mathematics

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An Improved Approach to Investigate the Oscillatory Properties of Third-Order Neutral Differential Equations

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Abstract: In this work, by considering a third-order differential equation with delay-neutral arguments, we investigate the oscillatory behavior of solutions. It is known that the relationships between the solution and its derivatives of different orders, as well as between the solution and its corresponding function, can help to obtain more efficient oscillation criteria for differential equations of neutral type. So, we deduce some new relationships of an iterative nature. Then, we test the effect of these relationships on the criteria that exclude positive solutions to the studied equation. By comparing our results with previous results in the literature, we show the importance and novelty of the new results.

Keywords: third-order differential equations; neutral type; monotonic characteristics; oscillatory criteria

MSC: 34C10; 34K11

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Citation: Moaaz, O.; Alnafisah, Y. An Improved Approach to Investigate the Oscillatory Properties of Third-Order Neutral Differential Equations. *Mathematics* **2023**, *11*, 2290. https://doi.org/10.3390/math11102290

Academic Editors: Osman Tunç, Vitalii Slynko and Sandra Pinelas

Received: 27 March 2023
Revised: 29 April 2023
Accepted: 1 May 2023
Published: 15 May 2023

1. Introduction

Differential equations (DE) play a key role in linking mathematics to various applied sciences. DEs have applications in engineering, physics, biology, economics, and even the social sciences. In the past, many applications and phenomena were modeled on the principle of causality; that is, the future state of the system is independent of its past state and is affected only by the present. One is often thinking of either ordinary or partial DEs if it is also assumed that the system is governed by an equation involving the state and rate of change of the state. A closer examination reveals that the causation principle frequently just approximates reality and that a more accurate model would incorporate some of the earlier stages of the system. Also, it is pointless in some situations to not be dependent on the past. Therefore, the use of delay differential equations (DDE) in modeling phenomena contributes significantly to understanding and analyzing these problems better than ODEs.

One of the topics of qualitative theory, which is essentially concerned with analyzing the qualitative features of DEs, is the study of the oscillatory properties of solutions to DEs. The investigation of the oscillatory features of DDEs has advanced significantly over the past ten years. This is because there are many applications for DDEs. Additionally, oscillation theory is filled with intriguing theoretical issues that call for mathematical analysis methods. There was a lot of interest in the study of the oscillatory behavior of delay differential equation solutions, but this research was mostly centered on equations of even order.

The aim of this study is to create new conditions for evaluating the oscillatory behavior of solutions to the third-order neutral differential equation (NDE)

$$\left(r_2(t)\left(r_1(t)[x(t)+p(t)x(\tau(t))]'\right)'\right)'+q(t)x(\sigma(t))=0, \tag{1}$$

where $t \geq t_0$, and the following assumptions are satisfied:

(A1) $r_i \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$, and $\eta_i(t_0,\infty) = \infty$, for $i = 1,2$, where

$$\eta_i(h,k) := \int_h^k r_i^{-1}(s)\mathrm{d}s;$$

(A2) $p \in \mathbf{C}([t_0,\infty),[0,\infty))$ and $p(t) \leq p_0$, where p_0 is a constant;
(A3) $q \in \mathbf{C}([t_0,\infty),[0,\infty))$ and q does not vanish eventually;
(A4) $\tau,\sigma \in \mathbf{C}^1([t_0,\infty),\mathbb{R})$, $\tau(t) < t$, $\sigma(t) < t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

The corresponding function of the solution x is defined as $z(t) := x(t) + p(t)x(\tau(t))$. For a solution of Equation (1), we mean a function $x \in \mathbf{C}([t_x,\infty),\mathbb{R})$ for $t_x \geq t_0$, which has the properties $z, r_1z', r_2(r_1z')' \in \mathbf{C}^1([t_x,\infty),\mathbb{R})$, and x satisfies (1) for $t \geq t_x$. We consider only those solutions of Equation (1) which are not not vanish eventually.

