mathematics

Article
Ricci Vector Fields

Hanan Alohali and Sharief Deshmukh *

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; halohali@ksu.edu.sa
* Correspondence: shariefd@ksu.edu.sa

Abstract: We introduce a special vector field ω on a Riemannian manifold (N^m, g), such that the Lie derivative of the metric g with respect to ω is equal to ρRic, where Ric is the Ricci tensor of (N^m, g) and ρ is a smooth function on N^m. We call this vector field a ρ-Ricci vector field. We use the ρ-Ricci vector field on a Riemannian manifold (N^m, g) and find two characterizations of the m-sphere $S^m(\alpha)$. In the first result, we show that an m-dimensional compact and connected Riemannian manifold (N^m, g) with nonzero scalar curvature admits a ρ-Ricci vector field ω such that ρ is a nonconstant function and the integral of $Ric(\omega, \omega)$ has a suitable lower bound that is necessary and sufficient for (N^m, g) to be isometric to m-sphere $S^m(\alpha)$. In the second result, we show that an m-dimensional complete and simply connected Riemannian manifold (N^m, g) of positive scalar curvature admits a ρ-Ricci vector field ω such that ρ is a nontrivial solution of the Fischer–Marsden equation and the squared length of the covariant derivative of ω has an appropriate upper bound, if and only if (N^m, g) is isometric to m-sphere $S^m(\alpha)$.

Keywords: ρ-Ricci vector fields; Fischer–Marsden equation; m-sphere; Ricci curvature

MSC: 53C20; 53C21; 53B50

Citation: Alohali, H.; Deshmukh, S. Ricci Vector Fields. *Mathematics* **2023**, *11*, 4622. https://doi.org/10.3390/math11224622

Academic Editor: Daniele Ettore Otera

Received: 25 October 2023
Revised: 9 November 2023
Accepted: 10 November 2023
Published: 12 November 2023

1. Introduction

An m-dimensional complete simply connected Riemannian manifold of constant curvature α is isometric to one of the following spaces: the m-sphere $S^m(\alpha)$, the Euclidean space R^m, or the hyperbolic space $H^m(\alpha)$, referred to as $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$, respectively (cf. [1]). Because of this classification, there has been an interest in obtaining necessary and sufficient conditions on complete Riemannian manifolds so that they are isometric to one of the three model spaces $S^m(\alpha)$, R^m, and $H^m(\alpha)$, respectively. One of most sought questions is about obtaining different characterizations of spheres $S^m(\alpha)$ among complete Riemannian manifolds. In obtaining these characterizations, most of the time, the conformal and Killing vector fields are used on an m-dimensional complete Riemannian manifold (N^m, g) (cf. [2–11]). A vector field $\mathbf{u}$ on m-Riemannian manifold (N^m, g) is a conformal vector field if the Lie derivative $\pounds_{\mathbf{u}} g$ has the expression

$$\pounds_{\mathbf{u}} g = 2fg,$$

where f is a smooth function called the conformal factor. If $f = 0$ in the above definition, then $\mathbf{u}$ is called a Killing vector field.

In this paper, we are interested in a vector field ω on an m-dimensional Riemannian manifold (N^m, g) that satisfies

$$\frac{1}{2}\pounds_{\omega} g = \rho Ric, \tag{1}$$

where $\pounds_{\omega} g$ is the Lie-derivative of the metric g with respect to ω, ρ is a smooth function, and Ric is the Ricci tensor of (N^m, g). We call ω satisfying Equation (1) a ρ-Ricci vector field on (N^m, g). Naturally, if (N^m, g) is an Einstein manifold, then a ρ-Ricci vector field

ω is a conformal vector field on (N^m, g) (cf. [3,4]). If, in Equation (1), we take $\rho = 0$, then the 0-Ricci vector field ω on (N^m, g) is a Killing vector field on (N^m, g) (cf. [12]). A ρ-Ricci vector field on (N^m, g) is also a particular form of a potential field of a generalized soliton (cf. [12]), with $\alpha = -\rho$ and $\beta = \gamma = 0$.

