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On the Space of G-Permutation Degree of Some Classes of Topological Spaces

Ljubiša D. R. Kočinac [1,2,*], Farkhod G. Mukhamadiev [3] and Anvar K. Sadullaev [4]

1 Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia
2 Department of Mathematics, State University of Novi Pazar, 36300 Novi Pazar, Serbia
3 Faculty of Mathematics, National University of Uzbekistan Named after Mirzo Ulugbek, Str. University 4, Tashkent 100174, Uzbekistan; farhod8717@mail.ru
4 Department of Exact Sciences, Yeoju Technical Institute in Tashkent, Str. Usman Nasyr 156, Tashkent 100121, Uzbekistan; anvars1997@mail.ru
* Correspondence: lkocinac@gmail.com

Abstract: In this paper, we study the space of G-permutation degree of some classes of topological spaces and the properties of the functor SP^n_G of G-permutation degree. In particular, we prove: (a) If a topological space X is developable, then so is $\mathsf{SP}^n_G X$; (b) If X is a Moore space, then so is $\mathsf{SP}^n_G X$; (c) If a topological space X is an M_1-space, then so is $\mathsf{SP}^n_G X$; (d) If a topological space X is an M_2-space, then so is $\mathsf{SP}^n_G X$.

Keywords: functor of permutation degree; developable space; Moore space; M_1-space; M_2-space; Nagata space

MSC: 18F60; 54B30; 54E99

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1. Introduction

Let F be a covariant functor acting on a class of topological spaces. The following natural general problem in the theory of covariant functors was posed by V. V. Fedorchuk at the Prague Topological Symposium in 1981 (see [1]):

> Let $\mathcal{P}$ be a topological property and F a covariant functor. If a topological space X has the property $\mathcal{P}$, then whether $\mathsf{F}(X)$ has the same property, and vice versa, if $\mathsf{F}(X)$ has the property $\mathcal{P}$, does the space X also have the property $\mathcal{P}$?

This paper deals with such questions.

Let G be a subgroup of the symmetric group S_n, $n \in \mathbb{N}$, of all permutations of the set $\{1, 2, \dots, n\}$, and let X be a topological space. On the space X^n, define the following equivalence relation r_G: for elements $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in X^n

$$\mathbf{x}\, r_G\, \mathbf{y} \Leftrightarrow \text{ there is } \sigma \in G \text{ with } y_i = x_{\sigma(i)}, 1 \le i \le n.$$

The relation r_G is called the *G-symmetric equivalence relation*. The equivalence class of an element $\mathbf{x} \in X^n$ is denoted by $[\mathbf{x}]_G$ or $[(x_1, x_2, \dots, x_n)]_G$. The quotient space X^n/r_G (equipped with the quotient topology of the topology on X^n) is called the *space of G-permutation degree of* X and is denoted by $\mathsf{SP}^n_G X$. The quotient mapping of X^n to this space is denoted by $\pi^s_{n,G}$; when $G = S_n$, one writes π^s_G.

Let $f : X \to Y$ be a continuous mapping. Define the mapping $\mathsf{SP}^n_G : \mathsf{SP}^n_G X \to \mathsf{SP}^n_G Y$ by

$$\mathsf{SP}^n_G f([\mathbf{x}]_G) = [(f(x_1), f(x_2), \dots, f(x_n))]_G, \ [\mathbf{x}]_G \in \mathsf{SP}^n_G X.$$

It is easy to verify that SP^n_G as defined is a functor in the category of compacta. This functor is called the *functor of G-permutation degree*.

In [1,2], V. V. Fedorchuk and V. V. Filippov investigated the functor of G-permutation degree, and it was proved that this functor is a normal functor in the category of compact spaces and their continuous mappings.

In recent years, a number of studies have investigated various covariant functors, in particular the functor of G-permutation degree, and their influence on some topological properties (see, for instance, [3–6]). In [3,4], the index of boundedness, uniform connectedness, and homotopy properties of the space of G-permutation degree have been studied, and it was shown in [4] that the functor $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}$ preserves the homotopy and the retraction of topological spaces. References [5,6] deal with certain tightness-type properties and Lindelöf-type properties of the space of G-permutation degree.

The current paper is devoted to the investigation of some classes of topological spaces (such as developable spaces, Moore spaces, M_1-spaces, M_2-spaces, Lašnev's and Nagata's spaces) in the space of G-permutation degree.

Throughout the paper, all spaces are assumed to be T_1.