Notation 1. *For facilitation, we refer to the category of eventually positive solutions whose corresponding function is increasing by $S_\uparrow$, and whose corresponding function is decreasing by $S_\downarrow$.*

The study of the oscillation of solutions of odd-order DDEs has and still contains many interesting analytical issues. In the canonical case, the positive solutions of third-order DDEs are classified as increasing solutions or decreasing solutions (called Kneser solutions). In the case of neutral equations, positive solutions are classified into those having an increasing corresponding function and those having a decreasing corresponding function.

Most studies have focused on excluding increasing positive solutions using several techniques, and also setting a condition that ensures that decreasing positive solutions converge to zero.

During the past few years, there has been a constant interest in obtaining sufficient conditions for oscillatory and non-oscillatory properties of different order differential equations. For some groups that developed equations of the second order see [1–4], for the fourth order see [5–8] and for higher-order we refer the reader to [9–12].

For third-order DDE, the oscillatory properties of solutions to these equations have been investigated with many different techniques, see for example [13–18]. Recently, Jadlovská et al. [19] improved the oscillation results for the linear DDE

$$\left(r_2(t)\left(r_1(t)x'(t)\right)'\right)' + q(t)x(\sigma(t)) = 0.$$

For the NDE, different forms of third-order equations have been studied, see for example [20–23]. In the following, we review some contributions to the development of the oscillation theory of third-order NDEs.

Using a condition of Hille and Nehari type, Baculikova and Dzurina [24] studied the NDE

$$\left(r_2(t)\left([x(t) + p(t)x(\tau(t))]''\right)^\alpha\right)' + q(t)x^\alpha(\sigma(t)) = 0,$$

where α is a quotient of odd positive integers and asserted that $S_\uparrow = \emptyset$ under the condition

$$\liminf_{t\to\infty}\frac{t^\alpha}{a(t)}\int_t^\infty q(\ell)\frac{\sigma^{2\alpha}(\ell)}{\ell^\alpha}\mathrm{d}\ell > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}(1-p_0)^\alpha}. \tag{2}$$

In [25], they also used the technique of comparison with first-order equations to ensure that $S_\uparrow = \emptyset$ if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^t q(\ell)\left(\frac{(\sigma(\ell)-t_0)^2(1-p(\sigma(\ell)))}{2r^{1/\alpha}(\ell)}\right)^\alpha \mathrm{d}\ell > \frac{1}{\mathrm{e}}.$$

By using Riccati technique, Thandapani and Li [26] proved that $S_{\uparrow} = \emptyset$ if

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{2^{\alpha-1}} \rho(\ell) Q(\ell) - \frac{(1+p_0^{\alpha}/\tau_0)}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(\ell))^{\alpha+1}}{(\rho(\ell)\eta_1(\sigma(\ell),t_0)\sigma'(\ell))^{\alpha}} \right) \mathrm{d}\ell = \infty, \tag{3}$$

where α is a quotient of odd positive integers, $\sigma'(t) > 0$, $\tau'(t) \geq \tau_0 > 0$, $Q(t) := \min\{q(t), q(\tau(t))\}$ and $\rho \in C([t_0,\infty),(0,\infty))$. Jiang and Li [27] used the integral averaging technique to establish a Philos-type criteria for oscillation of NDE

$$\left(r_2(t)[x(t) + p(t)x(\tau(t))]'' \right)' + \sum_{i=1}^{m} q_i(t) f_i(x(\sigma_i(t))) = 0.$$

In [28], Graef et al. discussed the asymptotic properties of solutions of NDE

$$\left(\left([x(t) + p(t)x(\tau(t))]'' \right)^{\alpha} \right)' + q(t)x^{\alpha}(\sigma(t)) = 0,$$

where α is a quotient of odd positive integers and for $p(t) > 1$.

On the other hand, many studies have recently focused on the criteria of the absence of Kneser solutions. Džurina et al. [29] developed criteria that confirm the oscillation of all solutions of Equation (1) by obtaining criteria that exclude Kneser solutions and combining them with the criteria obtained by Thandapani and Li in [26]. Later, Moaaz et al. [30,31] extended and generalized the results in [29] to odd-order equations in the half-linear and non-linear cases.

