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Characterizing q-Bessel Functions of the First Kind with Their New Summation and Integral Representations

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Abstract: As a powerful tool for models of quantum computing, q-calculus has drawn the attention of many researchers in the discipline of special functions. In this paper, we present new properties and characterize q-Bessel functions of the first kind using some identities of q-calculus. The results presented in this article help us to obtain new expression results related to q-special functions. New summation and integral representations for q-Bessel functions of the first kind are also established. A few examples are also provided to demonstrate the effectiveness of the proposed strategy.

Keywords: quantum calculus; Bessel function; q-shift factorial; q-Bessel function; q-hypergeometric function

MSC: 11B83; 33C10; 33E20

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1. Introduction

One of the most important generalizations of conventional calculus is quantum calculus, often known as q-calculus, because it has been shown to be applicable to quantum physics and various other fields of study, including number theory, combinations, orthogonal polynomials, etc. Initially, the theory of q-calculus was developed by Jackson [1]. The introduction of q-calculus opened the possibility for the introduction and study of the q-analogs of several elementary and special functions, for example, q-exponential, q-trigonometric, q-gamma, q-beta [2], q-hypergeometric [3], and q-Bessel functions [4,5]. In recent years, the theory of q-special functions has attracted additional attention due to their usefulness in different emerging branches of mathematics and the sciences.

There is a close relationship between Bessel functions and problems involving circular or cylindrical symmetry. The free vibrations of a circular membrane can be studied, and the temperature distribution along a cylinder can be calculated, as two examples. Not only are they essential in electromagnetic theory but also in countless other branches of physics and engineering. Because of their natural link with cylinder-shaped domains, "cylinder functions" are shorthand for all solutions to Bessel's equation.

We were motivated by the applications of Bessel functions in different fields of science and engineering. Bessel functions are also used to find separable solutions to Laplace's equation and the Helmholtz equation in spherical coordinates and are therefore particularly relevant for many situations involving wave propagation and static potentials. The q-analog of these functions has attracted the attention of several researchers, who developed the theory of q-Bessel functions. q-Bessel function $J_n^1(x;q)$ was introduced and studied by Jackson [6]. $J_n^2(x;q)$ was introduced and studied by Ismail [7] and also studied by Hahn [8]. Hahn and Exton introduced the third q-Bessel function [9–11]. Some well-known forms of q-Bessel functions $(J_n^1(x;q))$ of the first kind can be found in [4,12]. The theory of q-Bessel functions has been studied by many mathematicians and physicists. This theory has grown to include two variables [13] and generalized q-Bessel functions [14]. Srivastava [15] considered some recent developments with respect to the extension of the q-Bessel polynomials.

Recently, several researchers working in the field of q-special functions introduced and studied several properties of q-Bessel functions (see, e.g., [13,14,16–21]). The series definition of a q-Bessel function $J_n(x;q)$ is given by the following expression

$$\begin{aligned} J_n(x;q) &= \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(x/[2]_q\right)^{n+2k}}{(q;q)_k (q^{n+1};q)_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(x/[2]_q\right)^{n+2k}}{[n+k]_q! [k]_q!} \qquad \text{for } -\infty < x < \infty. \end{aligned} \tag{1}$$

which converges absolutely for $|x| < [2]_q$ and $\frac{(q^{n+1};q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_n}$. The following generating function also characterizes the first kind of q-Bessel functions ($J_n^1(x;q)$ or $J_n(x;q)$) (see, e.g., [4,12]):

$$e_q\left([2]_q^{-1} xt\right) e_q\left(-\left([2]_q t\right)^{-1} x\right) = \sum_{n=-\infty}^{\infty} J_n(x;q)\, t^n \quad \text{for } t \neq 0 \text{ and } -\infty < x < \infty. \tag{2}$$

For any real number ($r \geq 0$), in view of (1), we have the following series definition of the q-Bessel functions of the second kind:

$$J_{r,q}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(x/[2]_q\right)^{r+2k}}{\Gamma_q(r+k+1)[k]_q!} \qquad \text{for } -\infty < x < \infty.$$

In this paper, we present new properties of q-Bessel functions. Applying our results, we characterize q-Bessel functions of the first kind. New summation and integral representations for q-Bessel functions of the first kind are also established. In summary, some of the results presented in this paper are original; we also refer to results reported in the literature as special cases (see, e.g., [2–5,7,9–12,22,23] and references therein).

2. Preliminaries

In this section, we state some basic definitions, notations, and known results in quantum calculus that are needed for further discussion throughout this paper.