Observe that the space $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$ is related to the space $\exp_{\mathrm{n}}X$ of nonempty $\leq n$-element subsets of X equipped with the Vietoris topology whose base form the sets of the form

$$O\langle U_1, U_2, \ldots, U_k\rangle = \{F \in \exp_{\mathrm{n}}X : F \subset \cup_{i=1}^{k} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, \ldots, k\}$$

where $U_1, U_2, \ldots, U_k$ are open subsets of X [2].

Observe that the mapping $\pi^h_{n,G} : \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X \to \exp_n X$ assigning to each G-symmetric equivalence class $[(x_1, x_2, \ldots, x_n)]_G$ the hypersymmetric equivalence class $[(x_1, x_2, \ldots, x_n)]^{hc}$ containing it represents the functor $\exp_{\mathrm{n}}$ as the factor functor of the functor $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}$ [1,2].

Also, the spaces $\mathrm{SP}^2_{\mathrm{G}}X$ and $\exp_2 X$ are homeomorphic, while it is not the case for $n > 2$ [2].

2. Results

In this section, we present the results obtained in this study.

For an open cover γ of a space X and a subset A of X, *the star of A with respect to γ* is defined by $\mathrm{St}(A, \gamma) = \bigcup\{U \in \gamma : U \cap A \neq \emptyset\}$.

Let γ be an open cover of X. Obviously, $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma = \{\pi^s_{n,G}(U_1 \times \ldots \times U_n) = [U_1 \times \ldots \times U_n]_G : U_1, \ldots, U_n \in \gamma\}$ is an open cover of $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$.

Proposition 1. *Let $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma$ be an open cover of $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$. For each $[(x_1, \ldots, x_n)]_G \in \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$, we have*

$$\mathrm{St}([(x_1, \ldots, x_n)]_G, \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma) \subset [\mathrm{St}(x_1, \gamma) \times \ldots \times \mathrm{St}(x_n, \gamma)]_G.$$

Proof. Let $[(y_1, \ldots, y_n)]_G \in \mathrm{St}([(x_1, \ldots, x_n)]_G, \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma)$. Then, there exists $[U_1 \times \ldots \times U_n]_G \in \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma$ such that $[(y_1, \ldots, y_n)]_G \in [U_1 \times \ldots \times U_n]_G$. On the other hand, $[U_1 \times \ldots \times U_n]_G \subset [V_1 \times \ldots \times V_n]_G$ if and only if $\bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n} V_i$ and for every V_i, $i = 1, 2, \ldots, n$, there exists a permutation $\sigma \in G$ such that $U_{\sigma(i)} \subset V_i$. Hence, we obtain that $[(y_1, \ldots, y_n)]_G \in [U_1 \times \ldots \times U_n]_G \subset [\mathrm{St}(x_1, \gamma) \times \ldots \times \mathrm{St}(x_n, \gamma)]_G$. This means that $\mathrm{St}([(x_1, \ldots, x_n)]_G, \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\gamma) \subset [\mathrm{St}(x_1, \gamma) \times \ldots \times \mathrm{St}(x_n, \gamma)]_G$. □

Lemma 1. *Let $x_1, x_2, \ldots, x_n$ be points of X. For each $i = 1, 2, \ldots, n$, let $\{U_{im}\}_{m=1}^{\infty}$ be a decreasing sequence of nonempty subsets of X such that $\bigcap_{m=1}^{\infty} U_{im} = \{x_i\}$. Then,*

$$\bigcap_{m=1}^{\infty} [U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G = \{[(x_1, x_2, \ldots, x_n)]_G\}.$$

Proof. Let $i = 1, 2, \ldots, n$, and assume that $[y_1, y_2, \ldots, y_n]_G \in \bigcap_{m=1}^{\infty}[U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G$. Then, for each positive integer m, $[y_1, y_2, \ldots, y_n]_G \in [U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G$. This means that there exists a permutation $\sigma \in G$ such that $y_i \in U_{\sigma(i)m}$ for all $i = 1, 2, \ldots, n$. In addition, $y_i \in \bigcap_{m=1}^{\infty} U_{\sigma(i)m} = \{x_{\sigma(i)}\}$ for all $i = 1, 2, \ldots, n$. Consequently, it follows that $y_i = x_{\sigma(i)}$. This means that $[(y_1, y_2, \ldots, y_n)]_G = [(x_1, x_2, \ldots, x_n)]_G$. □