Although many results deal with the oscillatory behavior of solutions of third order NDEs, there are many open analytical issues related to these studies. For example, the traditional relationship between the solution and its corresponding function, which was used in all previous studies, is not standard and can be improved upon. Thus, the monotonic properties of positive increasing solutions can be improved. For Kneser's solutions, the problem of obtaining criteria without the need for constraints on delay functions is still under investigation. Another interesting problem is obtaining criteria for the case of $p_0 > 1$ without requiring that the conditions $\tau \circ \sigma = \sigma \circ \tau$ and $\tau'(t) > 0$. It is worth noting that we will address some of these problems during this study.

In this work, the oscillatory behavior of solutions of third-order differential equations with neutral-delay arguments was investigated. We derive some new inequalities and relationships between the solution and its corresponding function. We consider the two cases $p_0 < 1$ and $p_0 > 1$ without restrictions on the delay functions. Then, we obtain new monotonic characteristics for the positive solutions using an improved approach. By using these characteristics, we obtain more efficient criteria for testing the oscillation of the solutions of the studied equation.

2. Main Results

It is easy to see the significance of classifying the signs of derivatives of non-oscillatory solutions at the beginning of any study of the oscillatory features of solutions to NDEs. Based on Lemma 1.1 in [32], we find that the corresponding function z of any eventually positive solution to the studied equation is characterized by the following properties:

(P1) z and $(r_1 \cdot z')'$ are positive, and $\left(r_2 \cdot (r_1 \cdot z')' \right)'$ is nonpositive;

(P2) z' is of fixed sign.

2.1. Properties of Positive Solutions

2.1.1. Category $S_{\uparrow}$

For convenience, we define $G^{[0]}(t) := t$, $G^{[j]}(t) = G\left(G^{[j-1]}(t)\right)$, $G^{[-j]}(t) = G^{-1}\left(G^{[-j+1]}(t)\right)$, for $j = 1, 2, \ldots$,

$$\widehat{\eta}_0(t) := \int_{t_0}^{t} \frac{\eta_2(t_0, u)}{r_1(u)} \mathrm{d}u,$$

$$p_1(t; \ell, m) := \sum_{k=0}^{m} \left(\prod_{l=0}^{2k} p\left(\tau^{[l]}(t)\right) \right) \left[\frac{1}{p(\tau^{[2k]}(t))} - 1 \right] \frac{\widehat{\eta}_\ell\left(\tau^{[2k]}(t)\right)}{\widehat{\eta}_\ell(t)},$$

$$p_2(t; \ell, m) := \sum_{k=1}^{n} \left(\prod_{i=1}^{2k-1} \frac{1}{p(\tau^{[-i]}(t))} \right) \left[1 - \frac{1}{p(\tau^{[-2k]}(t))} \frac{\widehat{\eta}_\ell\left(\tau^{[-2k]}(t)\right)}{\widehat{\eta}_\ell(\tau^{[-2k+1]}(t))} \right]$$

and

$$\widehat{p}(t; \ell, m) := \begin{cases} 1 & \text{for } p_0 = 0, \\ p_1(t; \ell, m) & \text{for } p_0 < 1, \\ p_2(t; \ell, m) & \text{for } p_0 > \frac{\widehat{\eta}_0(t)}{\widehat{\eta}_0(\tau)}, \end{cases}$$

for $\ell = 0, 1, \ldots$, where m is a non-negative integer and $(\widehat{\eta}_\ell)$ is a functional sequence to be specified later.

Lemma 1. *([33] Lemma 1) Suppose that $x \in S_{\uparrow} \cup S_{\downarrow}$. Then, eventually,*

$$x(t) > \sum_{k=0}^{m} \left(\prod_{l=0}^{2k} p\left(\tau^{[l]}(t)\right) \right) \left[\frac{z\left(\tau^{[2k]}(t)\right)}{p(\tau^{[2k]}(t))} - z\left(\tau^{[2k+1]}(t)\right) \right], \tag{4}$$

for any integer $m \geq 0$.