The complex number w has a q-analog defined as follows (see, e.g., [24,25]):

$$[w]_q = \frac{1-q^w}{1-q} = \sum_{k=1}^{w} q^{k-1} \qquad \text{for } 0 < q < 1 \text{ and } w \in \mathbb{C}. \tag{3}$$

The q-factorial (see [24,25]) is defined as

$$[r]_q! = \begin{cases} \prod_{s=1}^{r} [s]_q = [1]_q [2]_q \ldots [r]_q, & 0 < q < 1 \quad r \geq 1 \\ 1, & \qquad\qquad r = 0, \end{cases}$$

which satisfies $[r+1]_q! = [r+1]_q [r]_q!$. For $a \in \mathbb{C}$ and $0 < q < 1$, the q-shift factorials ($(a;q)_s$) (see [24,25]) are defined by

$$(a;q)_0 = 1,$$

$$(a;q)_s = \prod_{r=0}^{s-1} (1 - q^r a) \qquad \text{for } s \in \mathbb{N}, \tag{4}$$

and

$$(a;q)_\infty = \lim_{s\to+\infty}(a;q)_s = \prod_{r=0}^{+\infty}(1-q^r a).$$

The following known property is important for computation (see [24]):

$$(a;q)_{r+s} = (a;q)_r(aq^r;q)_s \qquad \text{for } r,s \in \mathbb{N}.$$

For more information on the q-shift factorial ($(a;q)_n$), please refer to [24,25] and related references therein. It is known that the equivalent expression of Equation (4) is represented by (see [24,25])

$$\left([r]_q\right)_s = \frac{\Gamma_q(r+s)}{\Gamma_q(r)} = \begin{cases} [r]_q[r+1]_q[r+2]_q\ldots[r+s-1]_q, & s \geq 1 \\ 1, & s = 0, \end{cases} \tag{5}$$

where $\Gamma_q(r)$ is the q-gamma function (see, e.g., [2]) satisfying

$$\Gamma_q(r+1) = [r]_q!. \tag{6}$$

Gauss's q-binomial coefficient (see [24,25]) is given by

$$\begin{bmatrix} r \\ t \end{bmatrix}_q = \frac{[r]_q!}{[r-t_q!][t]_q!} \qquad \text{for } t = 0,1,\ldots,r.$$

For more details, we refer the reader to [24,25] and references therein.

In [24,25], the two q-exponential functions indicated by $e_q(x)$ and $E_q(x)$ were defined, respectively, by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \qquad \text{for } 0 < q < 1. \tag{7}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!} \qquad \text{for } 0 < q < 1. \tag{8}$$

The following is the relationship between $e_q(x)$ and $E_q(x)$:

Theorem 1 (see, e.g., [24,25])**.** *For $0 < q < 1$, we have*

$$e_q(x)E_q(-x) = 1. \tag{9}$$

In [26], when a function (f) was being differentiated with regard to an x value, the notation for the q-derivative is ($D_{q,x}f(x)$) was given by

$$D_{q,x}f(x) = \frac{f(qx)-f(x)}{qx-x} \qquad \text{for } 0 < q < 1 \text{ and } x \neq 0.$$

In particular, we have

$$D_{q,x}\, x^n = [n]_q x^{n-1}.$$

The kth-order q-derivatives of q-exponential functions (see [26]) are given by

$$D^k_{q,x}e_q(\alpha x) = \alpha^k e_q(\alpha x) \qquad \text{for } k \in \mathbb{N},\ 0 < q < 1 \text{ and } \alpha \in \mathbb{C}$$

and

$$D^k_{q,x}E_q(\alpha x) = \alpha^k q^{\binom{k}{2}} E_q(\alpha q^k x) \qquad \text{for } k \in \mathbb{N},\ 0 < q < 1 \text{ and } \alpha \in \mathbb{C},$$

where $D^k_{q,x}$ represents the kth-order q-derivative with regard to x. In particular, we use $D_{q,x}$ instead of $D^1_{q,x}$. The following formula is well known (see [26]):

$$D_{q,x}\big(f(x)g(x)\big) = f(x)D_{q,x}g(x) + g(qx)D_{q,x}f(x). \tag{10}$$

For any function ($f(x)$) the q-definite integral is defined as follows (see, e.g., [1]):

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n\, f(aq^n).$$

In particular, we have the following (see [1]):

$$\int D_qf(x)d_qx = f(x). \tag{11}$$

A q-definite integral of the q-derivative of a function (f) on $[0,a]$ is defined as follows (see, e.g., [1]):

$$\int_0^a D_qf(x)d_qx = f(a). \tag{12}$$

For more information on the q-derivative and q-definite integral, we refer the reader to [1] and references therein.

It is well known that the following generating function characterizes the Bessel functions of the first kind ($J_n(x)$) (see [27]):

$$exp\left[\frac{1}{2}x(t-\frac{1}{t})\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad \text{for } t \neq 0 \tag{13}$$

and the series definition

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{k!(n+k)!} \qquad \text{for } -\infty < x < \infty. \tag{14}$$