Proposition 2. *Let X be a space, and let $x_1, x_2, \ldots, x_n$ be points of X. For each $i = \overline{1,n}$, let $\mathcal{U}_i = \{\mathcal{U}_{im}\}_{m\in\mathbb{N}}$ be a local base of X at x_i. Then, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{U} = \{[U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G : U_{im} \in \mathcal{U}_i, i = \overline{1,n}\}_{m\in\mathbb{N}}$ is a local base of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ at $[(x_1, x_2, \ldots, x_n)]_G$.*

Proof. Without loss of the generality, suppose that $\mathcal{U}_{im+1} \subset \mathcal{U}_{im}$ for every positive integer m. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$ be an open subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ which contains $[(x_1, x_2, \ldots, x_n)]_G$. Then, there exist open subsets $V_1, V_2, \ldots, V_n$ of X such that $[(x_1, x_2, \ldots, x_n)]_G \in [V_1 \times V_2 \times \ldots \times V_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. Put $V_{x_i} = \cap\{V \in \{V_1, V_2, \ldots, V_n\} : x_i \in V\}$ for every $i = \overline{1,n}$. Then, $V_{x_1}, \ldots, V_{x_n}$ are open subsets of X such that $[(x_1, x_2, \ldots, x_n)]_G \in [V_{x_1} \times V_{x_2} \times \ldots \times V_{x_n}]_G \subset [V_1 \times V_2 \times \ldots \times V_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. Since $\mathcal{U}_i$ is a local base at x_i, there exists a positive integer m_i such that $x_i \in U_{m_i i} \subset V_{x_i}$. Let $m = \max\{m_1, \ldots, m_n\}$. Then, $x_i \in U_{mi} \subset V_{x_i}$. Consequently, $[U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{U}$ and $[(x_1, x_2, \ldots, x_n)]_G \in [U_{1m} \times U_{2m} \times \ldots \times U_{nm}]_G \subset [V_{x_1} \times V_{x_2} \times \ldots \times V_{x_n}] \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. Therefore, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{U}$ is a local base of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ at $[(x_1, x_2, \ldots, x_n)]_G$. □

A space X is *developable* [7,8] if there exists a sequence $\{\gamma_m : m \in \mathbb{N}\}$ of open covers of X such that, for each $x \in X$, $\{\mathrm{St}(x, \gamma_m) : m \in \mathbb{N}\}$ is a local base at x. Such a sequence of covers is called a *development* for X. It is well known that every metrizable space is developable, and every developable space is clearly first countable.

Remark 1. *Clearly, the above definition of the developable space is equivalent to the following:*

(a) For each $x \in X$ and for each positive integer m such that $\mathrm{St}(x, \gamma_m) \neq \varnothing$, $\mathrm{St}(x, \gamma_m)$ is a neighborhood of the point x, and

(b) For each $x \in X$ and for each open U containing x, there exists a positive integer m such that $x \in \mathrm{St}(x, \gamma_m) \subset U$.

Theorem 1. *If X is a developable space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Assume that X is a developable space and $\{\mu_m : m \in \mathbb{N}\}$ is a development for X. For every $m \in \mathbb{N}$, let

$$\gamma_m = \left\{\bigcap_{j=1}^{m} V_j : V_j \in \mu_j, j = \overline{1,n}\right\}.$$

Then, $\{\gamma_m\}_{m\in\mathbb{N}}$ is also a development for X such that $\mathrm{St}(x, \gamma_{m+1}) \subset \mathrm{St}(x, \gamma_m)$ for all $x \in X$ and every $m \in \mathbb{N}$. Put

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\gamma_m = \{[U_{m1} \times \ldots \times U_{mn}]_G : U_{m1}, \ldots, U_{mn} \in \gamma_m\}.$$

It can be easily checked that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\gamma_m$ is an open cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ for every $m \in \mathbb{N}$.

Now, we will prove that for each $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, $\{\mathrm{St}([(x_1, x_2, \ldots, x_n)]_G, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\gamma_m)\}_{m\in\mathbb{N}}$ is a local base at $[(x_1, x_2, \ldots, x_n)]_G$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U$ be an open subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ such that $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U$. Then, there exist open subsets $U_1, U_2, \ldots, U_n$ of X such that $[(x_1, x_2, \ldots, x_n)]_G \in [U_1 \times U_2 \times \ldots \times U_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U$. Since $\{\mathrm{St}(x_i, \gamma_m)\}_{m\in\mathbb{N}}$ is a local base at x_i for any $i = \overline{1,n}$, there exists a positive integer m_i such that $\mathrm{St}(x_i, \gamma_{m_i}) \subset U_{x_i} = \bigcap\{U_j : x_i \in U_j, j = \overline{1,n}\}$. Then ,there exists $m \geq \max\{m_1, m_2, \ldots, m_n\}$ such that $\mathrm{St}(x_i, \gamma_m) \subset \mathrm{St}(x_i, \gamma_{m_i})$ for all $i = \overline{1,n}$. By Proposition 1, we have

