mathematics

Review

Stability of Differential Systems with Impulsive Effects

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Abstract: In this paper, a brief survey on the stability of differential systems with impulsive effects is provided. A large number of research results on the stability of differential systems with impulsive effects are considered. These systems include impulsive differential systems, stochastic impulsive differential systems and differential systems with several specific impulses (non-instantaneous impulses, delayed impulses, impulses suffered by logic choice and impulse time windows). The stability issues as well as the applications in neural networks are discussed in detail.

Keywords: stability; differential systems; impulsive effects; stochastic systems; delay; neural networks

MSC: 34D20; 34H15; 93C27; 93D05

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1. Introduction

As is known to all, in the evolution process of many actual systems, variables actively or passively present the characteristics of sudden changes at some time, which can be described by impulses or impulsive control. One of the mathematical models with wide applicability is the impulsive differential system. In addition, time delay is common in practical systems. The variables of these systems are affected by both the present and the past states. As a result, impulsive delay differential systems have received a great deal of attention from early on. On the other hand, strictly speaking, any actual system inevitably has a variety of random factors, such as internal random parameters, external random interference and observation noise. Therefore, it is necessary to introduce stochastic effects into many mathematical models. Stochastic impulsive differential systems have become one of the most studied issues in recent years.

In the qualitative theory of dynamical systems, finding sufficient conditions to ensure the stability of the solutions is a common problem, through which many important results can be obtained [1,2]. It is noted that the essence of 'impulses' is to describe or trigger sudden changes in the state of the system. Therefore, on the one hand, we can describe or reflect some dynamic processes more realistically through systems with impulsive effects. On the other hand, we can design appropriate impulsive control according to the actual demand, which has a positive effect on the system, so as to achieve the desired stability and other purposes, such as make the unstable system become stable.

Due to the merging, bifurcation, the loss of autonomy and other characteristics of solutions, the theory of impulsive differential systems is more complex than that of non-impulsive differential systems. Therefore, researchers have been introducing and improving research methods over the years. Based on the objective reality, combined with the theoretical analysis and control requirements, by using the Lyapunov stability theory, Razumikhin technique, comparison principle, functional analysis theory and other methods, the study of the stability of impulsive differential equations has achieved a lot of results.

The study of stochastic systems is developed based on the research of deterministic systems. Some classical research methods have been successfully applied, such as the Lyapunov method, Lyapunov-Krasovskii functional, Razumikhin technique, Lie algebra, Matrix inequality, Itô's formula and so on. With the rapid development of research, scientists have proposed a variety of research methods, such as the Euler-Maruyama method, average dwell-time method [3], equivalent method [4], Runge-Kutta-Maruyama method [5], sinh-cosh method [6], sech-function method [7], variational method [8], Haar wavelet method [9] and so on. Through the application of these methods, researchers have obtained a significant number of results on stochastic systems.

By analyzing a considerable amount of stability research work, this paper presents a comprehensive survey on the stability of impulsive differential systems, including analysis methods, stability results and applications. The organizational structure of this survey is as follows:

- Section 1 Introduction
- Section 2 Stability of Impulsive Differential Systems
    - Section 2.1 Stability of Impulsive Differential Systems
    - Section 2.2 Stability of Impulsive Delay Differential Systems
    - Section 2.3 Stability of Impulsive Functional Differential Systems
- Section 3 Stability of Stochastic Impulsive Differential Systems
    - Section 3.1 Stability of Stochastic Impulsive Functional Differential Systems
    - Section 3.2 Stability of Stochastic Impulsive Delay Differential Systems
    - Section 3.3 Stability of Stochastic Impulsive Differential Systems with Markovian switching
- Section 4 Stability of Differential Systems with Several Specific Impulses
    - Section 4.1 Stability of Differential Systems with Non-instantaneous Impulses
    - Section 4.2 Stability of Differential Systems with Delayed Impulses
    - Section 4.3 Stability of Differential Systems with Impulses suffered by logic choice
    - Section 4.4 Stability of Differential Systems with Impulse Time Windows
- Section 5 Applications in Neural Networks
- Section 6 Discussion

Attention. In this paper, for the conditions that each system needs to meet and the corresponding definitions of various stability systems, please refer to the original reference literature for details. For the sake of brevity, this paper only describes the common parts of each system, summarizes the research methods, shows some research results, and finds some new trends.

Unless otherwise specified, the following notations are used throughout the paper.

Notations. Let R, R^+ and N denote the set of real numbers, non-negative real numbers and positive integers, respectively. Let R^n be the n-dimensional Euclidean space, and $\|\cdot\|$ denotes the Euclidean norm on R^n. Let $\mathbb{C}([-\tau,0],R^n)$ denote the family of $\phi:[-\tau,0]\to R^n$, ϕ is continuous everywhere except a finite number of points $\hat{t}$ at which $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist, and $\phi(\hat{t}^+)=\phi(\hat{t})$. Let $\{\Omega,F,\{F_t\}_{t\geq 0},P\}$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and F_0 containing all p-null sets). Let $w(t)=(w_1(t),w_2(t),\cdots,w_m(t))^T$ be an m-dimensional Brownian motion defined on $\{\Omega,F,\{F_t\}_{t\geq 0},P\}$. Let $PC([-\tau,0],R^n)$ denote all piecewise continuous functions on $[-\tau,0]$, with the norm $\|\xi\|=\sup\limits_{-\tau\leq s\leq 0}\|\xi(s)\|$. Let $PC^b_{F_0}([-\tau,0],R^n)$ denote the family of F_{t_0}-measurable bounded $PC([-\tau,0],R^n)$-valued random variables, satisfying $\sup\limits_{-\tau\leq s\leq 0}E\|\phi\|^p<\infty$. $E\xi$ denotes the expectation of stochastic process ξ. $\mathcal{K}$ denotes the class of continuous strictly increasing functions $a:R^+\to R^+$ such that $a(0)=0$.

2. Stability of Impulsive Differential Systems

In this section, we will review some research work on the stability of impulsive differential systems, which is divided into three parts: *Stability of Impulsive Differential Systems* (IDS) without delay and other effects, *Stability of Impulsive Delay Differential Systems* (IDDS), *Stability of Impulsive Functional Differential Systems* (IFDS). Furthermore, a brief remark on the stability of impulsive differential systems with impulses occurring at uncertain moments is given.

2.1. Stability of Impulsive Differential Systems

Consider the following impulsive differential systems:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq t_k \\ \Delta x(t) = I_k(x(t)), & t = t_k, \ \ k \in N \end{cases} \tag{1}$$

where $f : J \times R^n \to R^n$ is continuous, $J = [t_0, \infty)$, $I_k : R^n \to R^n$ is continuous, $\Delta x(t) = x(t^+) - x(t^-)$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

IDS in the form (1) was proposed and studied previously; the systematic treatment of the fundamental stability theory is offered in [1,2]. The relations between various types of Lipschitz stability notions and sufficient conditions for these stabilities of IDS (1) were obtained in [10]. By using Lyapunov's direct method, some stability criteria for $(h_0, h)-$asymptotically stable and $(h_0, h)-$unstable of IDS (1) were obtained and applied to some population growth models in [11,12].

From the control point of view, impulsive control is attractive, since the stability of chaotic systems can be obtained only by small control impulses at discrete points. By using the comparison theorem, the authors derived some conditions for the asymptotic stability of IDS (1) in [13] and the asymptotic set-stability of IDS (1) in [14]; both of these results can be used to design impulsive control laws to achieve stability. In [15], in addition to the comparison theorem, two Lyapunov-like functions are also used to provide the strict practical stability criteria of IDS (1). For illustration, we cite the following result from [13].

Definition 1 ([1,13])**.** *Let $V : R^+ \times R^n \to R^+$, then V is said to belong to class V_0, if*

(i) V is continuous in $(t_{k-1}, t_k] \times R^n$, and for each $x \in R^n$, $k \in N$, $\lim\limits_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$ exists;

(ii) V is locally Lipschizian in x.