In view of Equation (13), for $x = 0$, we have

$$\sum_{n=-\infty}^{\infty} J_n(0)t^n = 1.$$

By comparing the two sides of the previous equation, we find that

$$J_n(0) = 0 \qquad \text{for } n \neq 0,$$

and

$$J_0(0) = 1.$$

Again, based on Equation (13), we can deduce

$$J_0'(0) = 0 \qquad \text{for } n = 0.$$

Furthermore, inputting $n = 1$ and $x = 0$ into the q-derivative of Equation (14), we obtain

$$J_1'(0) = \frac{1}{2}.$$

For more details, we refer the reader to [27] and references therein. It is well known that

$$J_{-n}(x;q) = (-1)^n J_n(x;q).$$

It is obvious that if we take $x = 0$ in (2), we obtain

$$\sum_{n=-\infty}^{\infty} J_n(0;q)t^n = 1.$$

Comparing both sides of the above equation for equal powers of t, we obtain

$$J_n(0;q) = 0, \quad \text{for} \quad n \neq 0$$

and

$$J_0(0;q) = 1.$$

Taking $n = 0$ and $x = 0$ into the q-derivative of Equation (2), we have

$$J_0'(0;q) = 0. \tag{15}$$

Similarly, taking $n = 1$ and $x = 0$ into the q-derivative of Equation (2) yields

$$J_1'(0;q) = \frac{1}{[2]_q}. \tag{16}$$

Remark 1. *Replacing x with $-x$ in (1) yields*

$$J_n(-x;q) = (-1)^n J_n(x;q).$$

Theorem 2 (see, e.g., [27])**.** *For any Bessel function ($J_n(x)$), the following differential recurrence relations hold:*

$$D_x\{x^n J_n(x)\} = x^n J_{n-1}(x), \tag{17}$$

$$D_x\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x), \tag{18}$$

$$D_x\, J_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x), \tag{19}$$

$$D_x\, J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x), \tag{20}$$

and

$$D_x\, J_n(x) = \frac{1}{2}\Big(J_{n-1}(x) - J_{n+1}(x)\Big). \tag{21}$$

Remark 2. *Taking $n = 0$ in Equation (20) yields*

$$J_0'(x) = -J_1(x). \tag{22}$$

The following crucial formulas are used to prove our results.

Theorem 3 (see, e.g., [27])**.**

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} sinx \tag{23}$$

and

$$J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} cosx. \tag{24}$$

Theorem 4 (see, e.g., [23])**.**

$$\sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2}\int_0^x J_0(y)d_y. \tag{25}$$

3. Characteristics of q-Bessel Functions

In this section, we study several features of q-Bessel functions of the first kind, including their recurrence relations.

We now establish the following recurrence relations for a q-Bessel function ($J_n(x;q)$).

Theorem 5. *The q-derivative of the product of q-Bessel function $J_n(x;q)$ and power x^n can be expressed by the following recurrence relation*

$$D_{q,x}\{x^n J_n(x;q)\} = \frac{x^n}{[2]_q} J_{n-1}(x;q) + \frac{\left(\sqrt{q}\right)^{n+1} x^n}{[2]_q} J_{n-1}\left(\sqrt{q}x;q\right), \tag{26}$$

Proof. By multiplying each side of (1) by x^n, then taking the q-derivative of the product with regard to x, we obtain

$$D_{q,x}\{x^n J_n(x;q)\} = \sum_{k=0}^{\infty} \frac{(-1)^k [2n+2k]_q x^{2k+2n-1}}{[n+k]_q![k]_q!([2]_q)^{2k+n}}. \tag{27}$$

From Equation (3) for $[2n+2k]_q$, we have

$$\begin{aligned} [2n+2k]_q &= \frac{1-q^{2(n+k)}}{1-q} \\ &= \frac{1-q^{n+k}}{1-q}(1+q^{n+k}) \\ &= [n+k]_q(1+q^{n+k}). \end{aligned}$$

Incorporating the aforementioned formula into the right-hand side of Equation (27), we obtain

$$\begin{aligned} D_{q,x}\{x^n J_n(x;q)\} =& \frac{x^n}{[2]_q} \sum_{k=0}^{\infty} \frac{(-1)^k (1+q^{n+k}) x^{2k+n-1}}{[n+k-1]_q![k]_q!([2]_q)^{2k+n-1}} \\ =& \frac{x^n}{[2]_q} \sum_{k=0}^{\infty} \frac{(-1)^k x^{n-1+2k}}{[n-1+k]_q![k]_q!([2]_q)^{n-1+2k}} \\ &+ \frac{\left(\sqrt{q}\right)^{n+1} x^n}{[2]_q} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{q}x\right)^{n-1+2k}}{[n-1+k]_q![k]_q!([2]_q)^{n-1+2k}}. \end{aligned} \tag{28}$$

By using (1), we can prove (26). □

Theorem 6. *The q-derivative of the product of q-Bessel function $J_n(x;q)$ and power x^{-n} can be expressed by the following recurrence relation:*

$$D_{q,x}\{x^{-n} J_n(x;q)\} = -\frac{-x^{-n}}{[2]_q} J_{n+1}(x;q) - \frac{\left(\sqrt{q}\right)^{-n+1} x^{-n}}{[2]_q} J_{n+1}\left(\sqrt{q}x;q\right), \tag{29}$$