$$\begin{aligned}
[(x_1, x_2, \ldots, x_n)]_G &\in \mathrm{St}([(x_1, x_2, \ldots, x_n)]_G, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\gamma_m) \\
&\subset [\mathrm{St}(x_1, \gamma_{m_1}) \times \ldots \times \mathrm{St}(x_n, \gamma_{m_n})]_G \\
&\subset [U_{x_1} \times \ldots \times U_{x_n}]_G \subset [U_1 \times \ldots \times U_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U.
\end{aligned}$$

By Statement (b) of Remark 1, it means that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a developable space. □

A regular developable space is a *Moore space* [7,8].

Proposition 3. *If X is a Moore space, then so is $\mathsf{SP}^n_G X$.*

Proof. By Theorem 1, if X is a developable space, then the space $\mathsf{SP}^n_G X$ is also developable. On the other hand, it is well known from [9] that regularity is preserved under the closed-and-open mapping and Cartesian product. Therefore, if X is a regular space, then the space $\mathsf{SP}^n_G X$ is also regular. □

A family $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of a topological space is *closure preserving* [7,9] if $\overline{\bigcup_{\alpha\in\mathcal{A}_0} U_\alpha} = \bigcup_{\alpha\in\mathcal{A}_0} \overline{U_\alpha}$ for every $\mathcal{A}_0 \subset \mathcal{A}$.

Theorem 2. *If $\mathcal{U}$ is a closure-preserving family of subsets of X, then $\mathsf{SP}^n_G\mathcal{U} = \{[U_1 \times U_2 \times \ldots \times U_n]_G : U_1, U_2, \ldots, U_n \in \mathcal{U}\}$ is a closure-preserving family of subsets of $\mathsf{SP}^n_G X$.*

Proof. Let $\mathsf{SP}^n_G\mathcal{U}_0$ be a subfamily of $\mathsf{SP}^n_G\mathcal{U}$ and $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^n_G X \setminus \bigcup\{\overline{\mathsf{SP}^n_G W} : \mathsf{SP}^n_G W \in \mathsf{SP}^n_G\mathcal{U}_0\}$. Let $V_i = X \setminus \bigcup\{\overline{U} : x_i \in X \setminus \overline{U}, U \in \mathcal{U}\}$. Since $\mathcal{U}$ is a closure preserving family of subsets of X, we have that $V_i = X \setminus \overline{\bigcup\{U : x_i \in X \setminus \overline{U}, U \in \mathcal{U}\}}$. This means that V_i is an open subset of X and $x_i \in V_i$ for all $i = 1, 2, \ldots, n$. Let $\mathsf{SP}^n_G V = [V_1 \times V_2 \times \ldots \times V_n]_G$. Then, $\mathsf{SP}^n_G V$ is open subset of $\mathsf{SP}^n_G X$, $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^n_G V$ and $\mathsf{SP}^n_G V \bigcap \mathsf{SP}^n_G W = \varnothing$ for all $\mathsf{SP}^n_G W \in \mathsf{SP}^n_G\mathcal{U}_0$. Therefore, $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^n_G V \subset \mathsf{SP}^n_G X \setminus \bigcup\{\mathsf{SP}^n_G W : \mathsf{SP}^n_G W \in \mathsf{SP}^n_G\mathcal{U}_0\}$. It shows that $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^n_G X \setminus \overline{\bigcup\{\mathsf{SP}^n_G W : \mathsf{SP}^n_G W \in \mathsf{SP}^n_G\mathcal{U}_0\}}$. Hence, $\mathsf{SP}^n_G\mathcal{U}$ is a closure preserving family of subsets of $\mathsf{SP}^n_G X$. □

A family $\mathcal{U}$ is called *σ-closure preserving* [7] if it is represented as a union of countably many closure preserving subfamilies.

An *M_1-space* [7,8] is a regular space having a σ-closure preserving base.

