{J}}|H(t,t_k)||I_k^1(u_n(t_k),v_n(t_k))-I_k^1(u(t_k),v(t_k))|\\ &+\sum_{k=1}^m\sup_{t\in\mathcal{J}}|L(t,t_k)||\overline{I}_k^1(u_n(t_k),v_n(t_k))-\overline{I}_k^1(u(t_k),v(t_k))|\to 0,\end{aligned}$$

as $n \to \infty$. Similarly,

$$
\begin{aligned}
& |N_2(u_n, v_n)(t) - N_2(u,v)(t)| \\
\leq & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \int_0^{2\pi} |f_2(t, u_n(s), v_n(s), \theta) - f_2(t, u(s), v(s), \theta)| ds \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)| |I_k^2(u_n(t_k), v_n(t_k)) - I_k^2(u(t_k), v(t_k))| \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| |\bar{I}_k^2(u_n(t_k), v_n(t_k)) - \bar{I}_k^2(u(t_k), v(t_k))| \to 0,
\end{aligned}
$$

when $n \to \infty$. Then, N is continuous.

Step 2: N maps bounded sets into bounded sets in $\mathcal{PC} \times \mathcal{PC}$. It is enough to show that $\forall q > 0$, there exists $l > 0$, such that for each

$$
(u,v) \in B_q = \{(u,v) \in \mathcal{PC} \times \mathcal{PC} : \|u\|_{\mathcal{PC}} \leq q, \ \|v\| \leq q\}.
$$

We obtain

$$
\|N(u,v)\|_{\mathcal{PC}} \leq l = (l_1, l_2).
$$

Then, for each $t \in \mathcal{J}$, we obtain

$$
\begin{aligned}
& |N_1(u,v)(t)| \\
\leq & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \left(q^{\alpha_1} \int_0^{2\pi} p_{f_1}(s) ds + q^{\beta_1} \int_0^{2\pi} \bar{p}_{f_1}(s) ds \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)| \left(d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1} \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| \left(\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1} \right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \|N_1(u,v)\|_{\mathcal{PC}} \\
\leq & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \left(q^{\alpha_1} \|p_{f_1}\|_{L^1} + q^{\beta_1} \|\bar{p}_{f_1}\|_{L^1} \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)| \left(d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1} \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| \left(\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1} \right) = \ell_1.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \|N_2(u,v)\|_{\mathcal{PC}} \\
\leq & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \left(q^{\alpha_1} \|p_{f_2}\|_{L^1} + q^{\beta_1} \|\bar{p}_{f_2}\|_{L^1} \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)| \left(d_k^2 q^{\alpha_2} + d_k^3 q^{\beta_2} \right) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| \left(\bar{d}_k^2 q^{\alpha_2} + \bar{d}_k^3 q^{\beta_2} \right) = \ell_2.
\end{aligned}
$$

Step 3: N maps bounded sets into the equi-continuous one of $\mathcal{PC} \times \mathcal{PC}$. We set $t_1, t_2 \in \mathcal{J}, t_1, t_2 > 0$, and $t_1 < t_2$ with B_q as a bounded set of $\mathcal{PC} \times \mathcal{PC}$, the same as in Step 2. Let $u, v \in B_q$; then, for $t \in \mathcal{J}$, we obtain

$$\begin{aligned}
& |N_1(u,v)(t_2) - N_1(u,v)(t_1)| \\
\leq\ & q^{\alpha_1} \int_0^{t_1} |H(t_2,s) - H(t_1,s)| p_{f_1}(s) ds \\
& + q^{\beta_1} \int_0^{t_1} |H(t_2,s) - H(t_1,s)| \bar{p}_{f_1}(s) ds \\
& + q^{\alpha_1} \int_{t_1}^{t_2} |H(t_2,s)| p_{f_1}(s) ds + q^{\beta_1} \int_{t_1}^{t_2} |H(t_2,s)| \bar{p}_{f_1}(s) ds \\
& + \sum_{t_1 < t < t_2} |H(t_2,t_k) - H(t_1,t_k)| (d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1}) \\
& + \sum_{t_1 < t < t_2} |L(t_2,t_k) - L(t_1,t_k)| (\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1}),
\end{aligned}$$

and

$$\begin{aligned}
& |N_2(u,v)(t_2) - N_2(u,v)(t_1)| \\
\leq\ & q^{\alpha_1} \int_0^{t_1} |H(t_2,s) - H(t_1,s)| p_{f_2}(s) ds \\
& + q^{\beta_1} \int_0^{t_1} |H(t_2,s) - H(t_1,s)| \bar{p}_{f_2}(s) ds \\
& + q^{\alpha_1} \int_{t_1}^{t_2} |H(t_2,s)| p_{f_2}(s) ds + q^{\beta_1} \int_{t_1}^{t_2} |H(t_2,s)| \bar{p}_{f_2}(s) ds \\
& + \sum_{t_1 < t < t_2} |H(t_2,t_k) - H(t_1,t_k)| (d_k^2 q^{\alpha_2} + d_k^3 q^{\beta_2}) \\
& + \sum_{t_1 < t < t_2} |L(t_2,t_k) - L(t_1,t_k)| (\bar{d}_k^2 q^{\alpha_2} + \bar{d}_k^3 q^{\beta_2}).
\end{aligned}$$

The term in the RHS tends to 0, as $t_2 - t_1$ goes to 0.