Proof. By multiplying both sides of Equation (1) by x^{-n} and taking the q-derivative of both sides of the result with respect to x yields

$$D_{q,x}\{x^{-n} J_n(x;q)\} = \sum_{k=0}^{\infty} \frac{(-1)^k [2k]_q x^{2k-1}}{[n+k]_q![k]_q!([2]_q)^{2k+n}}. \tag{30}$$

Based on Equation (3), for $[2k]_q$, we have

$$\begin{aligned}[2k]_q &= \frac{1-q^{2k}}{1-q} \\ &= [k]_q(1+q^k).\end{aligned}$$

Taking the preceding formula into the right-hand side of Equation (30) yields

$$D_{q,x}\{x^{-n}J_n(x;q)\} = \frac{x^{-n}}{[2]_q}\sum_{k=1}^{\infty}\frac{(-1)^k(1+q^k)x^{2k+n-1}}{[n+k]_q![k-1]_q!([2]_q)^{2k+n-1}},$$

or, equivalently,

$$\begin{aligned}D_{q,x}\{x^{-n}J_n(x;q)\} &= \frac{-x^{-n}}{[2]_q}\sum_{k=0}^{\infty}\frac{(-1)^k(1+q^{k+1})x^{2k+n+1}}{[n+k+1]_q![k]_q!([2]_q)^{2k+n+1}} \\ &= \frac{-x^{-n}}{[2]_q}\sum_{k=0}^{\infty}\frac{(-1)^k x^{n+1+2k}}{[n+1+k]_q![k]_q!([2]_q)^{n+1+2k}} \\ &\quad - \frac{\left(\sqrt{q}\right)^{-n+1}x^{-n}}{[2]_q}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\sqrt{q}x\right)^{n+1+2k}}{[n+1+k]_q![k]_q!([2]_q)^{n+1+2k}}.\end{aligned}$$

Making use of (1), we arrive at (29). □

Theorem 7. *The q-derivative of q-Bessel function $J_n(x;q)$ can be expressed by the following recurrence relation:*

$$D_{q,x}\,J_n(x;q) = \frac{1}{q^n[2]_q}J_{n-1}(x;q) + \frac{\left(\sqrt{q}\right)^{n+1}}{q^n[2]_q}J_{n-1}\left(\sqrt{q}x;q\right) - \frac{[n]_q}{q^n x}J_n(x;q), \tag{31}$$

Proof. Applying (10) to the left side of Equation (26), we have

$$[n]_q x^{n-1}J_n(x;q) + (qx)^n J_n'(x;q) = \frac{x^n}{[2]_q}J_{n-1}(x;q) + \frac{\left(\sqrt{q}\right)^{n+1}x^n}{[2]_q}J_{n-1}\left(\sqrt{q}x;q\right), \tag{32}$$

or, equivalently,

$$J_n'(x;q) = \frac{1}{q^n[2]_q}J_{n-1}(x;q) + \frac{\left(\sqrt{q}\right)^{n+1}}{q^n[2]_q}J_{n-1}\left(\sqrt{q}x;q\right) - \frac{[n]_q}{q^n x}J_n(x;q), \tag{33}$$

which can be rewritten as (31). □

Inputting (10) into the left-hand side of (29), we obtain the following theorem.

Theorem 8. *The q-derivative of q-Bessel function $J_n(x;q)$ can be expressed by the following recurrence relation:*

$$D_{q,x}\,J_n(x;q) = \frac{-q^n}{[2]_q}J_{n+1}(x;q) - \frac{\left(\sqrt{q}\right)^{n+1}}{[2]_q}J_{n+1}\left(\sqrt{q}x;q\right) + \frac{[n]_q q^n}{x}J_n(x;q). \tag{34}$$

Theorem 9. *The q-derivative of q-Bessel function $J_n(x;q)$ can be expressed by the following recurrence relation:*

$$D_{q,x}\, J_n(x;q) = \frac{1}{[2]_q}\left(\frac{1}{q^{n-1}[2]_q} J_{n-1}(x;q) - \frac{q^n}{[2]_q} J_{n+1}(x;q) + \frac{(\sqrt{q})^{n+1}}{q^{n-1}[2]_q} J_{n-1}(\sqrt{q}x;q) - \frac{(\sqrt{q})^{n+1}}{[2]_q} J_{n+1}(\sqrt{q}x;q) + \frac{[n]_q q^n}{x} J_n(x;q) - \frac{[n]_q}{q^{n-1}x} J_{n,q}(x;q)\right) \tag{35}$$

Proof. Multiplying Equation (31) by q and adding the result to Equation (34), we obtain (35). □

Theorem 10. *The q-derivative of the product of q-Bessel function $J_n(x;q)$ and $\frac{[n]_q}{q^n x} + \frac{[n]_q q^n}{x}$ can be expressed by the following recurrence relation:*

$$\left(\frac{[n]_q}{q^n x} + \frac{[n]_q q^n}{x}\right) J_n(x;q) = \frac{1}{q^{n-1}[2]_q} J_{n-1}(x;q) + \frac{q^n}{[2]_q} J_{n+1}(x;q) + \frac{(\sqrt{q})^{n+1}}{q^{n-1}[2]_q} J_{n-1}(\sqrt{q}x;q) + \frac{(\sqrt{q})^{n+1}}{[2]_q} J_{n+1}(\sqrt{q}x;q). \tag{36}$$

Proof. The difference between Equation (31) and Equation (34) yields (36). □

As direct consequences of Theorems 5–10, we establish the following identities for q-Bessel function $J_n(x;q)$.