Example 1. *Let $\mathbb{Q}$ denote the set of rational numbers. For $x \in \mathbb{R}$, put $L_x = \{(x, y) : (x, y) \in \mathbb{R}^2, y > 0\}$ and $X = \mathbb{R} \cup (\bigcup\{L_x : x \in \mathbb{R}\})$. Define a base for a topology on X as follows: for any $s, t \in \mathbb{Q}$ and $z = (x, w) \in L_x$ such that $0 < s < w < t$, we put $\mathcal{U}^x_{s,t}(z) = \{(x, y) : s < y < t\}$, and let $\mathcal{U}$ be the set of all such $\mathcal{U}^x_{s,t}(z)$. For all $r, s, t \in \mathbb{Q}$ and $z \in \mathbb{R}$ such that $s < z < t$ and $r > 0$, we put*

$$\mathcal{V}_{r,s,t}(z) = (s, t) \cup (\bigcup\{(w, y) : 0 < y < r, w \in (s, t) \setminus \{z\}\})$$

, and let $\mathcal{V}$ be the set of all $V_{r,s,t}(z)$. Now, put $\mathcal{B} = \mathcal{U} \cup \mathcal{V}$. Then one can check that $\mathcal{B}$ is a σ-closure preserving base for X. It shows that X is an M_1-space. Moreover, the space X is a first countable, but non-metrizable space.

Theorem 3. *If X is an M_1-space, then so is $\mathsf{SP}^n_G X$.*

Proof. Let X be an M_1-space and $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ be a σ-closure preserving base in X. Since the union of two closure preserving family of subsets of X is also closure preserving, we assume that $\mathcal{U}_i \subset \mathcal{U}_{i+1}$ for each i. For every positive integer i, set $\mathsf{SP}^n_G\mathcal{U}_i = \{[U_1 \times U_2 \times \ldots \times U_n]_G : U_1, U_2, \ldots, U_n \in \mathcal{U}_i\}$. Obviously, $\mathsf{SP}^n_G\mathcal{U}_i \subset \mathsf{SP}^n_G\mathcal{U}_{i+1}$ for all positive integers i. By Theorem 2, $\mathcal{U}_i$ is a closure preserving family of subsets of $\mathsf{SP}^n_G X$, and at the same time $\mathcal{U}_i$ is a family of open subsets of $\mathsf{SP}^n_G X$. Therefore, $\mathsf{SP}^n_G\mathcal{U} = \bigcup_{i=1}^{\infty} \mathsf{SP}^n_G\mathcal{U}_i$ is a σ-closure preserving family of open subsets of $\mathsf{SP}^n_G X$.

Now, we will show that $\mathsf{SP}^n_G\mathcal{U}$ is a base for $\mathsf{SP}^n_G X$. Let $[(x_1, x_2, \ldots, x_n)]_G$ be an arbitrary element of $\mathsf{SP}^n_G X$ and $\mathsf{SP}^n_G U$ be an open subset of $\mathsf{SP}^n_G X$ such that $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^n_G U$. Since $\mathcal{U}$ is a base for X, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $[(x_1, x_2, \ldots, x_n)]_G \in [U_1 \times U_2 \times \ldots \times U_n]_G \subset \mathsf{SP}^n_G U$. Since $\mathcal{U}_i \subset \mathcal{U}_{i+1}$ for each positive integer i, there exists i_0 such that $U_1, U_2, \ldots, U_n \in \mathcal{U}_{i_0}$. Then it follows that $[U_1 \times U_2 \times \ldots \times U_n]_G \in \mathsf{SP}^n_G\mathcal{U}_{i_0}$. Therefore, $\mathsf{SP}^n_G\mathcal{U}$ is a base for $\mathsf{SP}^n_G X$. This means that $\mathsf{SP}^n_G X$ is an M_1-space. □

A collection $\mathcal{B}$ of (not necessarily open) subsets of a regular space X is a *quasi-base* in X [7] if whenever $x \in X$ and U is a neighborhood of x, there exists a $B \in \mathcal{B}$ such that $x \in \mathrm{Int}B \subset B \subset U$.

An M_2-*space* [7,8] is a regular space having a σ-closure preserving quasi-base.

Theorem 4. *If X is an M_2-space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Suppose that X is an M_2-space and $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ-closure preserving quasi-base. Since the union of two closure-preserving family of subsets of X is also closure preserving, we assume that $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ for each i. For each positive integer i, put $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}_i = \{[B_1 \times B_2 \times \ldots \times B_n]_G : B_1, B_2, \ldots, B_n \in \mathcal{B}_i\}$. Obviously, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}_i \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}_{i+1}$ for all i. By Theorem 2, $\mathcal{B}_i$ is a closure preserving family of subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Therefore, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B} = \bigcup_{i=1}^{\infty} \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}_i$ is a σ-closure preserving family of subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.