This proves the equi-continuity for the case where $t \neq t_i$, $i \in [1, m+1]_{\mathbb{N}}$. It remains to examine the equi-continuity at $t = t_i$. First, we prove equi-continuity at $t = t_i^-$. Fix $\nu_1 > 0$, such that $\{t_k : k \neq i\} \cap [t_i - \nu_1, t_i + \nu_1] = \emptyset$. For $0 < h < \nu_1$, we obtain

$$\begin{aligned}
& |N_1(u,v)(t_i) - N_1(u,v)(t_i - h)| \\
\leq\ & q^{\alpha_1} \int_0^{t_i - h} |H(t_i,s) - H(t_i - h,s)| p_{f_1}(s) ds \\
& + q^{\beta_1} \int_0^{t_i - h} |H(t_i,s) - H(t_i - h,s)| \bar{p}_{f_1}(s) ds \\
& + q^{\alpha_1} \int_{t_i - h}^{t_i} |H(t_i,s)| p_{f_1}(s) ds + q^{\beta_1} \int_{t_i - h}^{t_i} |H(t_i,s)| \bar{p}_{f_1}(s) ds \\
& + \sum_{k=1}^{i-1} |H(t_i,t_k) - H(t_i - h,t_k)| (d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1}) \\
& + \sum_{k=1}^{i-1} |L(t_i,t_k) - L(t_i - h,t_k)| (\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1}).
\end{aligned}$$

The RHS tends to 0, as h goes to 0.

Next, we prove equi-continuity at $t = t_i^+$. We fix $\nu_2 > 0$, so that

$$\{t_k : k \neq i\} \cap [t_i - \nu_2, t_i + \nu_2] = \emptyset.$$

For $0 < h < \nu_2$, we obtain

$$\begin{aligned}
& |N_1(u,v)(t_i+h) - N_1(u,v)(t_i)| \\
\leq \ & q^{\alpha_1} \int_0^{t_i} |H(t_i+h,s) - H(t_i,s)| p_{f_1}(s) ds \\
& + q^{\beta_1} \int_0^{t_i} |H(t_i+h,s) - H(t_i,s)| \bar{p}_{f_1}(s) ds \\
& + q^{\alpha_1} \int_{t_i}^{t_i+h} |H(t_i+h,s)| p_{f_1}(s) ds + q^{\beta_1} \int_{t_i}^{t_i+h} |H(t_i+h,s)| \bar{p}_{f_1}(s) ds \\
& + \sum_{0<t_k\leq t_i} |H(t_i+h,t_k) - G(t_i,t_k)| (d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1}) \\
& + \sum_{t_i<t\leq t_i+h} |H(t_i+h,t_k)| (d_k^1 q^{\alpha_1} + d_k^2 q^{\beta_1}) \\
& + \sum_{0<t_k\leq t_i} |L(t_i+h,t_k) - L(t_i,t_k)| (\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1}) \\
& + \sum_{t_i<t\leq t_i+h} |L(t_i+h,t_k)| \bar{d}_k (\bar{d}_k^1 q^{\alpha_1} + \bar{d}_k^2 q^{\beta_1}).
\end{aligned}$$

The RHS tends to 0, as h goes to 0.

As a consequence of Steps 1 to 3 and the Arzela–Ascoli Theorem, we can conclude that $N : \mathcal{PC} \times \mathcal{PC} \to \mathcal{PC} \times \mathcal{PC}$ is a completely continuous operator.

Step 4: Now, it remains to show that the set

$$\mathcal{E} = \{(u,v) \in \mathcal{PC} \times \mathcal{PC} : u = \mu N_1(u,v), \quad v = \mu N_2(u,v), \text{ for some } 0 < \mu < 1\}$$

is bounded. Let $u, v \in \mathcal{E}$; then, $u = \mu N_1(u,v)$ and $v = \mu N_2(u,v)$, for $0 < \mu < 1$. Thus, for $t \in \mathcal{J}$, we obtain

$$\begin{aligned}
|u(t)| \ \leq \ & \int_0^{2\pi} \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \Big(p_{f_1}(s)|u(s)|^{\alpha_1} + \bar{p}_{f_1}(s)|v(s)|^{\beta_1} \Big) ds \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)| \Big(d_k^1 |u(t_k)|^{\alpha_1} + d_k^2 |v(t_k)|^{\beta_1} \Big) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| \Big(\bar{d}_k^1 |u(t_k)|^{\alpha_1} + \bar{d}_k^2 |v(t_k)|^{\beta_1} \Big).
\end{aligned}$$