Lemma 2. *Suppose that $x \in S_{\uparrow}$. Then, eventually,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{r_1(t) z'(t)}{\eta_2(t_0, t)} \right) \leq 0 \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{z(t)}{\widehat{\eta}_0(t)} \right) \leq 0. \tag{6}$$

Proof. Assume that $x \in S_{\uparrow}$. We have

$$r_1(t) z'(t) \geq \int_{t_0}^{t} \frac{r_2(u) [r_1(u) z'(u)]'}{r_2(u)} \mathrm{d}u \geq \eta_2(t_0, t) r_2(t) [r_1(t) z'(t)]', \tag{7}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{r_1 z'}{\eta_2} \right) = \frac{1}{r_2 \eta_2^2} \left[\eta_2 r_2 [r_1 z']' - r_1 z' \right] \leq 0.$$

Using this fact, we find

$$z(t) \geq \int_{t_0}^{t} \frac{r_1(u) z'(u)}{\eta_2(t_0, u)} \frac{\eta_2(t_0, u)}{r_1(u)} \mathrm{d}u \geq \frac{r_1(t) z'(t)}{\eta_2(t_0, t)} \widehat{\eta}_0(t), \tag{8}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{z}{\widehat{\eta}_0} \right) = \frac{\eta_2}{r_1 \widehat{\eta}_0^2} \left[\frac{r_1}{\eta_2} z' \widehat{\eta}_0 - z \right] \leq 0.$$

Hence, the proof ends. □

Lemma 3. *Suppose that $x \in S_{\uparrow}$. Then, eventually, $x(t) > \widehat{p}(t;0,m)z(t)$, and Equation (1) turns into*

$$\left(r_2(t)\left(r_1(t)z'(t)\right)'\right)' + q(t)\widehat{p}(\sigma(t);0,m)z(\sigma(t)) \le 0. \tag{9}$$

Proof. Suppose that $x \in S_{\uparrow}$.

Assume that $p_0 < 1$. It follows from Lemma 1 that (4) holds. From the facts that $\tau^{[2k+1]}(t) \le \tau^{[2k]}(t) \le t$, $z'(t) > 0$ and $(z(t)/\widehat{\eta}_0(t)) \le 0$, we arrive at

$$z\left(\tau^{[2k]}(t)\right) \ge z\left(\tau^{[2k+1]}(t)\right)$$

and

$$z\left(\tau^{[2k]}(t)\right) \ge \frac{\widehat{\eta}_0\left(\tau^{[2k]}(t)\right)}{\widehat{\eta}_0(t)} z(t),$$

for $k = 0, 1, \ldots$. Thus, inequality (4) becomes

$$x(t) > z(t) \sum_{k=0}^{m} \left(\prod_{l=0}^{2k} p\left(\tau^{[l]}(t)\right) \right) \left[\frac{1}{p(\tau^{[2k]}(t))} - 1 \right] \frac{\widehat{\eta}_0\left(\tau^{[2k]}(t)\right)}{\widehat{\eta}_0(t)},$$

which together with (1) gives (9).

On the other hand, assume that $p_0 > 1$. It follows from the definition of z that

$$\begin{aligned} p\left(\tau^{-1}\right)x(t) &= z\left(\tau^{-1}\right) - x\left(\tau^{-1}\right) \\ &= z\left(\tau^{-1}\right) - \frac{1}{p(\tau^{[-2]})}\left[z\left(\tau^{[-2]}\right) - x\left(\tau^{[-2]}\right)\right] \\ &= z\left(\tau^{-1}\right) - z\left(\tau^{[-2]}\right)\prod_{i=2}^{2}\frac{1}{p(\tau^{[-i]})} + \left[z\left(\tau^{[-3]}\right) - x\left(\tau^{[-3]}\right)\right]\prod_{i=2}^{3}\frac{1}{p(\tau^{[-i]})}, \end{aligned}$$

and so on. Hence, we arrive at

$$x(t) > \sum_{k=1}^{n} \left(\prod_{i=1}^{2k-1} \frac{1}{p(\tau^{[-i]}(t))} \right) \left[z\left(\tau^{[-2k+1]}(t)\right) - \frac{1}{p(\tau^{[-2k]}(t))} z\left(\tau^{[-2k]}(t)\right) \right]. \tag{10}$$