We could also approach to Equation (1) in another context (cf. [13]). On the m-dimensional Riemannian manifold (N^m, g), take a smooth function ρ and consider a 1-parameter family of metrics $g(t)$ satisfying the generalized Ricci flow (or ρ-Ricci flow) equation

$$\partial_t g = 2\rho Ric, \quad g(0) = g. \tag{2}$$

To reach a solution of above flow, we take a 1-parameter family of diffeomorphisms $\varphi_t : N^m \to N^m$ generated by the family of vector fields $\mathbf{W}(t)$ and let $\sigma(t)$ be a scale factor. Then, we are interested in a solution of flow (2) of the form

$$g(t) = \sigma(t)\varphi_t^*(g).$$

Differentiating the above equation with respect to t and substituting $t = 0$, while assuming $\sigma(0) = 1$, $\dot{\sigma}(0) = 0$, $\mathbf{W}(0) = \omega$, and using $\varphi_0 = id$, we obtain

$$\pounds_\omega g - 2\rho Ric = 0,$$

which is Equation (1). Thus, a ρ-Ricci vector field ω on (N^m, g) can be considered as stable solution of the flow (2).

We see that as a trivial example on the Euclidean space R^m, a constant vector field $\mathbf{a}$ is a ρ-Ricci vector field for any smooth function ρ on R^m. Similarly on the complex Euclidean space C^m with complex structure J and the vector field

$$\xi = \sum_{i=1}^{m} z^i \frac{\partial}{\partial z^i},$$

where $z^1, \ldots, z^m$ are Euclidean coordinates, the vector field $\omega = J\xi$ is a ρ-Ricci vector field for any smooth function ρ on C^m.

Next, we show that on the sphere $S^m(\alpha)$ of constant curvature α, there are many ρ-Ricci vector fields. With the embedding $i : S^m(\alpha) \to R^{m+1}$ and unit normal ξ and shape operator $-\sqrt{\alpha}I$, upon taking a nonzero constant vector field $\mathbf{b}$ on the Euclidean space R^{m+1}, we have $\mathbf{b} = \omega + f\xi$, where $f = \langle \mathbf{b}, \xi \rangle$ and ω is the tangential component of $\mathbf{b}$ to the sphere $S^m(\alpha)$. We denote the induced metric on the sphere $S^m(\alpha)$ by g and the Riemannian connection by D. Then, differentiating the above equation with respect to the vector field X on $S^m(\alpha)$, we have

$$D_X\omega = -\sqrt{\alpha} f X, \quad \nabla f = \sqrt{\alpha}\omega, \tag{3}$$

where ∇f is the gradient of f. Using the first equation in (3), it follows that

$$\pounds_\omega g = -2\sqrt{\alpha} f g$$

and the Ricci tensor of the sphere $S^m(\alpha)$ is given by

$$Ric = (m-1)\alpha g.$$

Thus, we see that the vector field ω on the sphere $S^m(\alpha)$ satisfies

$$\frac{1}{2}\pounds_\omega g = \rho Ric, \quad \rho = -\frac{1}{(m-1)\sqrt{\alpha}} f, \tag{4}$$

that is, ω is a ρ-Ricci vector field on the sphere $S^m(\alpha)$. Indeed, for each nonzero constant vector field on the Euclidean space R^{m+1}, there is a ρ-Ricci vector field on the sphere $S^m(\alpha)$.

The above example naturally leads to a question: Under what conditions is a compact and connected m-dimensional Riemannian manifold (N^m, g) admitting a ρ-Ricci vector field ω isometric to a m-sphere $S^m(\alpha)$?