Definition 2 ([2,13])**.** *Let $V \in V_0$ and assume that*

$$\begin{cases} D^+V(t, x) \leq g(t, V(t, x)), & t \neq t_k \\ V(t, x + I_k(x)) \leq \Psi_k(V(t, x)), & t = t_k \end{cases}$$

where $D^+V(t, x) = \lim\limits_{h\to 0^+} \sup \frac{1}{h}[V(t + h, x + hf(t, x)) - V(t, x)]$, $g : R^+ \times R^+ \to R$ is continuous, and $\Psi_k : R^+ \to R^+$ is nondecreasing. Then, the following system is the comparison system of IDS (1):

$$\begin{cases} \dot{\omega} = g(t, \omega), & t \neq t_k \\ \omega(t_k^+) = \Psi_k(\omega(t_k)), & k \in \ N \\ \omega(t_0^+) = \omega_0 \geq 0 \end{cases}$$

Theorem 1 ([13])**.** *Assume that the following conditions hold.*

(i) $V \in V_0$, $K(t)D^+V(t, x) + D^+K(t)V(t, x) \leq g(t, K(t)V(t, x))$, $t \neq t_k$. Where, $K(t) \geq m > 0$, $\lim\limits_{t\to t_k^-} K(t) = K(t_k)$, $k \in N$, and $D^+K(t) = \lim\limits_{h\to 0^+} \sup \frac{1}{h}[K(t + h) - K(t)]$; g is continuous in $(t_{k-1}, t_k] \times R^n$ for each $x \in R^n$, $g(t, 0) = 0$, and $\lim\limits_{(t,y)\to(t_k^+,x)} g(t, y) = g(t_k^+, x)$ exists.

(ii) $K(t_k^+)V(t_k^+, x + I_k(x)) \leq \Psi_k(K(t_k)V(t_k, x)), K \in N$.

(iii) $V(t, 0) = 0$ and $\alpha(\|x\|) \leq V(t, x)$ on $R^+ \times R^n$, where $\alpha \in \mathcal{K}$.

Then, the global asymptotic stability of the trivial solution $\omega = 0$ of comparison system implies global asymptotic stability of the trivial solution of IDS (1).

Based on Lyapunov theory, numerous stability results of IDS (1) are presented. Some criteria for the strict stability of IDS (1) were obtained by using Lyapunov functions in [16], and the conditions for the uniform stability and the uniform asymptotic stability of IDS (1) were obtained in [17]; these results show that impulses do contribute to the stability behavior of IDS (1).

In the subsequent research, the restrictions on IDS have become less and less conservative, and the research methods have become more and more diverse. For instance, ref. [18] discussed IDS (1), where $J = [0, T], T > 0$ and $t_k, k = 1, 2, \cdots m$. By using the Gronwall type integral inequality of piecewise continuous functions, four Ulam's type stability notations and several stability results are established. Refs. [19,20] investigated the fixed-time stability of IDS (1), where the function f in IDS (1) is essentially locally-bounded and may be discontinuous on variable x. In [19], the authors proposed an improved Lyapunov method, and presented a fixed-time stability criterion, which is effective for both stabilizing and destabilizing impulses. In [20], by applying the generalized Lyapunov functional method and some inequality techniques, some fixed-time stability criteria were obtained. The fixed-time stability was also studied in [21]. Based on the impulsive control theory, several Lyapunov theorems of IDS (1) were established, where $f = f(x(t))$. In addition, the 'average dwell time' condition has been well applied in the stability study of IDS, such as in [22,23].

2.2. Stability of Impulsive Delay Differential Systems

Generally, impulsive differential systems with constant delay can be described as:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau)), & t \neq t_k \\ \Delta x(t) = I_k(x(t)), & t = t_k, \ k \in N \end{cases} \tag{2}$$

where $f : J \times R^n \times R^n \to R^n$, $J = [t_0, \infty)$, $f(t, 0, 0) \equiv 0$, $I_k : R^n \to R^n$ and satisfying $I_k(0) \equiv 0$, $\tau \geq 0$ is the time delay, $\Delta x(t) = x(t^+) - x(t^-)$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

Nowadays, the stability of IDDS (2) has been extensively studied. In [24], the author proposed some sufficient conditions for the stability and uniform stability of IDDS (2), and showed that the stability of unstable delay differential systems can be caused by impulses. In [25], the authors focused on the linear form of IDDS (2), where $f = Ax(t) + Bx(t-\tau)$; they presented several asymptotic stability and exponential stability criteria by using Lyapunov functions and the comparison principle method. In [26–28], the authors investigated linear and nonlinear IDDS (2), respectively, where $\Delta x \mid_{t=t_k} = r_k x(t_k)$, $r_k \in R$ and $t_{k+1} - t_k = \tau$. Some theorems are established such that the stability of IDDS can be transformed into the stability of the corresponding delay differential equations without impulsive effects, and the convergent numerical processes are also proposed to calculate numerical solutions of IDDS.

Next, let us discuss the stability of impulsive differential systems with time-varying delay. The following linear impulsive delay differential system is considered in [29]:

$$\begin{cases} \dot{y}(t) + \sum\limits_{i=1}^{n} p_i(t) y(t - \tau_i(t)) = 0, & t \neq t_k \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in N \end{cases} \tag{3}$$

where $p_i : J \times R \to R$ are locally summable functions, $\tau_i : J \times R \to R^+$ are Lebesgue measurable functions, and $t - \tau_i(t) \to \infty (t \to \infty)$, $J = [t_0, \infty)$, $i = 1, 2, \cdots, n$; $b_k \in R$ and $b_k \neq -1$ are constants, $k \in N$; the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

In [29], the authors proposed an interesting 'equivalent method', which enables us to reduce the stability problem of IDDS (3) to the stability problem of a corresponding delay differential system without impulses as follows:

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) \prod_{t-\tau_i(t)\le t_k<t} (1+b_k)^{-1} x(t-\tau_i(t)) = 0, \quad t \ge t_0. \tag{4}$$

Theorem 2 ([29])**.** *For any $\sigma \ge t_0$, there exists a positive constant $M(\sigma)$ such that, for any $t \ge \sigma$,*

$$|\prod_{\sigma\le t_k<t} (1+b_k)| \le M(\sigma).$$

(i) If the zero solution of (4) is stable, then the zero solution of (3) is also stable.

(ii) Assume that $M(\sigma)$ is independent of t, i.e., $M(\sigma) = M$ is a constant. If the zero solution of (4) is uniformly stable, then the zero solution of (3) is also uniformly stable.

(iii) If the zero solution of (4) is asymptotically stable, then the zero solution of (3) is also asymptotically stable.

This 'equivalent method' has attracted the attention of many researchers, such as [4,5,26–28,30,31]. Nowadays, impulsive delay differential systems have been widely studied, and a variety of applications have been found. For instance, by using the Lyapunov-Razumikhin method, some sufficient conditions for the uniform asymptotic stability of n-species Lotka-Volterra cooperation systems were obtained in [32]. In [33], time-delay is state-dependent, i.e., $\tau = \tau(t, x(t))$, in addition to providing some Lyapunov-based sufficient conditions for the exponential stability, these results are also applied to the stability analysis of impulsive gene regulatory networks by using the linear matrix inequalities technique. More information can be found in the references cited in [24–29,32,33] and the studies in which they are cited.

2.3. Stability of Impulsive Functional Differential Systems

Over the years, the stability research of impulsive functional differential systems has received a lot of attention due to its great application potential. For the early research work, please refer to [34] and its references.

Consider the following general impulsive functional differential systems:

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \ne t_k \\ \Delta x(t) = I_k(x(t^-)), & t = t_k, \ \ k \in N \end{cases} \tag{5}$$

where $f : R^+ \times D \to R^n$, D is an open set in $\mathbb{C}([-\tau, 0], R^n)$, $f(t, 0) \equiv 0$, $I_k : R^n \to R^n$ and satisfying $I_k(0) \equiv 0$, $\Delta x(t) = x(t) - x(t^-)$, x_t is defined by $x_t(s) = x(t+s)$, $-\tau \le s \le 0$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$.