This implies by (H_3)–(H_5) that for each $t \in \mathcal{J}$ and $v \in \Gamma(N)$, we have

$$\begin{aligned}
\|u\|_{\mathcal{PC}} \ \leq \ & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)| \Big(\|p_{f_1}\|_{L^1} \|u\|_{\mathcal{PC}}^{\gamma} + \|\bar{p}_{f_1}\|_{L^1} \|v\|_{\mathcal{PC}}^{\gamma} \Big) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |G(t,t_k)| \Big(d_k^1 \|u\|_{\mathcal{PC}}^{\gamma} + d_k^2 \|v\|_{\mathcal{PC}}^{\gamma} \Big) \\
& + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)| \Big(\bar{d}_k^1 \|u\|_{\mathcal{PC}}^{\gamma} + \bar{d}_k^2 \|v\|_{\mathcal{PC}}^{\gamma} \Big).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|v\|_{\mathcal{PC}} \leq & \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)|\Big(\|p_{f_2}\|_{L^1}\|u\|^{\gamma}_{\mathcal{PC}} + \|\bar{p}_{f_2}\|_{L^1}\|v\|^{\gamma}_{\mathcal{PC}}\Big) \\ & + \sum_{k=1}^{m}\sup_{t\in\mathcal{J}}|H(t,t_k)|\Big(d_k^2\|u\|^{\gamma}_{\mathcal{PC}} + d_k^3\|v\|^{\gamma}_{\mathcal{PC}}\Big) \\ & + \sum_{k=1}^{m}\sup_{t\in\mathcal{J}}|L(t,t_k)|\Big(\bar{d}_k^2\|u\|^{\gamma}_{\mathcal{PC}} + \bar{d}_k^3\|v\|^{\gamma}_{\mathcal{PC}}\Big).\end{aligned}$$

Thus,

$$\begin{aligned} & \|u\|_{\mathcal{PC}} + \|v\|_{\mathcal{PC}} \\ \leq & \Big(H^*(\|p_{f_1}\|_{L^1} + \|p_{f_2}\|_{L^1}) + G^*(d_k^1 + d_k^2) + L^*(\bar{d}_k^1 + \bar{d}_k^2)\Big)\|u\|^{\gamma}_{\mathcal{PC}} \\ & + \Big(H^*(\|\bar{p}_{f_1}\|_{L^1} + \|\bar{p}_{f_2}\|_{L^1}) + G^*(d_k^2 + d_k^3) + L^*(\bar{d}_k^2 + \bar{d}_k^3)\Big)\|v\|^{\gamma}_{\mathcal{PC}} \\ \leq & K^*(\|u\|^{\gamma}_{\mathcal{PC}} + \|v\|^{\gamma}_{\mathcal{PC}}) \\ \leq & K^*(\|u\|_{\mathcal{PC}} + \|v\|_{\mathcal{PC}})^{\gamma},\end{aligned}$$

where

$$\gamma = \max\{\alpha_i, \beta_i\},$$

and

$$K_1 = H^*(\|p_{f_1}\|_{L^1} + \|p_{f_2}\|_{L^1}) + G^*(d_k^1 + d_k^2) + L^*(\bar{d}_k^1 + \bar{d}_k^2),$$

and

$$K_2 = H^*(\|\bar{p}_{f_1}\|_{L^1} + \|\bar{p}_{f_2}\|_{L^1}) + G^*(d_k^2 + d_k^3) + L^*(\bar{d}_k^2 + \bar{d}_k^3),$$

and

$$K_* = \max\{K_1, K_2\}.$$

If

$$\|u\|_{\mathcal{PC}} + \|v\|_{\mathcal{PC}} > 1,$$

we obtain

$$\|u\|_{\mathcal{PC}} + \|v\|_{\mathcal{PC}} \leq K_*^{\frac{1}{1-\gamma}} = \psi_*.$$

Consequently,

$$\|u\|_{\mathcal{PC}} \leq \max\{1, \psi_*\} = \bar{d}_* \quad \text{and} \quad \|v\|_{\mathcal{PC}} \leq \max\{1, \psi_*\} = \bar{d}_*.$$

This proves that $\mathcal{E}$ is bounded. By Theorem 2, we deduce that N has a fixed point (u, v), which is a solution to (1).