There are two well-known differential equations on a Riemannian manifold (N^m, g). The first is Obata's differential equation, namely (cf. [6,7]),

$$Hess(\sigma) = -\alpha\sigma g, \tag{5}$$

where σ is a non-constant smooth function, α is a positive constant, and $Hess(\sigma)$ is the Hessian of σ defined by

$$Hess(\sigma)(X, Y) = g(D_X \nabla\sigma, Y),$$

for smooth vector fields X, Y on N^m. Obata proved that a necessary and sufficient condition for a complete and simply connected Riemannian manifold (N^m, g) to admit a nontrivial solution of differential Equation (5) is that (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [6,7]). The other differential equation on (N^m, g) is the Fischer–Marsden equation (cf. [14–19])

$$(\Delta\sigma)g + \sigma Ric = Hess(\sigma), \tag{6}$$

where σ is a smooth function on N^m and $\Delta\sigma = div(\nabla\sigma)$ is the Laplacian of σ. We shall abbreviate the above Fischer–Marsden equation as FM-equation. Taking trace in the FM-Equation (6), we obtain

$$\Delta\sigma = -\frac{\tau}{m-1}\sigma, \tag{7}$$

where $\tau = TrRic$ is the scalar curvature of the Riemannian manifold (N^m, g). It is known that if (N^m, g) admits a nontrivial solution to the FM-equation, then the scalar curvature τ is necessarily constant (cf. [14]).

Note that by Equation (3), the smooth function f on the sphere $S^m(\alpha)$ has the Hessian

$$Hess(f)(X, Y) = g(D_X \nabla f, Y) = \sqrt{\alpha} g(D_X \omega, Y) = -\alpha f g(X, Y),$$

the Laplacian $\Delta f = div\left(\sqrt{\alpha}\omega\right) = -m\alpha f$, and $Ric = (m-1)\alpha g$. Consequently, on $S^m(\alpha)$, we see that

$$(\Delta f)g + fRic = Hess(f), \tag{8}$$

that is, f is a solution of the FM-equation on the sphere $S^m(\alpha)$. If we combine the two, namely a Riemannian manifold (N^m, g) admits a ρ-Ricci vector field ω such that ρ is a nontrivial solution of the FM-equation on (N^m, g), and seek an additional condition under which (N^m, g) is isometric to $S^m(\alpha)$, we can notice that the ρ-Ricci vector field ω on the sphere $S^m(\alpha)$ is a closed vector field. Therefore, in this paper, we use the closed ρ-Ricci vector field ω on a Riemannian manifold (N^m, g) and answer these two question in Section 3, where we find two characterizations of the sphere $S^m(\alpha)$.

In respect to first question raised above, in Section 3, we show that if a closed ρ-Ricci vector field ω on an m-dimensional compact and connected Riemannian manifold (N^m, g), $m > 2$ with scalar curvature $\tau \neq 0$, and nonzero nonconstant function ρ satisfies

$$\int_M Ric(\omega, \omega) \geq \frac{m-1}{m} \int_M (div\omega)^2,$$

then the scalar curvature τ is a positive constant $\tau = m(m-1)\alpha$, and (N^m, g) is isometric to $S^m(\alpha)$ (cf. Theorem 1). Also, the converse holds. Moreover, in respect to the second question raised above, we prove that if an m-dimensional complete and simply connected Riemannian manifold (N^m, g) with scalar curvature $\tau > 0$ admits a closed ρ-Ricci vector field ω such that the function ρ is a nontrivial solution of the FM-equation and the length of covariant derivative of ω satisfies

$$\|\nabla\omega\|^2 \leq \frac{1}{m}\tau^2\rho^2,$$

then τ is a positive constant $\tau = m(m-1)\alpha$ and (N^m, g) is isometric to $S^m(\alpha)$ (cf. Theorem 2), and the converse also holds.