IFDS in the form (5) has been well studied, and the Lyapunov functions and Razumikhin techniques are widely used in the stability study of it. For instance, by using Lyapunov functions and Razumikhin techniques, ref. [35] obtained some sufficient conditions for the uniform practical stability of IFDS (5), ref. [36] proposed several strict stability criteria of IFDS (5), ref. [37] developed some global exponential stability theorems of IFDS (5), and ref. [38] investigated the uniform-integral stability in terms of two measures. Furthermore, if $I_k = I_k(x(t^-))$ is extended to the form of $I_k = I_k(t, x(t^-))$ in IFDS (5), the Lyapunov functions and Razumikhin techniques are also effectively applied in the stability study. For example, ref. [39] discussed uniformly stability and asymptotically stability of sets, and ref. [40] put forward some asymptotic stability and uniformly asymptotic stability criteria. For illustration, we cite the following result from [36].

Theorem 3 ([36])**.** *Assume that the hypotheses in [36] and the following conditions hold:*

(i) There exists $V_1 \in V_0$, such that $b_1(\|x\|) \leq V_1(t,x) \leq a_1(\|x\|)$, where $a_1, b_1 \in \mathcal{K}$.

(ii) For any solution $x(t)$ of IFDS (5), $V_1(t+s, x(t+s)) \leq V_1(t, x(t))$ for $s \in [-\tau, 0]$, implies that $D^+V_1(t, x(t)) \leq 0$.

Also, for all $k \in N$ and $x \in S(\rho) = \{x \in R^n : \|x\| < \rho\}$, $V_1(t_k, x(t_k^-) + I_k(x(t_k^-))) \leq (1+d_k)V_1(t_k^-, x(t_k^-))$, where $d_k \geq 0$ and $\sum_{k=1}^{\infty} d_k < \infty$.

(iii) There exists $V_2 \in V_0$, such that $b_2(\|x\|) \leq V_2(t,x) \leq a_2(\|x\|)$, where $a_2, b_2 \in \mathcal{K}$.

(iv) For any solution $x(t)$ of IFDS (5), $V_2(t+s, x(t+s)) \geq V_2(t, x(t))$ for $s \in [-\tau, 0]$, implies that $D^+V_2(t, x(t)) \geq 0$.

Also, for all $k \in N$ and $x \in S(\rho)$, $V_2(t_k, x(t_k^-) + I_k(x(t_k^-))) \geq (1-c_k)V_2(t_k^-, x(t_k^-))$, where $0 \leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$.

Then, the trivial solution of IFDS (5) is strictly uniformly stable.

In addition to the cases of IFDS with finite delay, the theory of IFDS with infinite delay has also received considerable attention. In [41], the authors examined IFDS (5) with infinite delay, in which $x_t(s) = x(t+s)$, $-\infty < s \leq 0$. Unlike most researchers using Lyapunov-type functions, they used the techniques described in [42] to establish the existence conditions of global solutions, sufficient conditions for global asymptotic stability and global exponential stability of the concerned system, and applied the results to impulsive generalized Cohen–Grossberg systems.

Next, consider impulsive functional differential systems with infinite delay, whose differential equation part can be described as:

$$\dot{x}(t) = f(t, x(\cdot)), \quad t \neq t_k \tag{6}$$

In [43], the author discussed IFDS (6) with impulsive effects $\Delta x = I_k(t_k, x_{t_k^-})$. By using Lyapunov functions and Razumikhin techniques, the author obtained the uniform asymptotic stability and global stability criteria for a class of IFDS with infinite delay, the results showed that an unstable system can become stable under appropriate impulsive effects. In [44], the authors discussed IFDS (6) with impulsive effects $\Delta x = I_k(x(t_k^-))$, and some uniformly stable and uniformly asymptotically stable results are also obtained by using Lyapunov functions and Razumikhin techniques. In [45], the authors discussed IFDS (6) with impulsive effects $\Delta x = I_k(t_k, x(t_k^-))$; by establishing some new Razumikhin conditions, the authors put forward several uniform stability criteria. Furthermore, more information on the stability of IFDS can be found in [46], which is a comprehensive review for IFDS, including the basic theory and stability analysis of IFDS.

Impulsive differential systems with impulses occurring at fixed times have been extensively investigated and developed, as mentioned above. However, impulsive differential systems with impulses occurring at uncertain moments are more suitable and realistic to model some problems in the fields of chemistry, finance, biology, etc. The following remark gives a brief survey of impulsive differential systems with impulses occurring at uncertain moments, including the characteristics of several uncertain impulses and the corresponding stability analysis.

Remark 1. *Impulsive differential systems with impulses occurring at uncertain moments.*

One of the typical types of uncertain impulses is that the occurrence of impulses is related to the state variables. In this case, the impulsive points can be described as $\tau_k(x)$, $\tau_k(x) < \tau_{k+1}(x)$ and $\lim_{k\to\infty} \tau_k(x) = \infty$, see [47–50] and references therein.

In [47], some sufficient conditions for the uniform stability and uniform asymptotic stability were obtained by using Lyapunov functions and the Razumikhin technique. In [48], a (h_0, h)-stability criterion is obtained via establishing a comparison principle by vector Lyapunov functions. In [49], the authors studied the variational stability and variational asymptotic stability of the concerned system by using the Lyapunov functional. In [50], several Lyapunov-like theorems equipped with novel dwell time conditions for global asymptotic stability are obtained.

Random impulses constitute another typical type of uncertain impulses. Roughly speaking, the current research is mainly divided into two categories. One is that the impulsive instants and the number of impulses are random, such as [51–54]. The other is that the impulsive intensity is random, such as [55].

Let random variables τ_k $(K \in N)$ denote the waiting time between two consecutive impulsive moments, and meet $\sum_{k=1}^{\infty} \tau_k = \infty$ with probability 1. In [51], random variables $\tau_k \in \Gamma(\alpha_k, \lambda)$ $(K \in N)$, that is, τ_k are independent Gamma-distributed random variables. The authors obtained some p-moment exponential stability criteria. In [52], random variables τ_k are independent exponentially distributed random variables. By using Lyapunov functions and the Razumikhin technique, the authors proposed some extended exponential and weakly exponential stability results. Ref. [53] also discussed the case that random variables τ_k are independent exponentially distributed random variables, and gave some sufficient conditions for p-moment exponential stability. In [54], random variables τ_k are assumed to follow Erlang distribution. By using the contraction mapping principle, several Hyers-Ulam-Rassias stability and exponential stability theorems were obtained. In [55], the random impulsive intensity is determined by an arbitrary random sequence or an irreducible aperiodic Markov chain. By using the (mode-dependent) average impulsive interval method, the authors put forward some criteria on global asymptotic stability in probability and exponential stability in the mth moment.

3. Stability of Stochastic Impulsive Differential Systems

In this section, we will review some research work on the stability of stochastic impulsive differential systems; the section is divided into three parts: *Stability of Stochastic Impulsive Functional Differential Systems* (SIFDS), *Stability of Stochastic Impulsive Delay Differential Systems* (SIDDS) and *Stability of Stochastic Impulsive Differential Systems with Markovian switching* (SIDSM). Furthermore, a brief remark on the stability of stochastic impulsive differential systems with random impulses is given.

3.1. Stability of Stochastic Impulsive Functional Differential Systems

As we all know, stochastic impulsive functional differential systems have been widely studied, and we briefly discuss two general classes of SIFDS below.

Consider stochastic impulsive functional differential systems of the following form:

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dw(t), & t \neq t_k \\ x(t_k) = I_k(t_k, x(t_k^-)), & k \in N \end{cases} \tag{7}$$

where $f : R^+ \times PC^b_{F_0}([-\tau, 0], R^n) \to R^n$, $g : R^+ \times PC^b_{F_0}([-\tau, 0], R^n) \to R^{n\times m}$, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $I_k : R^+ \times R^n \to R^n$ and satisfying $I_k(t, 0) \equiv 0$, $\Delta x(t_k) = x(t_k) - x(t_k^-)$, $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

SIFDS in the form (7) has attracted extensive attention. Many methods have been proposed and improved to analyze the stability problems, such as Lyapunov functions, vector Lyapunov functions, Razumikhin technique, the average dwell time method and so on. For instance, in [56,57], based on the Razumikhin techniques and Lyapunov functions, the authors proposed some criteria on pth moment uniformly asymptotic stability, global exponential stability and instability of SIFDS (7), respectively. And the results showed that the impulses do make a contribution to the stability of SIFDS, even if the systems are unstable. In [58], both destabilizing impulses and stabilizing impulses are considered in SIFDS (7), by using the average impulsive interval method, the lower/upper bound of average impulsive interval can be derived, and some pth moment and almost sure exponential stability criteria were obtained. In [3], based on the vector Lyapunov function, combining Razumikhin techniques and the average dwell-time method, several pth moment exponential stability criteria of SIFDS (7) are obtained.