Corollary 1. *The following identities hold:*

$$qxJ_1'(x;q) = \frac{x}{[2]_q} J_0(x;q) + \frac{qx}{[2]_q} J_0(\sqrt{q}x;q) - J_1(x;q), \tag{37}$$

$$(xq)^{-1} J_1'(x;q) = x^{-2} J_1(x;q) - \frac{-x^{-1}}{[2]_q} J_2(x;q) - \frac{x^{-1}}{[2]_q} J_2(\sqrt{q}x;q), \tag{38}$$

$$J_0'(x;q) = \frac{-1}{[2]_q} J_1(x;q) - \frac{(\sqrt{q})}{[2]_q} J_1(\sqrt{q}x;q), \tag{39}$$

and

$$J_0''(x;q) = \frac{-1}{[2]_q} J_1'(x;q) - \frac{(\sqrt{q})}{[2]_q} J_1'(\sqrt{q}x;q). \tag{40}$$

Proof.

(i) Taking $n = 1$ in Equations (26) and (29), we obtain

$$qxJ_1'(x;q) + J_1(x;q) = \frac{x}{[2]_q} J_0(x;q) + \frac{qx}{[2]_q} J_0(\sqrt{q}x;q),$$

which is equivalent to (37).

(ii) Taking $n = 1$ in Equation (29), we have

$$(qx)^{-1} J_1'(x;q) - x^{-2} J_1(x;q) = -\frac{-x^{-1}}{[2]_q} J_2(x;q) - \frac{x^{-1}}{[2]_q} J_2(\sqrt{q}x;q),$$

which is equivalent to (38).

(iii) Taking $n = 0$ in Equation (34), we show (39).

(iv) Differentiating both sides of Equation (39) with respect to x, we prove (40).

□

Now, we represent some integral representations of a q-Bessel function ($J_n(x;q)$). In view of Equations (15), (39) and (40) and using Equations (11) and (16), we obtain the following:

$$\int \frac{1}{[2]_q} d_q x = J_1(0;q) + C, \tag{41}$$

$$\int_0^1 \frac{-1}{[2]_q}(1+q^{k+1}) J_1(x;q) d_q x = J_0(1;q) - 1 \tag{42}$$

and

$$\int_0^1 \frac{-1}{[2]_q}(1+q^{k+1}) J_1''(x;q) d_q x = J_0'(1;q) - J_0'(0;q) = J_0'(1;q). \tag{43}$$

Substituting n with $\frac{1}{2}$ in (2) and substituting n with $-\frac{1}{2}$ in (2), since $\Gamma_q(1/2) = \sqrt{\pi_q}$ (see [18]), we establish the following formulas for q-Bessel functions of the first kind:

Corollary 2. *Let $0 < q < 1$. Then*

$$J_{\frac{1}{2}}(x;q) = \sqrt{\frac{[2]_q}{x\pi_q}} \left(\frac{x}{[2]_q[1/2]_q} - \frac{x^3}{[2]_q^3[3/2]_q[1/2]_q} + \frac{x^5}{[2]_q^6[5/2]_q[3/2]_q[1/2]_q} - \dots \right) \tag{44}$$

and

$$J_{-\frac{1}{2}}(x;q) = \sqrt{\frac{[2]_q}{x\pi_q}} \left(1 - \frac{x^2}{[2]_q^2[1/2]_q} + \frac{x^4}{[2]_q^5[3/2]_q[1/2]_q} - \dots \right). \tag{45}$$

Remark 3. *(1) Equations (17)–(22) can be obtained by substituting $q \to 1^-$ in Equations (26)–(36), respectively;*

(2) Equations (23) and (24) can be obtained by substituting $q \to 1^-$ in Equations (44) and (45), respectively.

Here, we provide various examples illustrating the efficacy of the outcomes achieved in this part.

Example 1. *(1) Taking $n = 2$ into Equations (26) and (29), we obtain the following recurrence relations of the q-derivatives of the products of $J_2(x;q)$ with power x^2 and $J_{-2}(x;q)$ with power x^{-2}:*

$$D_{q,x}\{x^2 J_2(x;q)\} = \frac{x^2}{[2]_q} J_1(x;q) + \frac{q\sqrt{q}x^n}{[2]_q} J_1(\sqrt{q}x;q)$$

and

$$D_{q,x}\{x^{-2} J_2(x;q)\} = -\frac{-x^{-2}}{[2]_q} J_3(x;q) - \frac{x^{-2}}{[2]_q\sqrt{q}} J_3(\sqrt{q}x;q).$$