2. Preliminaries

Let ω be a closed ρ-Ricci vector field on an m-dimensional Riemannian manifold (N^m, g). If β is the 1-form dual to ω, that is,

$$\beta(X) = g(\omega, X), \quad X \in \Theta(TN^m), \tag{9}$$

where $\Theta(TN^m)$ is the space of smooth sections of the tangent bundle TN^m, then we have $d\beta = 0$. We denote by ∇_X the covariant derivative operator with respect to the Riemannian connection on (N^m, g) and notice that for the closed ρ-Ricci vector field ω, we have

$$\begin{aligned} 2g(\nabla_X\omega, Y) &= g(\nabla_X\omega, Y) + g(\nabla_Y\omega, X) + g(\nabla_X\omega, Y) - g(\nabla_Y\omega, X) \\ &= (\pounds_\omega g)(X, Y) + d\beta(X, Y) = 2\rho Ric(X, Y). \end{aligned}$$

Thus, for a closed ρ-Ricci vector field ω, we have

$$\nabla_X\omega = \rho TX, \quad X \in \Theta(TN^m), \tag{10}$$

where T is a symmetric operator called the Ricci operator given by

$$Ric(X, Y) = g(TX, Y).$$

Using the expression for the curvature tensor field R of (N^m, g)

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X.Y]}Z, \quad X, Y, Z \in \Theta(TN^m),$$

and Equation (10), we obtain

$$R(X, Y)\omega = X(\rho)TY - Y(\rho)TX + \rho((\nabla_X T)(Y) - (\nabla_Y T)(X)), \tag{11}$$

$X, Y \in \Theta(TN^m)$, where $(\nabla_X T)(Y) = \nabla_X TY - T(\nabla_X Y)$. The scalar curvature τ of (N^m, g) is given by $\tau = TrT$, where TrT is the trace of the symmetric operator T. Choosing a local frame $\{F_1, \ldots, F_m\}$ and using the definition of the Ricci tensor Ric

$$Ric(X, Y) = \sum_{j=1}^{m} g\big(R(F_j, X)Y, F_j\big),$$

together with Equation (3), we conclude that

$$Ric(Y, \omega) = Ric(Y, \nabla\rho) - \tau Y(\rho) + \rho g\left(Y, \sum_{j=1}^{m}\Big(\nabla_{F_j}T\Big)(F_j)\right) - \rho Y(\tau), \tag{12}$$

where $\nabla\rho$ is the gradient of ρ. It is known that the gradient of scalar curvature τ satisfies (cf. [1])

$$\frac{1}{2}\nabla\tau = \sum_{j=1}^{m}\Big(\nabla_{F_j}T\Big)(F_j). \tag{13}$$

Consequently, Equation (12) takes the form

$$Ric(Y, \omega) = Ric(Y, \nabla\rho) - \tau Y(\rho) - \frac{1}{2}\rho Y(\tau) \tag{14}$$

and we have

$$T(\omega) = T(\nabla\rho) - \tau\nabla\rho - \frac{1}{2}\rho\nabla\tau. \tag{15}$$

3. Characterizing Spheres via ρ-Ricci Fields

Let ω be a closed ρ-Ricci vector field on an m-dimensional Riemannian manifold (N^m, g). We shall use ρ-Ricci vector field and find two characterizations of m-sphere $\mathbf{S}^m(\alpha)$. In our first result, we prove the following result:

Theorem 1. *A closed ρ-Ricci vector field ω on an m-dimensional compact and connected Riemannian manifold (N^m, g), $m > 2$ with scalar curvature $\tau \neq 0$ and nonzero nonconstant function ρ satisfies*

$$\int_M Ric(\omega, \omega) \geq \frac{m-1}{m} \int_M (div\omega)^2,$$

if and only if, τ is a positive constant $m(m-1)\alpha$, and (N^m, g) is isometric to $S^m(\alpha)$.

Proof. Let (N^m, g) be an m-dimensional compact and connected Riemannian manifold, $m > 2$ with scalar curvature $\tau \neq 0$ and ω be a closed ρ-Ricci vector field defined on (N^m, g) with nonzero and nonconstant function ρ satisfying