Obviously, the Razumikhin technique is a very important and widely used method in the study of SIFDS, but the conditions in Razumikhin theorems are relatively restrictive, e.g., the time-derivatives of Razumikhin functions always need to be negative definite for all t. This problem has attracted the attention of many researchers, and some results have been obtained in improving the traditional Ruzumikhin method/theorems. In [59], based on the stochastic analysis theory, the authors proposed several improved Razumikhin stability criteria by means of Razumikhin and average dwell time method, where the time-derivatives of Razumikhin functions are allowed to be indefinite.

Definition 3 ([59])**.** *The function $V(t,x): R^+ \times R^n \to R^+$ belongs to class Ψ if it is continuously twice differentiable with respect to x and once differentiable with respect to t.*

Define an operator $\mathcal{L}V$, see [59], as follows:

$$\mathcal{L}V(t,\varphi) = V_t(t,\varphi(0)) + V_x(t,\varphi(0))f(t,\varphi) + \frac{1}{2}trace[g^T(t,\varphi)V_{xx}(t,\varphi(0))g(t,\varphi)].$$

Theorem 4 ([59])**.** *Let $p, c_1, c_2, \eta, \tau, q > 1$ all be positive numbers. If there exists a function $V \in \Psi$ and constants $\rho, \beta \in (0,1)$ such that the following conditions hold:*

(i) For all $x \in R^n$, $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$.

(ii) For all $t \geq t_0$ and $t \neq t_k$, $k \in N$, $E\mathcal{L}V(t,x_t) \leq \mu(t)EV(t,x(t))$ if $EV(t+\theta, x(t+\theta)) \leq qEV(t,x(t))$ for all $\theta \in [-\tau, 0]$.

(iii) $EV(t_k, x(t_k)) \leq \beta EV(t_k^-, x(t_k^-))$.

(iv) The function $\mu(\cdot)$ satisfies: $(\beta \vee q^{-1})e^{\varphi\mu} \leq \rho$, where $\beta \vee q^{-1} = \max(\beta, q^{-1})$.

Then, SIFDS (7) is pth moment exponentially stable.

Next, consider stochastic impulsive functional differential systems as below:

$$\begin{cases} dx(t) = f(t,x(t),x_t)dt + g(t,x(t),x_t)dw(t), & t \neq t_k \\ \Delta x(t_k) = I_k(t_k, x(t_k^-)), & k \in N \end{cases} \tag{8}$$

where $f: R^+ \times R^n \times PC^b_{F_0}([-\tau,0],R^n) \to R^n$, $g: R^+ \times R^n \times PC^b_{F_0}([-\tau,0],R^n) \to R^{n\times m}$, $f(t,0,0) \equiv 0$, $g(t,0,0) \equiv 0$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$, $I_k: R^+ \times R^n \to R^n$ and satisfying $I_k(t,0) \equiv 0$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

So far, although the complexity of the system has brought some difficulties to the research, a lot of results have been obtained regarding SIFDS. In [60], by using the Lyapunov method and inequality techniques, the authors obtained some results on pth exponential stability and almost sure exponential stability of SIFDS (8). In [61], by combining the parameters of the variation formula and the comparison principle, the authors obtained some sufficient Lyapunov-type conditions for the pth stability of SIFDS (8). These conditions depend on the integral average value of the time-varying coefficients and the average impulsive interval, which are quite different from most existing results. In [62], based on the Lyapunov direct method and the comparison principle, combined with the Razumikhin-type conditions and the discussion of the properties of the corresponding non-impulsive systems, the sufficient conditions for global pth moment exponentially ultimate boundedness and the global pth moment exponential stability are given. The results showed that the unbounded or unstable SIFDS (8) can become bounded or stable by adding appropriate impulsive perturbations. Based on stochastic analysis theory, the authors in [63] obtained several fixed-time stability theorems of SIFD by using the Lyapunov method.

3.2. Stability of Stochastic Impulsive Delay Differential Systems

Due to the needs of practical applications or theoretical research, there are many kinds of SIDDS, and a large number of research results have been reported, see [4,5,30,31,63–66] and references therein. In this part, the work of several types of systems will be discussed; for more information please refer to the original papers.

It is well known that Itô's formula is effective and extensive in the study of stochastic differential systems, and many results have been obtained based on it. However, Itô's formula cannot be directly extended to the study of stochastic impulsive differential systems, because it is difficult to deal with integral intervals containing impulsive points. In order to solve this problem, refs. [4,30,31] developed the 'equivalent method' in [29], established the relationship between SIDDS and a corresponding non-impulsive stochastic delay differential system, and obtained some stability results. Moreover, the 'equivalent method' has also been effectively applied to the stability of numerical solutions, such as [5,26–28].

The following SIDDS is established in [5]:

$$\begin{cases} dx(t) = f_1(t, x(t), x(t-\delta))dt + g_1(t, x(t), x(t-\delta))dw(t), & t \neq \delta_r, t \geq 0 \\ \Delta x(\delta_r) = J_r(t, x(\delta_r^-)), & r = 0, 1, 2 \cdots \end{cases} \tag{9}$$

where the constant time delay $\delta > 0$, $\delta_r = r\delta$, $r = 0, 1, 2, \cdots, 0 \leq \delta_0 < \delta_1 < \cdots < \delta_r < \cdots$, and $\lim_{r\to\infty} \delta_r = \infty$. $f_1, g_1 : R^+ \times R \times R \to R$ are continuous, $\Delta x(\delta_r) = x(\delta_r) - x(\delta_r^-)$. $J_r : [-\delta, \infty) \times R \to R$ is continuous bounded function and satisfying $J_r(t, x(\delta_r^-)) \neq -x(\delta_r^-)$ when $x(\delta_r^-) \neq 0$, $J_r(t, 0) = 0$.

A corresponding non-impulsive stochastic delay differential system (SDDS) is proposed as follows:

$$dz(t) = \left(z(t)\frac{v_r'(t)}{v_r(t)} + v_r(t)f_1\left(t, \frac{z(t)}{v_r(t)}, \frac{z(t-\delta)}{v_r(t)}\right)\right)dt + v_r(t)g_1\left(t, \frac{z(t)}{v_r(t)}, \frac{z(t-\delta)}{v_r(t)}\right)dw(t), \ \ t \geq 0 \tag{10}$$

where $v_r(t) : [-\delta, \infty) \to R$ is defined by $v_r(t) = 1 + \frac{J_r(t, x(r\delta^-))}{x(r\delta^-)}, r = 0, 1, 2, \cdots$.

In [5], by using the Modified Runge-Kutta-Maruyama (MRKM) method, the authors put forward the numerical approximation for SDDS (10), which is suitably applied for SIDDS (9) too. Briefly, they defined the four-stage RKM method for the approximate solution of SDDS (10) as:

$$Z_{n+1} = Z_n + h\sum_{i=1}^{4} \gamma_i f_2(t, \mathcal{Z}_n^i, \bar{\mathcal{Z}}_n^i) + \sum_{i=1}^{4} \delta_i g_2(t, \mathcal{Z}_n^i, \bar{\mathcal{Z}}_n^i)\Delta W_n,$$

then the numerical approximation of the solution of the SIDDS (9) can be defined by

$$X_n = \frac{Z_n}{v_r(nh)}, \ \ n = 0, 1, 2, \cdots.$$

For information about the symbols, please refer to [5].