Step 5: Now, we show that the set

$$S = \{(u,v) \in \mathcal{PC} \times \mathcal{PC} : (u,v) \text{ is solution of } (1)\},$$

and it is compact. Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a sequence in S; we put $B = \{(u_n, v_n) : n \in \mathbb{N}\} \subseteq \mathcal{PC} \times \mathcal{PC}$. Then, from earlier parts of the proof of this Theorem, we conclude that B is bounded and equicontinuous. Then, from the Ascoli–Arzela Theorem, we can conclude that B is compact. Hence, $(u_n, v_n)_{n\in\mathbb{N}}$ has a subsequence $(u_n, v_n)_{n_m\in\mathbb{N}} \subseteq S$, such that (u_{n_m}, v_{n_m}) converges to (u, v). Let

$$u_0(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_1(t, u(s), v(s), \theta) ds \\ - \sum\limits_{k=1}^{m} \big(H(t,t_k) I_k^1(u(t_k), v(t_k)) + L(t,t_k)\bar{I}_k^1(u(t_k), v(t_k))\big), \quad t \in \mathcal{J}, \end{cases}$$

and

$$v_0(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_2(t, u(s), v(s), \theta) ds \\ -\sum_{k=1}^{m} \left(H(t,t_k) I_k^2(u(t_k), v(t_k)) + L(t,t_k) \bar{I}_k^2(u(t_k), v(t_k)) \right), \quad t \in \mathcal{J}, \end{cases}$$

and

$$\begin{aligned} & |u_{n_m}(t) - u_0(t)| \\ \leq\ & \int_0^{2\pi} |H(t,s)| |f_1(t, u_{n_m}(s), v_{n_m}(s), \theta) - f_1(t, u(s), v(s), \theta)| ds \\ & + \sum_{k=1}^{m} |H(t,t_k)| |I_k^1(u_{n_m}(s), v_{n_m}(t_k)) - I_k(u(t_k), v(t_k))| \\ & + \sum_{k=1}^{m} |L(t,t_k)| |\bar{I}_k^1(u_{n_m}(s), v_{n_m}(t_k)) - \bar{I}_k(u(t_k), v(t_k))|. \end{aligned}$$

As $n_m \to \infty$, $u_{n_m} \to u_0$, and then

$$u(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_1(t, u(s), v(s), \theta) ds \\ -\sum_{k=1}^{m} \left(H(t,t_k) I_k^1(u(t_k), v(t_k)) + L(t,t_k) \bar{I}_k^1(u(t_k), v(t_k)) \right), \quad t \in \mathcal{J}, \end{cases}$$

and

$$v(t) = \begin{cases} \int_0^{2\pi} H(t,s) f_2(t, u(s), v(s), \theta) ds \\ -\sum_{k=1}^{m} \left(H(t,t_k) I_k^2(u(t_k), v(t_k)) + L(t,t_k) \bar{I}_k^2(u(t_k), v(t_k)) \right), \quad t \in \mathcal{J}. \end{cases}$$

Thus, S is compact. This completes the proof. □

4. Positive Solutions

For $k \in [1, m]_{\mathbb{N}}$, we assume

$$\begin{cases} u'' - \lambda^2 u = -\theta g_1(t) h_1(u,v) = -f_1(t,u,v,\theta), & t \in \mathcal{J} = [0, 2\pi], t \neq t_k, \\ v'' - \lambda^2 v = -\theta g_2(t) h_2(u,v) = -f_2(t,u,v,\theta), & t \in \mathcal{J} = [0, 2\pi], t \neq t_k, \\ u(t_k^+) - u(t_k^-) = I_k^1(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ v(t_k^+) - v(t_k^-) = I_k^2(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ u'(t_k^+) - u'(t_k^-) = \bar{I}_k^1(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ v'(t_k^+) - v'(t_k^-) = \bar{I}_k^2(u(t_k^-), v(t_k^-)), & k \in [1,m]_{\mathbb{N}}, \\ u(t=0) = u(t=2\pi), \qquad u'(t=0) = u'(t=2\pi), \\ v(t=0) = v(t=2\pi), v'(t=0) = v'(t=2\pi), \end{cases} \tag{2}$$

where $(g_i)_{i=1,2} : \mathcal{J} \to \mathbb{R}$, $(h_i)_{i=1,2} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions. Our goal of this part is the existence of positive solutions of (2).

Lemma 2. *Assume that all the conditions of Theorem 4 are satisfied and*

(H_6) *For each $u, v \in \mathbb{R}$ and $k \in [1,m]_{\mathbb{N}}$, $I_k^1(u,v) \leq 0$,*
(H_7) *For each $u, v \in \mathbb{R}$, $i = 1, 2$, and $t \in \mathcal{J}$,*

$$g_i(t) h_i(u,v) \geq 0.$$

(H_8) *For each $u, v \in \mathbb{R}$, $t \in \mathcal{J}$, and $k \in [1,m]_{\mathbb{N}}$,*

$$L(t,t_k) \bar{I}_k^1(u,v) \leq 0$$

Then, problem (2) has a unique positive solution on $\mathcal{J}$.