$$\int_M Ric(\omega, \omega) \geq \frac{m-1}{m} \int_M (div\omega)^2. \tag{16}$$

Then using Equation (10), we have

$$div\omega = \rho\tau. \tag{17}$$

Choosing a local orthonormal frame $\{F_1, \ldots, F_m\}$ and using

$$\|T\|^2 = \sum_{j=1}^{m} g\big(TF_j, TF_j\big)$$

and an outcome of Equation (10) as

$$(\pounds_\omega g)(X, Y) = 2\rho g(TX, Y), \quad X, Y \in \Theta(TN^m),$$

we conclude

$$\frac{1}{2}|\pounds_\omega g|^2 = 2\rho^2\|T\|^2. \tag{18}$$

Note that, we have

$$\begin{aligned} \left\|T - \frac{\tau}{m}I\right\|^2 &= \sum_{j=1}^{m} g\Big(\Big(TE_j - \frac{\tau}{m}E_j\Big), \Big(TE_j - \frac{\tau}{m}E_j\Big)\Big) \\ &= \|T\|^2 + \frac{1}{m}\tau^2 - 2\sum_{j=1}^{m} g\Big(TE_j, \frac{\tau}{m}E_j\Big), \end{aligned}$$

that is,

$$\left\|T - \frac{\tau}{m}I\right\|^2 = \|T\|^2 - \frac{1}{m}\tau^2. \tag{19}$$

Now, using Equation (10), we have

$$\rho\Big(TX - \frac{\tau}{m}X\Big) = \Big(\nabla_X\omega - \frac{\tau}{m}\rho X\Big),$$

which in view of a local frame $\{F_1, \ldots, F_m\}$ on (N^m, g) implies

$$\begin{aligned}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2 &= \sum_{j=1}^{m} g\left(\rho\left(TE_j-\frac{\tau}{m}E_j\right),\rho\left(TE_j-\frac{\tau}{m}E_j\right)\right)\\ &= \sum_{j=1}^{m} g\left(\nabla_{E_j}\omega-\frac{\tau}{m}\rho E_j,\nabla_{E_j}\omega-\frac{\tau}{m}\rho E_j\right)\\ &= \|\nabla\omega\|^2+\frac{1}{m}\tau^2\rho^2-\frac{2}{m}\tau\rho div\omega.\end{aligned}$$

Using (17), in above equation, yields

$$\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\|\nabla\omega\|^2-\frac{1}{m}\tau^2\rho^2,$$

which upon integration gives

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\int_{N^m}\left(\|\nabla\omega\|^2-\frac{1}{m}\tau^2\rho^2\right). \tag{20}$$

Next, we recall the following integral formula (cf. [20])

$$\int_{N^m}\left(Ric(\omega,\omega)+\frac{1}{2}|\pounds_\omega g|^2-\|\nabla\omega\|^2-(div\omega)^2\right)=0,$$

and employing it in Equation (20), we conclude

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\int_{N^m}\left(Ric(\omega,\omega)+\frac{1}{2}|\pounds_\omega g|^2-(div\omega)^2-\frac{1}{m}\tau^2\rho^2\right).$$

Using Equations (17) and (18) in the above equation yields

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\int_{N^m}\left(Ric(\omega,\omega)+2\rho^2\|T\|^2-\tau^2\rho^2-\frac{1}{m}\tau^2\rho^2\right),$$

that is,

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\int_{N^m}\left(Ric(\omega,\omega)+2\rho^2\left(\|T\|^2-\frac{1}{m}\tau^2\rho^2\right)-\tau^2\rho^2+\frac{1}{m}\tau^2\rho^2\right).$$

In view of Equation (19), the above equation implies

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\int_{N^m}\left(\frac{m-1}{m}\tau^2\rho^2-Ric(\omega,\omega)\right)$$

and substituting from Equation (17), it yields

$$\int_{N^m}\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=\frac{m-1}{m}\int_{N^m}(div\omega)^2-\int_{N^m}Ric(\omega,\omega).$$