A class of general SIDDS with infinite delays is studied in [66]:

$$\begin{cases} dx(t) = f(t, x(t), x(t-\tau(t)))dt + g(t, x(t), x(t-\tau(t)))dw(t), & t \neq t_k \\ \Delta x(t_k) = I_k(t_k, x(t_k^-)), & k \in N \end{cases}$$

where $\tau : R^+ \to R^+$ is the infinite delay satisfying $\lim_{t\to\infty} t - \tau(t) = \infty$. Due to the variability and unboundedness, some methods and results of finite delay may no longer be applicable to the problem of SIDDS with infinite delay, which also exists in the use of the Razumikhin method. To overcome this difficulty, the authors in [66] constructed a positive function $q(t)$ determined by the infinite delay, in which the operator of the Lyapunov function is controlled by a time-varying sign-changing function rather than a constant, and then a

Razumikhin-type theorem different from the traditional stability of SIDDS with infinite delay is established.

3.3. Stability of Stochastic Impulsive Differential Systems with Markovian Switching

In recent years, a typical class of stochastic systems—Markovian switched systems—has attracted more and more attention, and many stability results have been reported, such as [67–73] and references therein. A Markovian switched system is governed by a Markovian process.

In [67], a general stochastic impulsive differential system with Markovian switching is described as:

$$\begin{cases} dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))dw(t), & t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k), x_{t_k}, t_k, r(t_k)), & k \in N \end{cases} \tag{11}$$

where $f : PC([-\tau, 0], R^n) \times R^+ \times \mathcal{S} \to R^n$, $g : PC([-\tau, 0], R^n) \times R^+ \times \mathcal{S} \to R^{n\times m}$, $I_k : R^n \times PC([-\tau, 0], R^n) \times R^+ \times \mathcal{S} \to R^n$, and $f(0, t, i) \equiv 0$, $g(0, t, i) \equiv 0$, $I_k(0, 0, t, i) \equiv 0$, $k = 1, 2, \cdots$. $\Delta x(t_k) = x(t_k^+) - x(t_k)$. The fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$. The $r(t)$ $(t > 0)$ is a right-continuous Markovian chain on the probability space taking values in a finite state space $\mathcal{S} = \{1, 2, , \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + \gamma_{ij}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j while $\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}$. Assume that the Markovian chain $r(t)$ is independent of the Brownian motion $w(t)$. In [67], by using the Razumikhin-type technique, the authors presented some pth moment exponential stability results of SIDSM (11). Furthermore, the obtained results are applied to a class of linear systems by using the M-matrix method and Lyapunov functions.

The research on SIDSM is usually carried out around different impulsive effects, and accordingly, the impulsive effects can be expressed in different forms. Taking SIDSM (11) as an example, under the condition that the expression of the differential equation is unchanged, the authors of [68] investigated it with the impulsive effects $\Delta x(t_k) = I_k(x(t_k^-), t_k)$. By using the Lyapunov second method and Razumikhin techniques, some sufficient conditions on pth moment exponential stability, almost exponential stability and instability were obtaied. Ref. [69] investigated it with the impulsive effects $\Delta x(t_k) = I_k(t_k, x(t_k^-), r(t_k)) + J_k(t_k, x(t_k^-), r(t_k))$. By means of the average impulsive interval and Lyapunov function method, the authors obtained some exponential stability theorems, which are more convenient to apply than the Razumikhin-type conditions in the previous results. Furthermore, the authors demonstrated that appropriate impulses can stabilize some unstable hybrid systems. Ref. [70] investigated it with delayed impulses $x(t_k) = h_k(t_k, x(t_k^-), x((t_k - d_k)^-), r(t_k))$. By using the Razumikhin technique and Lyapunov functions, some pth moment exponential stability criteria are established, and the impulsive controller is designed to stabilize the given systems. It is worth noting that the results show that the delay part of the impulses can contribute to the stability of the system. Refs. [71,72] focused on the generalized SIDSM, where

$$dx(t) = f(x(t), x_t, t, r(t))dt + g(x(t), x_t, t, r(t))dw(t),\ t \neq t_k.$$

When $t = t_k$, $t \in N$, the impulsive effects are expressed in $\Delta x(t_k) = I_k(x(t_k^-), t_k)$ and $x(t_k) = I_k(x(t_k^-), x((t_k - d_k)^-), t_k^-, r(t_k))$, respectively. Based on stochastic theory, Lyapunov functional method and Razumikhin technique, some stability theorems are derived.

Similar to impulsive differential systems, stochastic impulsive differential systems with random impulses have also been considered by some scholars. However, the results are still relatively few at present since the structure of stochastic systems is more complex, and the randomness of impulsive time brings great difficulties to deal with the systems.

Remark 2. *Stochastic impulsive differential systems with random impulses.*

As far as we know, up to now, there have been only a few research results of stochastic impulsive differential systems with random impulses. Among these research papers, an important type denotes $\{\tau_i\}_{i=1}^{\infty}$ *to be a sequence of independent exponentially distributed random variables, where* $\{\tau_i\}_{i=1}^{\infty}$ *are called impulsive waiting times, see [74–76]. Based on stochastic processes theory and stochastic analysis theory, by using the Lyaponov method and Razumikhin techniques, some stability theorems have been proposed.*

4. Stability of Differential Systems with Several Specific Impulses

According to the characteristics of the impulses, this section is divided into four parts: *Stability of Differential Systems with Non-Instantaneous Impulses* (DS-NI), *Stability of Differential Systems with Delayed Impulses* (DS-DI), *Stability of Differential Systems with Impulses Suffered by Logic Choice* (DS-IL), *Stability of Differential Systems with Impulse Time Windows* (DS-ITW).

4.1. Stability of Differential Systems with Non-Instantaneous Impulses

Generally, the characteristic of instantaneous impulses is that the sudden change in the state variables is very short, and this time can be regarded as instantaneous compared with the development process of the whole system. Non-instantaneous impulse refers to a generalization of instantaneous impulse, which is characterized by the fact that the time of sudden change in state variables is not negligible in the development process of the whole system, and it needs to be described as the changing behavior on finite time intervals.

The concept of 'non-instantaneous impulses' was first introduced in [77], where the authors proposed a class of DS-NI in the following form:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, m \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \cdots, m \end{cases} \tag{12}$$

where $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$ are pre-fixed numbers, $f : [0, T] \times R \to R$ is a suitable function, $g_i : (t_i, s_i] \times R \to R$ is continuous for all $i = 1, \cdots, m$.

After giving the concepts of the solution for DS-NI (12), the authors discussed the existence of mild and classical solutions in [77]. Subsequently, a large number of researchers have carried out research on the non-instantaneous impulsive systems. In [78], the authors developed the results in [77] and discussed existence problems in a fractional power space. In addition to the existence and uniqueness of solutions, stability is still an important problem in the qualitative study of non-instantaneous impulsive systems.

In [79–81], the following DS-NI is considered:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, m \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \cdots, m \end{cases} \tag{13}$$

where $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$ are pre-fixed numbers, $f : [0, T] \times R \to R$ is continuous, and $g_i : (t_i, s_i] \times R \to R$ $(i = 1, \cdots, m)$ is a suitable function.

In [79], the authors discussed the existence and uniqueness of the solution of DS-NI (13), introduced the definition of generalized Ulam-Hyers-Rassias stability and obtained some results. In [80], by using piecewise Lyapunov functions, both uniform stability and uniform asymptotic stability criteria of DS-NI (13) are established. In [81], more research methods have been comprehensively utilized: the perturbation technique, monotone iterative method and a new estimation technique of the measure of noncompactness are all employed to discuss the existence, uniqueness and Ulam–Hyers–Rassias stability of DS-NI (13).

With the passage of time, non-instantaneous impulsive systems are gradually developing. In [82], the authors considered a class of DS-NI as below:

$$\begin{cases} x'(t) = f(t, x_t), & t \in (t_k, s_{k+1}],\ k = 0, 1, \cdots \\ x(t) = \Phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_k],\ k = 1, 2, \cdots \end{cases} \tag{14}$$

where $0 = s_0 < s_i \le t_i < s_{i+1}, i = 1, 2, \cdots$, are given and $\lim_{k\to\infty} s_k = \infty$. $f : R \times R^n \to R^n$ and $\Phi_i : [t_i, s_i] \times R^n \times R^n \to R^n$ $(i = 1, \cdots, m)$ are suitable functions, $x_t = x(t + s),\ s \in [-r, 0]$.