Proof. Consider the operator

$$\begin{array}{rcl} N : \mathcal{PC} \times \mathcal{PC} & \to & \mathcal{PC} \times \mathcal{PC} \\ (u,v) & \to & (N_1(t,u,v), N_2(t,u,v)) \end{array}$$

$$N_1(t,u,v) = \begin{cases} \theta \int_0^{2\pi} H(t,s)g_1(s)h_1(u(s),v(s))ds \\ -\sum\limits_{k=1}^{m} \big(H(t,t_k)I_k^1(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^1(u(t_k,v(t_k))\big), \quad t \in \mathcal{J}, \end{cases}$$

and

$$N_2(t,u,v) = \begin{cases} \theta \int_0^{2\pi} H(t,s)g_1(s)h_2(u(s),v(s))ds \\ -\sum\limits_{k=1}^{m} \big(H(t,t_k)I_k^2(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^2(u(t_k,v(t_k))\big), \quad t \in \mathcal{J}. \end{cases}$$

We prove that the fixed points of N are positive solutions to (2). Indeed, assume that $(u,v) \in \mathcal{PC} \times \mathcal{PC}$ is a fixed point of N. It is clear that

$$u(t) = \begin{cases} \theta \int_0^{2\pi} H(t,s)g_1(s)h_1(u(s),v(s))ds \\ -\sum\limits_{k=1}^{m} \big(H(t,t_k)I_k^1(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^1(u(t_k,v(t_k))\big), \quad t \in \mathcal{J}, \end{cases}$$

and

$$v(t) = \begin{cases} \theta \int_0^{2\pi} H(t,s)g_2(s)h_2(u(s),v(s))ds \\ -\sum\limits_{k=1}^{m} \big(H(t,t_k)I_k^2(u(t_k),v(t_k)) + L(t,t_k)\bar{I}_k^2(u(t_k,v(t_k))\big), \quad t \in \mathcal{J}, \end{cases}$$

which imply that (u,v) is a solution of (2).

If (u,v) is a fixed point of N, then (H_6) through (H_8) imply that $(u(t),v(t)) \geq (0,0)$, for each $t \in \mathcal{J}$.

As in Theorem 3, we can show that N is a contraction; so, by Perov's, we conclude that N has a unique fixed point (u,v), which is a positive solution of problem (2). This completes the proof. □

Let X be a real generalized Banach space. A nonempty closed convex set $\mathcal{K} \subset E$ is a cone, if it satisfies the following two conditions

(i) If $v \in \mathcal{K}$ and $\lambda \geq 0$, then $\lambda v \in \mathcal{K}$,
(ii) If $v, -v \in \mathcal{K}$, then $v = 0$.

$\mathcal{K}$ is a solid cone, if $Int(\mathcal{K}) \neq \emptyset$, where $Int(\mathcal{K})$ is the interior of $\mathcal{K}$.

Remark 2. *Any cone $\mathcal{K} \subset X$ induces a partial ordering $<$ on E given by*

$$u < v \Leftrightarrow v - u \in \mathcal{K}.$$

Theorem 6 ([21])**.** *Let X be a real generalized Banach space, $\mathcal{K} \subset X$ a cone of X, and $R > 0$. Let $\mathcal{K}_R = \{u \in \mathcal{K} : ||u|| < R\}$, and let $N : \mathcal{K}_R \to C$ be a completely continuous operator, where $r \in (0,R)$. If*

(i) $||N(u)|| < ||u||, \forall u \in \partial\mathcal{K}_r$,
(ii) $||N(u)|| > ||u||, \forall u \in \partial\mathcal{K}_R$,

then N has at least two fixed points $u, v, \in \mathcal{K}_R$, such that

$$||u|| < r, \qquad r < ||v|| \leq R.$$

Let $\mathcal{K}_1, \mathcal{K}_2$ be a solid cone of a real Banach space X and

$$N : Int(\mathcal{K}_1) \times Int(\mathcal{K}_2) \to Int(\mathcal{K}_1)$$

be an operator; it is said that N is called an α-concave operator, if

$$N(tu, tv) \geq t^{\alpha} N(u, v), \text{ for any } (u, v) \in in(\mathcal{K}) \text{ and } 0 < t < 1, \alpha \in [0, 1).$$

Let $(\mathcal{K}_i)_{i=1,2}$ be a solid cone of a real Banach space X and

$$N : Int(\mathcal{K}_1) \times Int(\mathcal{K}_2) \to Int(\mathcal{K}_1) \times Int(\mathcal{K}_2)$$

be an operator; it is said that N is called an α-concave operator, if $\{N_i\}_{i=1,2}$ is called an α-concave operator, with $N = (N_1, N_2)$.

Lemma 3 ([22])**.** *Let $\mathcal{K}$ be a normal solid cone of a real Banach space X, and*

$$N : Int(\mathcal{K}) \times Int(\mathcal{K}) \to Int(\mathcal{K}) \times Int(\mathcal{K})$$

be α-concave increasing operator. Then, N has only one fixed point in $Int(\mathcal{K}) \times Int(\mathcal{K})$.

By Lemma 3, we deduce the following Corollary

Corollary 1. *Let $\mathcal{K}_1, \mathcal{K}_2$ are a solid cone of a real Banach space X and*

$$N : Int(\mathcal{K}_1) \times Int(\mathcal{K}_2) \to Int(\mathcal{K}_1) \times Int(\mathcal{K}_2),$$

is α-concave increasing operator. Then N has only one fixed point in $Int(\mathcal{K}_1) \times Int(\mathcal{K}_2)$.