Employing inequality (16) in the above equation, we conclude

$$\rho^2\left\|T-\frac{\tau}{m}I\right\|^2=0.$$

However, $\rho\neq 0$ on connected N^m, gives

$$T=\frac{\tau}{m}I. \tag{21}$$

Taking the covariant derivative in above equation, we have

$$(\nabla_X T)(Y) = \frac{1}{m} X(\tau) Y$$

and using a frame $\{F_1, \ldots, F_m\}$ on (N^m, g) in above equation, we have

$$\sum_{j=1}^{m} \left(\nabla_{E_j} T\right)(E_j) = \frac{1}{m} \nabla \tau.$$

Using Equation (13) in this equation, we arrive at

$$\frac{1}{2} \nabla \tau = \frac{1}{m} \nabla \tau$$

and as $m > 2$, we conclude $\nabla \tau = 0$. Hence, the scalar curvature τ is a constant, and it is a nonzero constant. Now, Equations (15) and (21) imply

$$\frac{\tau}{m} \omega = \frac{\tau}{m} \nabla \rho - \tau \nabla \rho,$$

that is,

$$\omega = -(m-1) \nabla \rho \tag{22}$$

and it gives $div\omega = -(m-1)\Delta\rho$, which, in view of Equation (17), implies $\tau\rho = -(m-1)\Delta\rho$, that is,

$$-(m-1)\rho\Delta\rho = \tau\rho^2.$$

Integrating the above equation by parts, we arrive at

$$(m-1) \int_{N^m} \|\nabla \rho\|^2 = \tau \int_{N^m} \rho^2.$$

Since ρ is a nonconstant, from the above equation, we conclude the constant $\tau > 0$. We put $\tau = m(m-1)\alpha$ for a positive constant α. Now, differentiating Equation (22) and using Equations (10) and (21), we conclude

$$\nabla_X \nabla \rho = -\alpha \rho X, \quad X \in \Theta(TN^m),$$

where ρ is a nonconstant function and $\alpha > 0$ is a constant. Hence, $Hess(\rho) = -\alpha\rho g$; that is, (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [6,7]).

Conversely, suppose that (N^m, g) is isometric to the sphere $S^m(\alpha)$. Then, we know that a nonzero constant vector field $\mathbf{b}$ on the ambient Euclidean space R^{m+1} induces a vector field ω on the sphere $S^m(\alpha)$, which, according to Equation (4), is a ρ-Ricci vector field. Clearly, the scalar curvature of $S^m(\alpha)$ is given by $\tau = m(m-1)\alpha \neq 0$. We claim that the function ρ is nonzero and nonconstant. If $\rho = 0$, then by Equation (4), we have $f = 0$, which, in view of Equation (3), implies $\omega = 0$, and this in turn will imply that the constant vector field $\mathbf{b} = 0$. This is contrary to the assumption that $\mathbf{b}$ is a nonzero constant vector field. Hence, $\rho \neq 0$. Now, suppose ρ is a constant; then, by Equation (4), f is a constant, and by Equation (3), we have $div\omega = -m\sqrt{\alpha} f$, which, by Stokes's Theorem on compact $S^m(\alpha)$, would imply $f = 0$. This in turn, by virtue of Equation (4), implies $\rho = 0$, which is a contradiction, as seen above. Hence, the function ρ is nonzero and nonconstant.

Next, using Equations (3) and (4), we have

$$div\omega = m(m-1)\alpha\rho \tag{23}$$

and it gives

$$\int_{S^m(\alpha)} (div\omega)^2 = m^2(m-1)^2\alpha^2 \int_{S^m(\alpha)} \rho^2. \tag{24}$$

Now, using Equation (4), we have

$$\nabla\rho = -\frac{1}{(m-1)\sqrt{\alpha}}\nabla f, \tag{25}$$

which, on using Equation (3), gives

$$\nabla\rho = -\frac{1}{m-1}\omega.$$

Taking divergence in the above equation and using Equation (23), we conclude $\Delta\rho = -m\alpha\rho$, that is, $\rho\Delta\rho = -m\alpha\rho^2$. Integrating this equation by parts, we conclude

$$\int_{S^m(\alpha)} \|\nabla\rho\|^2 = m\alpha \int_{S^m(\alpha)} \rho^2.$$

Treating this equation with Equation (24), we conclude

$$\int_{S^m(\alpha)} (div\omega)^2 = m(m-1)^2\alpha \int_{S^m(\alpha)} \|\nabla\rho\|^2. \tag{26}$$