In [82], based on Lyapunov-like functions and the Razumikhin technique, the stability, uniform stability and asymptotic uniform stability results of DS-NI (14) were obtained by using the comparison principle, where the comparison equations are nonlinear non-instantaneous impulsive differential equations without any delay.

In [83], the authors introduced the following DS-NI systems:

$$\begin{cases} x'(t) = Ax(t) + g(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots \\ x(t_i^+) = Bx(t_i^-) + b_i, & i = 1, 2, \cdots \\ x(t) = Bx(t_i^-) + b_i, & t \in (t_i, s_i],\ i = 1, 2, \cdots \\ x(s_i^+) = x(s_i^-), & i = 1, 2, \cdots \end{cases} \tag{15}$$

where $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} < \cdots,\ t_i \to \infty\ (i \to \infty)$, $b_i \in R^n,\ A,\ B$ are $n \times n$ constant matrixes and satisfying $AB = BA$, $g : R^+ \times R^n \to R^n$ is continuous.

In [83], based on the concept of non-instantaneous impulsive Cauchy matrix introduced by the authors, some useful criteria for asymptotic stability of linear DS-NI are derived. Furthermore, the existence, uniqueness and Ulam–Hyers–Rassias stability of nonlinear DS-NI (15) are discussed. In [84], the authors considered DS-NI that are similar to DS-NI (15) but more general. Based on the concept of non-instantaneous impulsive evolution operator introduced by the authors, sufficient conditions for the asymptotic stability of linear and semilinear systems are obtained.

4.2. Stability of Differential Systems with Delayed Impulses

Delayed impulses have very important theoretical significance and application value, so they have received a lot of attention and achieved many results. In this part, we discuss the stability of DS-DI from the perspective of the types of delayed impulses.

Case A. The delayed impulses are assumed containing the following term:

$$x(t_k^- - d_k) \ \text{ or } \ x((t_k - d_k)^-),\ \ k \in N.$$

where $d_k \ge 0,\ k \in N$ are the impulse delays and satisfying $d = \max_{k\in N}\{d_k\} < \infty$.

In [85], the delayed impulses are assumed to take the form of $x(t_k) = I_k(x(t_k^-)) + J_k(x(t_k^- - \tau))$ $(\tau > 0)$. By using Lyapunov functions and Razumikhin techniques, the authors provided some criteria on stability, asymptotic stability and practical stability of impulsive functional DS-DI, respectively. In [86], the delayed impulses are assumed to take in the form of $x(t_k) = I_k(t_k, x(t_k^-), x((t_k - d_k)^-))$. By using comparison principle and impulsive delay differential inequality techniques, some sufficient conditions ensuring the exponential stability and asymptotical stability criteria of stochastic functional DS-DI are derived. In [87], the delayed impulses are assumed to take the form of $x(t_k) = C_k x(t_k^-) + J_k(x(t_k^- - d_k))$ and t_k is not a fixed point. Based on Lyapunov theory, by using the comparison principle and inequality techniques, several quasi-uniformly asymptotic stability and quasi-exponential stability criteria of DS-DI are obtained.

Case B. The delayed impulses are assumed containing the following term:

$$I_k(x_{t_k^-}) \text{ or } I_k(t_k, x_{t_k^-}), \; k \in N.$$

where $x_{t^-} = x(t^- + s), \; -\tau \le s \le 0$.

In [88,89], the delayed impulses are assumed to take the form of $\Delta x(t_k) = I_k(x(t_k^-)) + J_k(x_{t_k^-})$ and $\Delta x(t_k) = I_k(x_{t_k^-})$, respectively. Consequently, using Lyapunov functions and Razumikhin techniques, some global exponential stability results were obtained.

In [90–92], the following general impulsive functional differential systems with delayed impulses (IFDS-DI) are considered:

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \neq t_k \\ \Delta x(t_k) = I_k(t_k, x_{t_k^-}), & k \in N \end{cases} \tag{16}$$

By using Lyapunov functions and Razumikhin techniques, ref. [90] put forward several criteria on the uniform asymptotic stability and [91] obtained several global exponential stability criteria of IFDS-DI (16), respectively. Ref. [92] is a relatively comprehensive review of recent work on delayed impulses, including the concepts of stability, the effects of impulses and some stability results, etc. The authors of [93] investigated a generalized IFDS-DI (16), in which $\Delta x(t_k) = I_k(t_k, x(t_k^-)) + J_k(t_k, x_{t_k^-})$, some sufficient conditions ensuring the global exponential stability are given in terms of Lyapunov functions and Razumikhin techniques too.

In [94–97], the following general stochastic impulsive functional differential systems with delayed impulses (SIFDS-DI) are considered:

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dw(t), & t \neq t_k \\ \Delta x(t_k) = I_k(t_k, x_{t_k^-}), & k \in N \end{cases} \tag{17}$$

The Lyapunov–Razumikhin method still plays an important role in the stability study of SIFDS-DI, and many results have been obtained by applying this method. For instance, in [94], several moment and almost sure exponential stability criteria of SIFDS-DI (17) are given, which can be applied to the cases of systems with arbitrary large time delays. Moreover, it is verified that an unstable stochastic delay system can be stabilized by impulses. In [95], the authors proposed several criteria on pth moment and almost sure exponential stability of SIFDS-DI (17). In [96], several criteria on the exponential stability and uniform stability of SIFDS-DI (17) in terms of two measures are derived. In [97], the authors proposed some exponential stability criteria of SIFDS-DI (17). The results showed that, if the frequency and amplitude of impulses are appropriately related to the increase/decrease in continuous flow, the concerned system can be stable.

Case C. State-dependent delayed impulses.

Take the following two articles as examples to illustrate the state-dependent delayed impulses.

In [98], the following impulsive differential system with state-dependent delayed impulses is considered:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \ge t_0, \; t \neq t_k \\ x(t_k) = I_k(t_k^- - \tau, x(t_k^- - \tau)), \; \tau = \tau(t_k, x(t_k^-)), & k \in N \end{cases} \tag{18}$$

In [99], the following stochastic impulsive differential system with state-dependent delayed impulses is considered:

$$\begin{cases} du(t) = f(t, u(t), u_t)dt + g(t, u(t), u_t)dw(t), & t \ge t_0, \; t \neq t_k \\ u(t_k^+) = I_k(t_k^-, u(t_k^-), u(t_k^- - \tau_k)), \; \tau_k = \tau(t_k, u(t_k^-)), & k \in N \end{cases} \tag{19}$$

It can be clearly seen that the time-delay which is state-dependent, does exist in the impulses of systems (18) and (19). To our knowledge, some classical stability analysis methods for time-delay systems, such as delay decomposition method and free-weighting matrix method, cannot be effectively applied to differential systems with state-dependent delays. Even the widely used Lyapunov-Razumikhin method and Lyapunov-Krasovskii method have difficulties in dealing with state-dependent delay differential systems too. In [98], based on impulsive control theory and some comparison results, an estimate of Lyapunov functions coupled with the effect of state delay is given, then some uniform stability, uniform asymptotic stability and exponential stability criteria are established. In [99], based on impulsive control theory and stochastic analysis theory, by using the classical Itô's formula, combined with the average impulsive interval and comparison properties, some effective conditions ensuring stability of system (19) are derived.

4.3. Stability of Differential Systems with Impulses Suffered by Logic Choice

There are various logical phenomena in the real world. However, due to the lack of mathematical tools that can effectively analyze logical functions, early research on logical systems mainly focused on topological structures, and qualitative analysis was very rare. This situation changed after the semi-tensor product method was proposed in [100], which can transform logical functions into equivalent algebraic expressions. Against this background, the differential systems with impulses suffered by logic choice have attracted the attention of some researchers in recent years, see [101–109].

Let δ_n^i denote the ith column of the identity matrix I_n, $i = 1, 2, \cdots, n$, and $\Delta_n = \{\delta_n^i | i = 1, 2, \cdots, n\}$. A matrix $L \in R^{n\times m}$ is called logical matrix, if $Col(L) \subset \Delta_n$. Moreover, we identify logical values with equivalent vectors as: $T = 1 \sim \delta_2^1$, $F = 0 \sim \delta_2^2$.