Theorem 7. *Assume (H_6)–(H_8) and the following conditions are satisfied:*

$$(f_i)_{i=1,2} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R},$$

is nondecreasing function and $I_k^i, \bar{I}_k^i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are decreasing functions, with

$$\begin{cases} \int_0^{2\pi} H(t,s) g_i(s) h_i(u(s), v(s)) ds > 0, \\ H(t, t_k) I_k^i(u(t_k), v(t_k)) + L(t, t_k) \bar{I}_k^i(u(t_k), v(t_k)) < 0, \\ f_i(\eta u, \eta v) \geq \eta^{\alpha} f_i(u, v), \\ I_k^i(\eta u, \eta v) \leq \eta^{\alpha} I_k^i(u, v), \\ \bar{I}_k(\eta u, \eta v) \leq \eta^{\alpha} \bar{I}_k^i(u, v), \end{cases}$$

$\forall u, v > 0, t \in \mathcal{J}, i = 1, 2, 0 < \eta < 1,$ *where* $0 \leq \alpha < 1, i = 1, 2.$

Then, problem (2) has a unique positive solution $(u_\theta(t), v_\theta(t))$.

Proof. We pose

$$\mathcal{K}_1 = \{u \in \mathcal{PC} : u(t) \geq 0 \text{ for } t \in \mathcal{J}\} \text{ and } \mathcal{K}_2 = \{v \in \mathcal{PC} : v(t) \geq 0, \text{ for } t \in \mathcal{J}\}. \quad (3)$$

Then, $\mathcal{K}_1, \mathcal{K}_2$ are cones in $\mathcal{PC}$, by (H_6) through (H_8), which imply that, $N_i(\mathcal{K}_1 \times \mathcal{K}_2) \subset \mathcal{K}_i$, for $i = 1, 2$. We assert that

$$N : Int(\mathcal{K}_1) \times Int(\mathcal{K}_2) \to Int(\mathcal{K}_1) \times Int(\mathcal{K}_2)$$

is an α-concave increasing operator. Indeed,

$$\begin{aligned}
& N_i(\eta u, \eta v) \\
= \; & \theta \int_0^{2\pi} H(t,s) g_i(s) h_i(\eta u(s), \eta v(s)) ds \\
- \; & \sum_{k=1}^{m} \Big(H(t,t_k) I_k^i(\eta u(t_k), \eta v(t_k)) + L(t,t_k) \bar{I}_k^i(\eta u(t_k), \eta v(t_k)) \Big) \\
\geq \; & \eta^\alpha \theta \int_0^{2\pi} H(t,s) g_i(s) h_i(u(s), v(s)) ds \\
& -\eta^\alpha \sum_{k=1}^{m} \Big(H(t,t_k) I_k^i(u(t_k), v(t_k)) + L(t,t_k) \bar{I}_k^i(u(t_k), v(t_k)) \Big) \\
\geq \; & \eta^\alpha N_i(u,v), \text{ for any } 0 < \eta < 1,
\end{aligned}$$

where $0 \leq \alpha < 1$. Since $f_i(u,v)$ is nondecreasing and I_k^i, $\bar{I}_k^i$ are decreasing, then

$$\begin{aligned}
& N_i(u_1, v_1)(t) \\
= \; & \theta \int_0^{2\pi} H(t,s) g_i(s) h_i(u_1(s), v_1(s)) ds \\
- \; & \sum_{k=1}^{m} \Big(H(t,t_k) I_k^i(u_1(t_k), v_1(t_k)) + L(t,t_k) \bar{I}_k^i(u_1(t_k), v_1(t_k)) \Big) \\
\leq \; & \theta \int_0^{2\pi} H(t,s) g_i(s) h_i(u_2(s), v_2(s)) ds \\
- \; & \sum_{k=1}^{m} \Big(H(t,t_k) I_k^i(u_2(t_k), v_2(t_k)) + L(t,t_k) \bar{I}_k^i(u_2(t_k), v_2(t_k)) \Big) \\
= \; & N_i(u_2, v_2)(t), \text{ for } (u_1, v_1) \leq (u_2, v_2), (u_1, v_1), (u_2, v_2) \in C \times C.
\end{aligned}$$

By Corollary 1, N has a unique fixed point $(u_\theta, v_\theta) \in Int(\mathcal{K}_1) \times Int(\mathcal{K}_2)$. This completes the proof of the Theorem. $\square$

Now, we seek a solution to problem (2) via the Krasnosel'skii twin fixed point Theorem