Also, using Equations (3) and (25), we have

$$\omega = -(m-1)\nabla\rho$$

and it changes Equation (26) to

$$\int_{S^m(\alpha)} (div\omega)^2 = m\alpha \int_{S^m(\alpha)} \|\omega\|^2.$$

Finally, using $Ric(\omega,\omega) = (m-1)\|\omega\|^2$ in the above equation, we conclude

$$\int_{S^m(\alpha)} Ric(\omega,\omega) = \frac{m-1}{m} \int_{S^m(\alpha)} (div\omega)^2$$

and this finishes the proof. □

Next, we consider a closed ρ-Ricci vector field on a compact and connected Riemannian manifold (N^m, g) such that the smooth function ρ is a nontrivial solution of the FM-equation and find yet another characterization of the sphere $S^m(\alpha)$. Indeed we prove the following theorem.

Theorem 2. *An m-dimensional complete and simply connected Riemannian manifold (N^m, g) with scalar curvature $\tau > 0$ admits a closed ρ-Ricci vector field ω such that the function ρ is a nontrivial solution of the FM-equation and the length of covariant derivative of ω satisfies*

$$\|\nabla\omega\|^2 \le \frac{1}{m}\tau^2\rho^2,$$

if and only if τ is a positive constant $\tau = m(m-1)\alpha$ and (N^m, g) is isometric to $S^m(\alpha)$.

Proof. Suppose (N^m, g) is an m-dimensional complete and simply connected Riemannian manifold with scalar curvature $\tau > 0$, and it admits a closed ρ-Ricci vector field ω, where ρ is a nontrivial solution of the FM-Equation (6) and the length of covariant derivative of ω satisfies

$$\|\nabla\omega\|^2 \le \frac{1}{m}\tau^2\rho^2. \tag{27}$$

For ρ, we define the operator B_ρ by

$$B_\rho X = \nabla_X \nabla \rho, \quad X \in \Theta(TN^m),$$

then B_ρ is a symmetric operator related to $Hess(\rho)$ by

$$Hess(\rho)(X,Y) = g\big(B_\rho X, Y\big), \quad X, Y \in \Theta(TN^m). \tag{28}$$

As ρ is a nontrivial solution of the FM-equation, using Equations (6) and (28), we have

$$\rho TX = B_\rho X - (\Delta\rho)X,$$

which, in view of Equation (7), becomes

$$B_\rho X = \rho TX - \frac{\tau}{m-1}\rho X. \tag{29}$$

Note that owing to the fact that ρ is a nontrivial solution of the FM-equation on (N^m, g), the scalar curvature τ is a constant and we put $\tau = m(m-1)\alpha$ for a constant α. Using Equation (29), we have

$$B_\rho X + \alpha\rho X = \rho TX - (m-1)\alpha\rho X, \quad X \in \Theta(TN^m).$$

Now, using Equation (10) in the above equation, we have

$$B_\rho X + \alpha\rho X = \nabla_X \omega - (m-1)\alpha\rho X, \quad X \in \Theta(TN^m).$$

Taking a local frame $\{F_1, \ldots, F_m\}$ on (N^m, g), by the above equation, we conclude

$$\begin{aligned}
\left\|B_\rho + \alpha\rho I\right\|^2 &= \sum_{j=1}^{m} g\big(B_\rho F_j + \alpha\rho F_j, B_\rho F_j + \alpha\rho F_j\big) \\
&= \sum_{j=1}^{m} g\Big(\nabla_{F_j}\omega - (m-1)\alpha\rho F_j, \nabla_{F_j}\omega - (m-1)\alpha\rho F_j\Big) \\
&= \|\nabla\omega\|^2 + m(m-1)^2\alpha^2\rho^2 - 2(m-1)\alpha\rho(div\omega).
\end{aligned}$$