(2) Taking $n = 3$ into Equations (31) and (34), we obtain the following recurrence relations for $J_3(x;q)$:

$$D_{q,x}\, J_3(x;q) = \frac{1}{q^3[2]_q} J_2(x;q) + \frac{(\sqrt{q})^4}{q^3[2]_q} J_2(\sqrt{q}x;q) - \frac{[3]_q}{q^3 x} J_3(x;q)$$

and

$$D_{q,x}\, J_3(x;q) = \frac{-q^3}{[2]_q} J_4(x;q) - \frac{q^2}{[2]_q} J_4(\sqrt{q}x;q) + \frac{[3]_q q^3}{x} J_3(x;q).$$

(3) *Taking $n = 2$ in Equation (35), we obtain the following recurrence relation of $J_2(x;q)$:*

$$D_{q,x}\, J_2(x;q) = \frac{1}{[2]_q}\left(\frac{1}{q\,[2]_q} J_1(x;q) - \frac{q^2}{[2]_q} J_3(x;q) + \frac{\sqrt{q}}{[2]_q} J_1\big(\sqrt{q}x;q\big) - \frac{q\sqrt{q}}{[2]_q} J_3\big(\sqrt{q}x;q\big) + \frac{[2]_q q^2}{x} J_2(x;q) - \frac{[2]_q}{qx} J_{2,q}(x;q)\right).$$

4. Summation and Integral Representations

In this section, we establish some summation and integral formulas for a q-Bessel function ($J_n(x;q)$) by utilizing the identities (7) and (9), as well as the generating function and series description, of $J_n(x;q)$.

Recall that the hypergeometric representation of ${}_r\phi_s$ was defined by (see [28])

$${}_r\phi_s\left(\begin{matrix} b_1, b_2, \cdots, b_r \\ c_1, c_2, \cdots, c_s \end{matrix}\ ;\ q, z\right) = \sum_{n=0}^{\infty} \frac{(b_1, b_2, \ldots, b_r; q)_n}{(c_1, c_2, \ldots, c_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} \frac{z^n}{(q;q)_n}$$

and

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\ ;\ q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n} \frac{z^n}{(q;q)_n}. \tag{46}$$

In particular, the following q-hypergeometric functions was given in (see [28]):

$${}_2\phi_1\left(\begin{matrix} -n, b \\ c \end{matrix}\ ;\ q, q^{c+n-b}\right) = \frac{\big([c-b]_q\big)_n}{\big([c]_q\big)_n}, \qquad |q^{c+n-b}| < 1. \tag{47}$$

Theorem 11. *The following summation formulas of q-Bessel function $J_n(x;q)$ hold:*

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} \big(x/[2]_q\big)^k J_{n+k}(x;q)}{[k]_q!} = \frac{\big(x/[2]_q\big)^n}{[n]_q!}. \tag{48}$$

Proof. In view of Equation (9), we have

$$e_q\Big([2]_q^{-1} xt\Big)\, e_q\Big(-\big([2]_q t\big)^{-1} x\Big)\, E_q\Big(\big([2]_q t\big)^{-1} x\Big) = e_q\Big([2]_q^{-1} xt\Big),$$

which, upon using Equations (8), (7) and (2), yields

$$\sum_{n=-\infty}^{\infty} J_n(x;q) t^n \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} \big(x/[2]_q\big)^k t^{-k}}{[k]_q!} = \sum_{n=0}^{\infty} \frac{\big(x/[2]_q\big)^n t^n}{[n]_q!},$$

or, equivalently,

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{J_n(x;q) q^{\binom{k}{2}} \big(x/[2]_q\big)^k t^{n-k}}{[k]_q!} = \sum_{n=0}^{\infty} \frac{\big(x/[2]_q\big)_q^n t^n}{[n]_q!}.$$

By changing the index of n to $n+k$ as the range of the values of n on $-\infty < n < \infty$, we obtain

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{J_{n+k}(x;q) q^{\binom{k}{2}} \big(x/[2]_q\big)^k t^n}{[k]_q!} = \sum_{n=-\infty}^{\infty} \frac{\big(x/[2]_q\big)_q^n t^n}{[n]_q!}.$$

Therefore, by comparing the equal powers of t on both sides, we obtain (48). □

Theorem 12. *The following summation formula of q-Bessel functions $J_n(x;q)$ holds:*

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \frac{q^{-\binom{k}{2}}}{[2]_q} \int_0^x J_0(y;q) d_q y. \tag{49}$$

Proof. By applying Equation (1), we have

$$\begin{aligned}\sum_{k=0}^{\infty} J_{2k+1}(x;q) &= \sum_{k,n=0}^{\infty} \frac{(-1)^n \left(x/[2]_q\right)^{2n+2k+1}}{[n]_q![n+2k+1]_q!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n-k}\left(x/[2]_q\right)^{2n+1}}{[n-k]_q![n+k+1]_q!}.\end{aligned}$$

By multiplying the right side of the above equation by $\frac{[n]_q!}{[n]_q!}$, $\frac{[k]_q!}{[k]_q!}$, $\frac{[n+1]_q!}{[n+1]_q!}$ and $\frac{q^{\binom{k}{2}-nk}}{q^{\binom{k}{2}-nk}}$, we obtain