Now, we will prove that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}$ is a quasi-base for $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Let $[(x_1, x_2, \ldots, x_n)]_G$ be an arbitrary element of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$ be an open subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ such that $[(x_1, x_2, \ldots, x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. Consequently, there exist open subsets $V_1, V_2, \ldots, V_n$ of X such that $[(x_1, x_2, \ldots, x_n)]_G \in [V_1 \times V_2 \times \ldots \times V_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. Since $\mathcal{B}$ is a quasi-base for X, there exist a permutation $\sigma \in G$ and $B_{\sigma(j)} \in \mathcal{B}_i$ such that $x_j \in \mathrm{Int}B_{\sigma(j)} \subset V_{\sigma(j)}$, where $j = 1, 2, \ldots, n$. Note that $[(x_1, x_2, \ldots, x_n)]_G \in [\mathrm{Int}B_1 \times \mathrm{Int}B_2 \times \ldots \times \mathrm{Int}B_n]_G \subset \mathrm{Int}([B_1 \times B_2 \times \ldots \times B_n]_G) \subset [B_1 \times B_2 \times \ldots \times B_n]_G \subset [V_1 \times V_2 \times \ldots \times V_n]_G \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V$. It shows that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathcal{B}$ is a quasi-base for $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. $\square$

Recall now that a space X is said to be stratifiable if f for every closed subset $F \subset X$ there is a sequence of open subsets $(U(F,k))_{k\in\mathbb{N}}$ such that (i) $F = \bigcap_{k\in\mathbb{N}} U(F,k) = \bigcap_{k\in\mathbb{N}} \overline{U(F,k)}$, and (ii) if $F_1 \subset F_2$, then $U(F_1,k) \subset U(F_2,k)$ for each $k \in \mathbb{N}$. In the paper [10] it was proved that a space is stratifiable if and only if it is M_2. Therefore, we obtain the following:

Corollary 1. *If a space X is stratifiable, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

A space X is a *Lašnev space* [7,8] if there exist a metric space Z and a continuous closed mapping from Z onto X. Lašnev spaces are known to be M_1-spaces.

Theorem 5. *Let X be a space, and let n be a positive integer. If X^n is a Lašnev space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Suppose that X^n is a Lašnev space. Then, there exist a metric space Z and a continuous closed mapping $g : Z \to X^n$. Since $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a closed, onto mapping, we obtain that the mapping $\pi^s_{n,G} \circ g : Z \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is also a closed mapping from the metric space Z onto the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. This means that the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a Lašnev space. $\square$

Theorem 6 ([8])**.** *Let X be a space. Then, X^2 is a Lašnev space if and only if $\exp_2 X$ is a Lašnev space.*

As we said in the Introduction, in Reference [2], it was shown that the spaces SP^2X and $\exp_2 X$ are homeomorphic. Hence, we obtain the following corollary.

Corollary 2. *Let X be a space. Then, X^2 is a Lašnev space if and only if SP^2X is a Lašnev space.*

A space X is a *Nagata space* [11] provided that for each $x \in X$, there exist sequences $\{U_m(x)\}_{m\in\mathbb{N}}$ and $\{V_m(x)\}_{m\in\mathbb{N}}$ of open neighborhoods of x such that for all $x, y \in X$:

(1) $\{U_m(x)\}_{m\in\mathbb{N}}$ is a local base at x;
(2) if $y \notin U_m(x)$, then $V_m(x) \cap V_m(y) = \varnothing$ (or equivalently, if $V_m(x) \cap V_m(y) \neq \varnothing$, then $x \in U_m(y)$).

The definition of the Nagata space is equivalent to the following [11,12]: a Nagata space is a first countable stratifiable space.

Corollary 3. *Let X be a space, and let n be a positive integer. If X is a Nagata space, then so is $\mathsf{SP}^n_G X$.*

3. Conclusions

This work is related to the following important question. Let F be a covariant functor and $\mathcal{P}$ a topological property. If a space X has the property $\mathcal{P}$, whether $\mathsf{F}(X)$ has the same or some other property. We studied the preservation of certain classes of spaces (developable spaces, Moore space, M_1- and M_2-spaces, Nagata spaces) under the influence of the functor SP^n_G of G-permutation degree. We proved that this functor preserves each mentioned class of spaces. It would be interesting to study the preservation of these and some other properties under the influence of other important functors.

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