Theorem 8. *Assume (H_3)–(H_8) and the following conditions are satisfied:*

(H_9) *There exist $R_1, R_2 > 0$ and $r_1, r_2 > 0$, with $r_1 < R_1$ and $r_2 < R_2$, such that*

$$\begin{cases}
\theta \sup\limits_{(t,s) \in \mathcal{J}^2} |H(t,s)| \|g_1\|_\infty h_1^*(r_1, r_2) + \sum\limits_{k=1}^{m} \sup\limits_{t \in \mathcal{J}} |H(t,t_k)| (d_k^1 r_1^{\alpha_1} + d_k^2 r_2^{\beta_1}) \\
+ \sum\limits_{k=1}^{m} \sup\limits_{t \in \mathcal{J}} |L(t,t_k)| (\bar{d}_k^1 r_1^{\alpha_1} + \bar{d}_k^2 r_2^{\beta_1}) < r_1 \\
\theta \sup\limits_{(t,s) \in \mathcal{J}^2} |H(t,s)| \|g_2\|_\infty h_2^*(r_1, r_2) + \sum\limits_{k=1}^{m} \sup\limits_{t \in \mathcal{J}} |H(t,t_k)| (d_k^2 r_1^{\alpha_2} + d_k^3 r_2^{\beta_2}) \\
+ \sum\limits_{k=1}^{m} \sup\limits_{t \in \mathcal{J}} |L(t,t_k)| (\bar{d}_k^2 r_1^{\alpha_2} + \bar{d}_k^3 r_2^{\beta_2}) < r_2,
\end{cases}$$

where

$$h_i^*(r_1, r_2) = \sup_{(u,v) \in (0, r_1] \times (0, r_2]} |h_i(u,v)|, \quad i = 1, 2,$$

$$\begin{cases} \min\limits_{t\in[0,2\pi]} \theta \int_0^{2\pi} H(t,s)g_1(s)h_1(\mu_1(s),\mu_2(s))ds \\ -\sum\limits_{k=1}^{m} \left(\left(H(t,t_k)I_k^1(\mu_1(t_k),\mu_2(t_k)) + L(t,t_k)\bar{I}_k^1(\mu_1(t_k),\mu_2(t_k)) \right) \right) > R_1, \\ \mathit{if}\ (r_1,r_2) < (\mu_1,\mu_2) \\ \\ \min\limits_{t\in[0,2\pi]} \theta \int_0^{2\pi} H(t,s)g_2(s)h_2(\mu_1(s),\mu_2(s))ds \\ -\sum\limits_{k=1}^{m} \left(H(t,t_k)I_k^2(\mu_1(t_k),\mu_2(t_k)) + L(t,t_k)\bar{I}_k^2(\mu_1(t_k),\mu_2(t_k)) \right) > R_2, \\ \mathit{if}\ (r_1,r_2) < (\mu_1,\mu_2). \end{cases}$$

Then, problem (2) *has at least two positive solutions* $(u_1,v_1),(u_2,v_2)$*, such that*

$$\|u_1\| < r_1, \|v_1\| < r_2 \text{ and } r_1 < \|u_2\| \le R_1, r_2 < \|v_2\| \le R_2.$$

Proof. Let $\mathcal{K}_1,\mathcal{K}_2$ be a cone defined in (3). Then, (H_6) through (H_8) imply that $N(\mathcal{K}_1,\mathcal{K}_2) \subset \mathcal{K}_1,\mathcal{K}_2$. For any $R_1,R_2 > 0$,

$$\mathcal{K}_{R_1} = \{u \in \mathcal{K}_1 : \|u\| < R_1\},$$

and

$$\mathcal{K}_{R_2} = \{v \in \mathcal{K} : \|v\| < R_2\}.$$

Using (H_4)–(H_7), we can show that $N \in C^1(\mathcal{K}_{R_1} \times \mathcal{K}_{R_2}, \mathcal{K}_1 \times \mathcal{K}_2$ is a completely continuous operator.

We should deal with the hypotheses of the Krasnosel'skii twin fixed point Theorem 6.

Claim 1: $(\|N_1(u,v)\|_{\mathcal{PC}}, \|N_2(u,v)\|_{\mathcal{PC}}) < (\|u\|_{\mathcal{PC}}, \|v\|_{\mathcal{PC}}), \forall (u,v) \in \partial\mathcal{K}_{r_1} \times \partial\mathcal{K}_{r_2}$, where

$$\mathcal{K}_{r_1} = \{u \in \mathcal{K} : \|u\| < r_1\} \text{ and } \mathcal{K}_{r_2} = \{v \in \mathcal{K} : \|v\| < r_2\}.$$