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{-k}[n]_q! q^{\binom{k}{2}-nk}}{[n-k]_q!} \frac{[n+1]_q![k]_q!}{[n+k+1]_q!} \frac{(-1)^n \left(x/[2]_q\right)^{2n+1}}{[k]_q![n]_q![n+1]_q! q^{\binom{k}{2}-nk}}.$$

Making further use of (5) and the following property (see [28])

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^{-k} q^{\binom{k}{2}-nk},$$

we arrive at

$$\sum_{k=0}^{\infty} J_{2k+1,q}(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(q^{-n};q)_k \left([1]_q\right)_k}{\left([n+2]_q\right)_k [k]_q!} \frac{(-1)^n \left(x/[2]_q\right)^{2n+1}}{[n]_q![n+1]_q! q^{\binom{k}{2}-nk}}.$$

Making use of Equation (46) yields

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \sum_{n=0}^{\infty} {}_2\phi_1\left(\begin{matrix} q^{-n}, q \\ q^{n+2} \end{matrix} ; q, 1 \right) \frac{q^{nk-\binom{k}{2}} (-1)^n \left(x/[2]_q\right)^{2n+1}}{[n]_q![n+1]_q!}.$$

By using Equation (47), we obtain

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \sum_{n=0}^{\infty} \frac{\left([n+1]_q\right)_n}{\left([n+2]_q\right)_n} \frac{q^{nk-\binom{k}{2}} (-1)^n \left(x/[2]_q\right)^{2n+1}}{[n]_q![n+1]_q!},$$

or, equivalently,

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \sum_{n=0}^{\infty} \frac{\Gamma_q(n+2)\Gamma_q(2n+1) q^{nk-\binom{k}{2}} (-1)^n \left(x/[2]_q\right)^{2n+1}}{\Gamma_q(2n+2)\Gamma_q(n+1)[n]_q![n+1]_q!}.$$

Therefore, making use of Equation (6), we obtain

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \sum_{n=0}^{\infty} \frac{q^{nk-\binom{k}{2}} (-1)^n \left(x/[2]_q\right)^{2n+1}}{[2n+1]_q [n]_q! [n]_q!}.$$

By applying Equation (12), we obtain

$$\sum_{k=0}^{\infty} J_{2k+1}(x;q) = \frac{q^{-\binom{k}{2}}}{[2]_q} \int_0^x \sum_{n=0}^{\infty} \frac{(-q^k)^n \left(y/[2]_q\right)^{2n}}{[n]_q![n]_q!} d_q y,$$

which can be simplified as (49). □

Theorem 13. *The following summation formula of q-Bessel functions ($J_n(x;q)$) holds:*

$$\sum_{k=-\infty}^{\infty} J_k(x;q) J_{n-k,q}(y) = J_n(x+y;q). \tag{50}$$

Proof. By replacing x with $x+y$ in the generating function (2), we have

$$e_q\left([2]_q^{-1} xt\right) e_q\left(-\left([2]_q t\right)^{-1} x\right) e_q\left([2]_q^{-1} yt\right) e_q\left(-\left([2]_q t\right)^{-1} y\right) = \sum_{n=-\infty}^{\infty} J_n(x+y;q)\, t^n.$$

By using Equation (2) again, we obtain

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(x;a) J_k(x;q)\, t^{n+k}. = \sum_{n=-\infty}^{\infty} J_n(x;q)\, t^n.$$

Equating the equal powers of t, we prove (50). □

Remark 4. *When $q \to 1^-$, Equation (49) reduces to Equation (25).*

Finally, we consider a few examples that show the effectiveness of the results that were attained in this section.

Example 2. *(1) Taking $n = 2$ in Equation (48), we obtain the following summation formula of q-Bessel function $J_2(x;q)$:*

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} \left(x/[2]_q\right)^k J_{2+k}(x;q)}{[k]_q!} = \frac{\left(x/[2]_q\right)^2}{[3]_q [2]_q}.$$

(2) Taking $n = 3$ in Equation (50), we obtain the following summation formula of q-Bessel function $J_3(x;q)$:

$$\sum_{k=-\infty}^{\infty} J_k(x;q) J_{2-k,q}(y) = J_3(x+y;q).$$

5. Conclusions and Recommendations for Future Work

Recently, many special q-functions have been used to study quantum calculus. It is amazing that this led to the presentation of new properties of q-Bessel functions of the first kind using some identities of q-calculus. The generating function and series definition of the q-Bessel functions of the first kind ($J_n(x;q)$) are the most important part of all of these tasks. In this paper, we characterized the properties of q-Bessel functions of the first kind and obtained some of their recurrence relations, summation formulas, and integral representations. We also presented a few examples to demonstrate the effectiveness of the proposed strategy.