Definition 4 ([100])**.** *For two matrices $A \in R^{n\times m}$ and $B \in R^{p\times q}$, the semi-tensor product of A and B is:*

$$A \ltimes B = (A \otimes I_{\alpha/m})(B \otimes I_{\alpha/p}),$$

where $\alpha = lcm(m, p)$ denotes the least common multiple of m and p, $\otimes$ represents the Kronecker product of matrices.

Lemma 1 ([100])**.** *Given a logical function $f(p_1, p_2, \cdots, p_r) \in \Delta_2$ with logical variables $p_1, \cdots, p_r \in \Delta_2$, there exists a unique 2×2^r logical matrix M_f called the structure matrix of f, such that*

$$f(p_1, p_2, \cdots, p_r) = M_f \ltimes p_1 \ltimes p_2 \ltimes \cdots \ltimes p_r = M_f \ltimes_{i=1}^{r} p_i$$

Moreover, $Col(M_f) \subset \Delta_2$. We note that $\ltimes_{i=1}^{r} p_i \in \Delta_{2^r}$.

'Impulses suffered by logic choice' was first introduced in [101], where the authors proposed a class of DS-IL as follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq t_k \\ \Delta x(t_k) = \Phi_k(x(t_k)), & k \in N \end{cases} \tag{20}$$

where $f : R^+ \times R^n \to R^n$ is continuous and $f(t, 0) \equiv 0$, $\Delta x(t) = (\Delta x_1(t), \cdots, \Delta x_n(t))^T$, the fixed impulsive points $\{t_k\}_{k=1}^{\infty}$ satisfying $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

The impulses $\Phi_k(x(t_k))$ are affected by the logical relationship between $x_i(t_k)$, $i = 1, 2, \cdots n$, $k = 1, 2, 3 \cdots$, and can be described as:

$$\begin{aligned} \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k) \\ &= u_i(p_1(x_1(t_k)), \cdots, p_n(x_n(t_k)))I_k(x_i(t_k)) + \overline{u_i(p_1(x_1(t_k)), \cdots, p_n(x_n(t_k)))}J_k(x_i(t_k)). \end{aligned}$$

where $u_i : \{\delta_2^1, \delta_2^2\}^n \to \{0,1\}$ is a logical function and $\bar{u}_i$ denotes the negation logical function of u_i. For $\forall k \in N$, I_k and J_k are continuous and satisfying $I_k(0) = J_k(0) = 0$. Define the piecewise function $p_i : R \to \{0,1\}$ as follows:

$$p_i(s) = \begin{cases} \delta_2^2 \sim 0, & |s| \geq c_i \\ \delta_2^1 \sim 1, & |s| < c_i \end{cases}$$

where $c_i > 0$ is the threshold.

Obviously, the impulses in DS-IL (20) are affected by logic effects, and the changes in state variables at the impulsive times will be selected from I_k and J_k under the influence of logic functions.

Based on the concepts of semi-tensor product and the structure matrix of logical function, the impulses of DS-IL (20) can be transformed into the following equivalent algebraic expression:

$$\Delta x(t_k) = \begin{pmatrix} I_k(x_1(t_k)) & J_k(x_1(t_k)) & & \\ & \ddots & & \\ & & I_k(x_n(t_k)) & J_k(x_n(t_k)) \end{pmatrix} Mp(x(t_k)).$$

where $p(x(t_k)) = \ltimes_{i=1}^{n} p_i(x_i(t_k))$ is the semi-tensor product of $p_i(x_i(t_k)),\ i = 1, 2, \cdots, n$. $M = (M_1^T, M_2^T, \cdots, M_n^T)^T \in R^{2n \times 2^n}$, M_i is the unique 2×2^n structural matrix of logical functions u_i and $\bar{u}_i$.

After the impulses suffered by logic choice are successfully transformed into the algebraic form, some classical differential equation research methods can be applied to the systems with impulses suffered by logic choice. For instance, ref. [101] proposed the following asymptotic stability theorem by means of Lyapunov functions.

Theorem 5 (see [101]). *Assume that*

(i) $f(t,x)$ satisfies the QUAD condition and $\lambda_{max}(Q - \varepsilon P^{-1}) < 0$.

QUAD condition: for some $\varepsilon > 0$, $\forall x, y \in R^n, x \neq y$, $\exists$ a positive definite diagonal matrix P and a diagonal matrix $Q \ni (x-y)^T P\{[f(t,x) - f(t,y)] - Q(x-y)\} \leq -\varepsilon(x-y)^T(x-y)$.

(ii) $\exists$ two functions $\phi_{1k}, \phi_{2k} \in \mathcal{K}$ such that $|I_k(x_i)| \leq \phi_{1k}(|x_i|)$, $|J_k(x_i)| \leq \phi_{2k}(|x_i|)$.

(iii) $\exists\, d > 0$ such that for $z \in (0,d)$, $\lambda_{max}(Q - \varepsilon P^{-1})(t_k - t_{k-1}) + \ln(\psi_k(z)/z) \leq -\gamma_k$, where $\gamma_k \geq 0$, $\sum_{k=1}^{\infty} \gamma_k = \infty$.

Then the trivial solution of DS-IL (20) is asymptotically stable.

Inspired by [101], some researchers have carried out qualitative research on impulsive systems with logical effects, especially on impulses suffered by logic choice, and obtained some stability results. In [102], the authors proposed several criteria on finite-time stability of a class of nonlinear DS-IL. The authors in [103] extended the method in [101] to a discrete system and obtained some stability results. In [104], the author presented some exponential stability criteria of linear delay DS-IL, and the results showed that the constraint of the coefficient functions can be reduced by logic impulsive control. Ref. [105] focused on the stability analysis of a general DS-IL; some stability results are given by using the 'equivalent method'. Furthermore, refs. [106,107] extended the research work of the impulses suffered by logic choice to stochastic systems, and the sufficient conditions for stability in [107] are relatively easy to verify because they do not contain Lyapunov functions.

4.4. Stability of Differential Systems with Impulse Time Windows

The occurrence time τ_k of impulses may occur at a little range of time $[\tau_k^l, \tau_k^r)$, which can be considered as the time error of impulsive control (see [110,111]). That is, the impulsive times τ_k can stochastically occur in a small time interval $[\tau_k^l, \tau_k^r)$, which is called the impulse time windows. The concept of 'impulse time window' was firstly introduced in [112]. In practical applications, impulsive control with impulse time windows may be more effective than impulsive control with fixed times. Until now, stability research on systems with impulse time windows has yielded some results, see [109–116]. For illustration, we cite the following result from [114].

In [114], the following linear DF-ITW is considered:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq \theta,\ t \neq \tau_k,\ \tau_k \in [\tau_k^l, \tau_k^r) \\ \Delta x(t) = x(t) - x(t^-) = Cx(t^-), & t = \tau_k,\ K \in N \end{cases} \tag{21}$$

where $x \in R^n$, A, B, $C \in R^{n\times n}$, τ=const.> 0, $\theta \geq t_0$, $\{\tau_k^l\}_{k=1}^{\infty}$ and $\{\tau_k^r\}_{k=1}^{\infty}$ are fixed points, $0 \leq t_0 = \tau_0^l = \tau_0 = \tau_0^r \leq \tau_1^l < \tau_1^r \leq \tau_2^l < \tau_2^r \leq \cdots \leq \tau_k^l < \tau_k^r \leq \cdots$, $\lim_{k\to\infty} \tau_k^r = \infty$, the impulse time τ_k is any value of $[\tau_k^l, \tau_k^r)$.

Because the impulses can take any value in the impulse time windows, they are still more difficult to deal with than the fixed impulsive points system. Based on Lyapunov functions and Razumikhin technique, by classifying the values of impulse time τ_k, the following uniform stability theorem was obtained in [114].