For $(u,v) \in \partial\mathcal{K}_{r_1} \times \partial\mathcal{K}_{r_2}$, from (H_6), (H_7) and (H_9), we have

$$\begin{aligned} & |N_1(u,v)(t)| \\ \le\ & \theta \int_0^{2\pi} |H(t,s)||g_1(s)h_1(u(s),v(s))|ds \\ & + \sum_{k=1}^{m} |H(t,t_k)||I_k^1(u(t_k),v(t_k))| + \sum_{k=1}^{m} |L(t,t_k)||\bar{I}_k^1(u(t_k),v(t_k))| \\ \le\ & \theta \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)|\|g_1\|_\infty h_1^*(r_1,r_2) + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)|(d_k^1 r_1^{\alpha_1} + d_k^2 r_2^{\beta_1}) \\ & + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)|(\bar{d}_k^1 r_1^{\alpha_1} + \bar{d}_k^2 r_2^{\beta_1}) < r_1 = \|u\|_{\mathcal{PC}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & |N_2(u,v)(t)| \\ \le\ & \theta \sup_{(t,s)\in\mathcal{J}^2} |H(t,s)|\|g_2\|_\infty h_2^*(r_1,r_2) + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |H(t,t_k)|(d_k^2 r_1^{\alpha_2} + d_k^3 r_2^{\beta_2}) \\ & + \sum_{k=1}^{m} \sup_{t\in\mathcal{J}} |L(t,t_k)|(\bar{d}_k^2 r_1^{\alpha_2} + \bar{d}_k^3 r_2^{\beta_2}) < r_2 = \|v\|_{\mathcal{PC}}. \end{aligned}$$

Thus,

$$\|N_1(u,v)\|_{\mathcal{PC}} < \|u\|_{\mathcal{PC}} \text{ and } \|N_2(u,v)\|_{\mathcal{PC}} < \|v\|_{\mathcal{PC}}, \text{ for each } (u,v) \in \partial\mathcal{K}_{r_1} \times \partial C_{r_2}.$$

Claim 2: $(\|N_1(u,v)\|_{\mathcal{PC}}, \|N_2(u,v)\|_{\mathcal{PC}}) > (\|u\|_{\mathcal{PC}}, \|v\|_{\mathcal{PC}}), \forall (u,v) \in (\partial C_{R_1}, \partial C_{R_2})$, for $(u,v) \in \partial \mathcal{K}_{R_1} \times \partial \mathcal{K}_{R_2}$, and we have $\|u\|_{\mathcal{PC}} = R_1$, $\|v\|_{\mathcal{PC}} = R_2$; then, $r_1 < \|u\|_{\mathcal{PC}} = R_1, r_2 < \|v\|_{\mathcal{PC}} = R_2$, and from (H_9), we have

$$\begin{aligned} N_1(u,v)(t) &\geq \min_{t\in[0,2\pi]} \theta \int_0^{2\pi} H(t,s) g_1(s) h_1(\mu_1(s), \mu_2(s)) ds \\ &\quad - \sum_{k=1}^{m} \Big(H(t,t_k) I_k^1(\mu_1(t_k), \mu_2(t_k)) + L(t,t_k) \bar{I}_k^1(\mu_1(t_k), \mu_2(t_k)) \Big) \\ &> R_1 = \|u\|_{\mathcal{PC}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} N_2(u,v)(t) &\geq \min_{t\in[0,2\pi]} \theta \int_0^{2\pi} H(t,s) g_2(s) h_2(\mu_1(s), \mu_2(s)) ds \\ &\quad - \sum_{k=1}^{m} \Big(H(t,t_k) I_k^2(\mu_1(t_k), \mu_2(t_k)) + L(t,t_k) \bar{I}_k^2(\mu_1(t_k), \mu_2(t_k)) \Big) \\ &> R_2 = \|v\|_{\mathcal{PC}}. \end{aligned}$$

Thus,

$$\|N_1(u,v)\|_{\mathcal{PC}} > \|u\|_{\mathcal{PC}} \text{ and } \|N_2(u,v)\|_{\mathcal{PC}} > \|v\|_{\mathcal{PC}}, \text{ for each } (u,v) \in \partial \mathcal{K}_{R_1} \times \partial \mathcal{K}_{R_2}.$$

Then, problem (2) has at least two positive solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{K}_{R_1} \times \mathcal{K}_{R_2}$, such that

$$\begin{cases} \|u_1\| < r_1, \|v_1\| < r_2 \\ r_1 < \|u_2\| \leq R_1, r_2 < \|v_2\| \leq R_2. \end{cases}$$

This completes the proof. □

5. Conclusions and Discussion of the Results

Sufficient conditions for the existence of solutions to systems of nonlinear second-order differential equations with periodic impulse action are constructed. The proposed Schaefer's fixed point theorem is quite effective in studying in such cases for systems of nonlinear differential equations. Necessary and sufficient conditions for the existence of positive solutions are also established.

Practical Significance

Transient processes in electrical circuits are modeled using degenerate differential equations or differential algebraic equations. Taking into account pulsed effects on currents and voltages in an electrical circuit at fixed times significantly complicates the study of transient regimes. The obtained theoretical results make it possible to indicate the necessary and sufficient conditions for the existence of the stated problem (1), which can model electrical circuits with pulsed effects. The development of the mathematical theory of this type of system is associated with practical problems in the theory of control for complex systems and the widespread introduction of digital technologies, see [23–25].

Extending these results to consider the question of stability (Qualitative studies) will make it possible to advance the study in this direction.

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