We now outline our proposed main avenues for future research, starting with the work presented here. The results presented in this paper suggest several ideas to characterize properties of q-Bessel functions and other special q-functions of the second and third kinds and obtain recurrence relations, summation formulas, and integral representations. Moreover, the results established in this paper can help us obtain new expression results related to other special q-functions in future studies.

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References

1. Jackson, D.O.; Fukuda, T.; Dunn, O.; Majors, E. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
2. Askey, R. The q-Gamma and q-Beta functions. *Appl. Anal.* **1978**, *8*, 125–141. [CrossRef]
3. Andrews, G.E. q-hypergeometric and related functions. In *NIST Handbook of Mathematical Functions*; U.S. Department of Commerce: Washington, DC, USA, 2010; pp. 419–433.
4. Dattoli, G.; Torre, A. q-Bessel functions: The point of view of the generating function method. *Rend. Mat. Appl.* **1997**, *17*, 329–345.
5. Štampach, F.; Šťovíček, P. The Hahn-Exton q-Bessel function as the characteristic function of a Jacobi matrix. *Spec. Matrices* **2014**, *2*, 20141014. [CrossRef]
6. Jackson, F.H. The application of basic numbers to Bessel's and Legendre's functions. *Proc. Lond. Math. Soc.* **1905**, *2*, 192–220. [CrossRef]
7. Berg, C.; Ismail, M.E.H. q-Hermite polynomials and classical orthognonal polynomials. *J. Math.* **1996**, *48*, 43–63.
8. Hahn, W. Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation. *Math. Nachrichten* **1949**, *2*, 340–379. [CrossRef]
9. Exton, H. A basic analogue of the Bessel-Clifford equation. *Jñānābha* **1978**, *8*, 49–56.
10. Exton, H. *q-Hypergeometric Functions and Applications*; Ellis Horwood Ltd.: Chichester, UK; Halsted Press [John Wiley & Sons, Inc.]: New York, NY, USA, 1983.
11. Hahn, W. Die mechanische Deutung einer geometrischen Differenzgleichung. *Z. Angew. Math. Mech.* **1953**, *33*, 270–272. [CrossRef]
12. Dattoli, G.; Torre, A. Symmetric q-Bessel functions. *Matematiche* **1996**, *51*, 153–167.
13. Riyasat, M.; Khan, S.; Nahid, T. Quantum algebra εq (2) and 2D q-Bessel functions. *Rep. Math. Phys.* **2019**, *93* 191–206. [CrossRef]
14. Riyasat, M.; Khan, S.; Haneef, M. Generalized 2D Extension of the q-Bessel polynomials. *Rep. Math. Phys.* **2022**, *90*, 63–80. [CrossRef]
15. Srivastava, H.M. An introductory overview of Bessel polynomials, the generalized Bessel polynomials and the q-Bessel polynomials. *Symmetry* **2023**, *15*, 822. [CrossRef]
16. Cao, J.; Srivastava, H.M.; Zhou, H.L.; Arjika, S. Generalized q-Difference Equations for q-Hypergeometric Polynomials with Double q-Binomial Coefficients. *Mathematics* **2022**, *10*, 556. [CrossRef]
17. Cao, J.; Huang, J.-Y.; Fadel, M.; Arjika, S. A review on q-difference equations for Al-Salam–Carlitz polynomials and applications to U(n+1) type generating functions and Ramanujan's integrals. *Mathematics* **2023**, *11*, 1655. [CrossRef]
18. Dattoli, G.; Germano, B; Górska, K.; Martinelli, M.R. Can Umbral and q-calculus be merged? *arXiv* **2019**, arXiv:1909.00058.
19. Raza, N.; Fadel, M.; Nisar, K.S.; Zakarya, M. On 2-variable q-Hermite polynomials. *Aims Math.* **2021**, *6*, 8705–8727. [CrossRef]
20. Srivastava, H.M.; Yasmin, G.; Muhyi, A.; Araci, S. Certain results for the twice-iterated 2D q-Appell polynomials. *Symmetry* **2019**, *11*, 1307. [CrossRef]
21. Wolter, G. $3nj$-symbols and identities for q-Bessel functions. *Ramanujan J.* **2018**, *47*, 317–337.
22. Florenini, R.; Vinet, L. Quantum algebras and q-special functions. *Ann. Phys.* **1993**, *221*, 53–70. [CrossRef]
23. Rainville, E.D. *Special Functions*; The Macmillan Company: New York, NY, USA, 1960.
24. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions of Encyclopedia Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1999; Volume 71.
25. Kac, V.G.; Pokman C. *Quantum Calculus*; Springer: New York, NY, USA, 2002; Volume 113.
26. Jackson, F.H. XI.—On q-functions and a certain difference operator. *Earth Environ. Sci. Trans. Royal Soc.* **1909**, *46*, 253–281. [CrossRef]
27. Andrews, L.C. *Special Functions of Mathematics for Engineers*, 2nd ed.; McGraw-Hill, Inc.: New York, NY, USA, 1992.
28. Gasper, G.; Rahman, M. Basic Hypergeometric Series. In *Encyclopedia of Mathematics and Its Applications*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004; Volume 96.