Theorem 6 (see [114])**.** *Let $\lambda_1, \lambda_2 > 0$ be the smallest and the largest eigenvalues of symmetric and positive matrix P, respectively, λ_3 and λ_4 be the largest eigenvalues of $P^{-1}(A^T P + PA + PP)$ and $P^{-1}B^T B$, respectively, $0 < \lambda_5 < 1$ be the largest eigenvalues of $P^{-1}[(I + C)^T P(I + C)]$, where I is the identity matrix. Then, if $(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau_k^r - \tau_{k-1}^r) < -\ln \lambda_5$, the trivial solution of DF-ITW (21) is uniformly stable.*

In [112], the research method in [114] is extended to a class of general impulsive delay functional differential systems with impulse time windows, and some sufficient conditions for the global exponential stability were obtained. Different from [112,114], the authors in [110] obtained some stable and asymptotically stable comparison theorems by employing the comparison principle rather than the Lyapunov-Razumikhin method. Recently, a class of uncertain sandwich control system with impulse time windows was proposed in [111], and several exponential stability criteria were obtained in terms of linear matrix inequalities and inequality techniques. In [109], the authors proposed a class of linear (uncertain) delay impulsive differential systems with impulse time windows and logic choice. By using the semi-tensor product method and Lyapunov-Razumikhin technique, some uniform stability criteria were obtained.

5. Applications in Neural Networks

The impulsive system can simulate the working process of biological neurons more accurately, so as to improve the effect of the neural network in practical applications, such as for robot control and biomedical engineering. Therefore, it is of great significance to study the application of impulsive systems in neural networks, which has received a lot of attention. Based on the structure of impulsive neural networks, we review some recent work on stability below.

Consider the following impulsive neural networks:

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i, & t \neq \tau_k \\ \Delta x_i(t) = H_{ik}(\cdot), & t = t_k \end{cases} \tag{22}$$

where f_j, g_j are the activation functions of the neurons, $c_i > 0$ denotes the decay rate, a_{ij}, b_{ij} denote the weights of the neuron interconnections, τ_{ij} denotes the transmission delay and I_i is the external bias.

In [117–121], the authors investigated the stability of neural networks in the form of system (22) or similar. For more information about the functions in system (22), refer to [117–121], respectively. In order to better study the neural network system, various techniques and methods have been constantly proposed or developed. In [117], the impulsive term in system (22) is described as $x(t_k) = h_k(x(t_k^-)) + w_k(x((t_k - \tau(t_k))^-))$ $(k \in N)$, based on the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, the authors established a differential inequality which overcomes the difficulty in application of (generalized) Halanay inequality, and several global exponential stability criteria were provided. In [118], in terms of nonsmooth analysis, some globally asymptotically stability results of a class of impulsive delay differential equations were proposed, which can be applied to a class of neural network like (22), where $c_i x_i(t)$ is replaced by $c_i(x_i(t))$. In [119], system (22) with uncertainty is considered, that is, A, B, C is replaced by $A + \Delta A, B + \Delta B, C + \Delta C$, respectively. By employing Lyapunov-Razumikhin functions together with differential inequalities, some sufficient conditions of robust exponential stability were obtained. In [120], proportional delay is considered in neural networks, to put it simply, $x(t - \tau(t))$ is replaced by $x(qt)$, where $q \in (0, 1)$ is proportional delay factor and $qt = t - (1 - q)t$, in which $(1 - q)t$ corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell. By using the improved Lyapunov-Razumikhin method, some sufficient conditions ensuring the existence, uniqueness and globally asymptotic stability of the concerned neural networks are presented. In [121], the authors considered the neural networks with state-dependent impulsive effects, where the state jump $\Delta x|_{t=\xi_k}$ satisfying $\xi_k = \theta_k + \tau_k(x(\xi_k))$. By using B-equivalence method (reduce a state-dependent impulsive system to a fixed-time impulsive system), a comparison principle can be applied and some exponential stability results were established. In addition, unbounded continuously distributed delays are also considered in neural networks, and some stability results were obtained, such as [122,123].

Based on the nonlinear dynamic characteristics and stochastic properties of neurons, scholars have constructed a variety of stochastic impulsive neural network models (see [124–128] and references therein). While providing important tools for us to better understand the way neurons operate, it also provides us with more efficient methods in the fields of pattern recognition, data classification, and control systems, etc.

In [124], the authors considered stochastic impulsive Hopfield neural networks as follows:

$$\begin{cases} dx_i(t) = [-c_i x(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{l=1}^{N} \sum_{j=1}^{n} b_{ij}^{(l)} g_j(x_j(t - \rho_l(t)))]dt \\ \qquad\qquad + \sum_{s=1}^{m} \sigma_{is}(t, x_i(t), x_i(t - \rho_1(t)), \cdots, x_i(t - \rho_N(t)))dw_s(t), & t \neq \tau_k \\ \Delta x_i(t_k) = H_{ik}(x_i(t_k^-)), & K \in N \end{cases} \tag{23}$$

where σ_{is} denotes the diffusion-coefficient of stochastic effects. By employing the extended Razumikhin method and Lyapunov functions, some pth moment exponential stability theorems of stochastic impulsive Hopfield neural networks (23) are presented. In [125], the authors investigated stochastic neural networks with mixed delays and hybrid impulses, some sufficient conditions of input-to-state stability were obtained in terms of average impulsive interval approach. Bidirectional associative memory (BAM) neural networks have also attracted the interest of many researchers, and some stability results of stochastic impulsive BAM neural networks have been reported. For instance, in [126], the authors investigated the pth asymptotical/exponential input-to-state stability of a class of stochastic impulsive BAM neural networks and presented some criteria by establishing integral-differential inequalities with time-varying inputs. Meanwhile, Cohen-Grossberg neural networks are another active research topic. In [127], by using the 'equivalent method', some sufficient conditions ensuring the exponential p-stability of a class of stochastic impulsive reaction-diffusion Cohen-Grossberg neural networks are derived.

In view of the needs of scientific research and practical work, in addition to the above-mentioned impulsive neural network systems, more and more types of impulsive neural network systems have been established, classical research methods have been continuously improved, and new research methods have been continuously proposed. In [129], a novel concept of finite-time stable function pair is proposed to investigate the finite-time stability of impulsive differential inclusion, and the results are applied to study discontinuous impulsive neural networks. In [128], a new concept of average stochastic impulsive gain is proposed. By using Dupire Itô′s formula and the Lyapunov method, some criteria on the mean-square exponential stability of multi-linked stochastic delayed complex networks with stochastic hybrid impulses are derived. In [130], an impulsive disturbed neural network model with delays is constructed in quaternion space, and the exponential stability conditions of the delayed system are derived by utilizing generalized norms. In [131,132], impulsive reaction–diffusion neural networks are considered, a new analysis method and a novel vector inequality are developed, respectively, and some stability results are obtained.

6. Discussion

In recent years, differential systems with impulsive effects have attracted much attention due to their important role in theoretical research and engineering practice. A traditional and important problem is to study the stability of differential systems with impulsive effects. In this paper, a large number of studies have been reviewed. We present a survey on the stability of differential systems with impulsive effects, and focus on the following topics:

(i) Stability of impulsive differential systems (IDS, IDDS, IFDS).

We present three general impulsive differential systems (Systems (1), (2) and (5)) with emphasis on the application of classical stability methods, such as the comparison method (Theorem 1 [13]) and the Lyapunov-Razumikhin techniques (Theorem 3 [36]), etc. In addition, the research on the stability of differential systems with uncertain impulsive moments is combed.

(ii) Stability of stochastic impulsive differential systems (SIFDS, SIDDS, SIDSM).

We present three general stochastic impulsive differential systems (Systems (7), (8) and (11)) with emphasis on the improvement of classical methods and the introduction of new methods. For example, the time-derivatives of Razumikhin functions are allowed to be indefinite (Theorem 4 [59]), and the MRKM method is introduced for the numerical approximation ([5]). In addition, a brief remark on the stability of stochastic impulsive differential systems with random impulses is given.

(iii) Stability of differential systems with specific impulses (DS-NI, DS-DI, DS-IL, DS-ITW).

For these four types of systems, we describe their main characteristics and development process, analyze their stability studies, and give some examples and stability results for illustration.

From this survey, we can see that the stability of differential systems with impulsive effects has been well studied. Furthermore, many studies have demonstrated that unstable systems can become stable under impulsive control and the design laws of impulsive control are given. However, at the same time, we can also see that the structure of the differential system with impulsive effects is very complex, and the coupling, mixing and synthesis of various variables bring challenges to the research work.

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