Now, using Equation (10), we have $div\omega = \tau\rho = m(m-1)\alpha\rho$, and inserting it in the above equation, we arrive at

$$\left\|B_\rho + \alpha\rho I\right\|^2 = \|\nabla\omega\|^2 - m(m-1)^2\alpha^2\rho^2,$$

that is,

$$\left\|B_\rho + \alpha\rho I\right\|^2 = \|\nabla\omega\|^2 - \frac{1}{m}\tau^2\rho^2.$$

Using inequality (27) in the above equation results in

$$B_\rho = -\alpha\rho I,$$

that is,

$$Hess(\rho) = -\alpha\rho g. \tag{30}$$

Note that as $\tau > 0$, the constant $\alpha > 0$, and ρ is a nontrivial solution, ρ is a nonconstant function. Hence, by Equation (30), the complete and simply connected Riemannian manifold (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [6,7]).

Conversely, suppose that (N^m, g) is isometric to the sphere $S^m(\alpha)$. Then, by Equation (7), the function f is a solution of FM-equation on the sphere $S^m(\alpha)$, which has a closed ρ-Ricci vector field ω. The solution f of the FM-equation is related to ρ by Equation (4), that is,

$$f = -(m-1)\sqrt{\alpha}\rho. \tag{31}$$

In the proof of Theorem 1, we have seen that ρ is a nonconstant function on $S^m(\alpha)$. Moreover, using Equation (31), we have

$$\Delta f = -(m-1)\sqrt{\alpha}\Delta\rho, \quad Hess(f) = -(m-1)\sqrt{\alpha}Hess(\rho)$$

and the Equation (7) takes the form

$$-(m-1)\sqrt{\alpha}(\Delta\rho)g + fRic = -(m-1)\sqrt{\alpha}Hess(\rho),$$

which, in view of Equation (31), changes to

$$(\Delta\rho)g + \rho Ric = Hess(\rho).$$

Hence, ρ is a nontrivial solution of the FM-equation on the sphere $S^m(\alpha)$. Now, the Ricci operator T of the sphere $S^m(\alpha)$ is given by $T = (m-1)\alpha I$ and, therefore, Equation (10) on $S^m(\alpha)$ is

$$\nabla_X\omega = (m-1)\alpha\rho X, \quad X \in \Theta(TS^m(\alpha)).$$

Using the expression for the scalar curvature $\tau = m(m-1)\alpha$ for the sphere $S^m(\alpha)$, we have

$$\nabla_X\omega = \frac{\tau}{m}\rho X, \quad X \in \Theta(TS^m(\alpha)).$$

This proves

$$\|\nabla\omega\|^2 = \frac{1}{m}\tau^2\rho^2$$

and completes the proof. □

4. Conclusions

In the previous section, we used a closed ρ-Ricci vector field ω on an m-dimensional Riemannian manifold (N^m, g) to find two different characterizations of an m-sphere $S^m(\alpha)$. The scope of studying ρ-Ricci vector fields on a Riemannian manifold is quite modest. We observe that, in the previous section, we restricted the ρ-Ricci vector field ω to be closed, which simplified the expression for the covariant derivative of ω. It will be interesting to investigate whether we could achieve similar results after removing the restriction that the ρ-Ricci vector field ω is closed. It will be an interesting future topic to study the geometry of an m-dimensional Riemannian manifold (N^m, g) that admits a ρ-Ricci vector field ω, which needs not be closed. In order to simplify the findings on an m-dimensional Riemannian manifold (N^m, g) admitting a ρ-Ricci vector field ω which is not necessarily closed, we could impose the restriction on the Ricci operator T of (N^m, g) to be a Codazzi-type tensor, such that it satisfies

$$(\nabla_X T)(Y) = (\nabla_Y T)(X), \quad X, Y \in \Theta(TN^m).$$

Note that above restriction on (N^m, g) is slightly stronger than demanding the scalar curvature be a constant.

Author Contributions: Writing—original draft preparation, S.D.; writing—review and editing, H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors would like to extend their sincere appreciations to Supporting project number (RSPD2023R860) King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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