From the facts that $t \le \tau^{[-2k+1]}(t) \le \tau^{[-2k]}(t)$, $z'(t) > 0$ and $(z(t)/\widehat{\eta}_0(t))' \le 0$, we obtain

$$z\left(\tau^{[-2k]}(t)\right) \le \frac{\widehat{\eta}_0\left(\tau^{[-2k]}(t)\right)}{\widehat{\eta}_0(\tau^{[-2k+1]}(t))} z\left(\tau^{[-2k+1]}(t)\right)$$

and

$$z\left(\tau^{[-2k+1]}(t)\right) \ge z(t).$$

Thus, inequality (10) becomes

$$x(t) > z(t) \sum_{k=1}^{n} \left(\prod_{i=1}^{2k-1} \frac{1}{p(\tau^{[-i]}(t))} \right) \left[1 - \frac{1}{p(\tau^{[-2k]}(t))} \frac{\widehat{\eta}_0\left(\tau^{[-2k]}(t)\right)}{\widehat{\eta}_0(\tau^{[-2k+1]}(t))} \right],$$

which together with (1) gives (9).

Hence, the proof ends. □

Theorem 1. *Suppose that there is a* $\varphi \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\varphi(u)\widehat{p}(\sigma(u);0,m)q(u)\frac{\widehat{\eta}_0(\sigma(u))}{\widehat{\eta}_0(u)}-\frac{r_1(u)(\varphi'(u))^2}{4\eta_2(t_0,u)\varphi(u)}\right]\mathrm{d}u=\infty. \tag{11}$$

Then, $S_{\uparrow}=\emptyset$.

Proof. Suppose the contrary that $x\in S_{\uparrow}$. We define

$$w:=\varphi\cdot\frac{r_2\cdot[r_1\cdot z']'}{z}>0.$$

Then,

$$w'=\frac{\varphi'}{\varphi}w+\varphi\cdot\frac{1}{z}\Big(r_2\cdot[r_1\cdot z']'\Big)'-\varphi\cdot\frac{r_2\cdot[r_1\cdot z']'}{z^2}z'. \tag{12}$$

As in the proof of Lemma 2, we obtain that (7) holds. It follows from (7) and (9) that (12) becomes

$$w'\leq\frac{\varphi'}{\varphi}w-\varphi\cdot\widehat{p}(\sigma;0,m)\cdot q\cdot\frac{z(\sigma)}{z}-\varphi\cdot\frac{\eta_2}{r_1}\left(\frac{r_2\cdot[r_1\cdot z']'}{z}\right)^2, \tag{13}$$

which, with the fact that $(z(t)/\widehat{\eta}_0(t))'\leq 0$, gives

$$\begin{aligned} w' &\leq \frac{\varphi'}{\varphi}w-\varphi\cdot\widehat{p}(\sigma;0,m)\cdot q\cdot\frac{\widehat{\eta}_0(\sigma)}{\widehat{\eta}_0(t)}-\frac{\eta_2}{\varphi\cdot r_1}\cdot w^2\\ &= -\varphi\cdot\widehat{p}(\sigma;0,m)\cdot q\cdot\frac{\widehat{\eta}_0(\sigma)}{\widehat{\eta}_0(t)}+\frac{r_1\cdot(\varphi')^2}{4\eta_2\cdot\varphi}-\frac{\eta_2}{\varphi\cdot r_1}\left(w-\frac{r_1\cdot\varphi'}{2\eta_2}\right)^2\\ &\leq -\varphi\cdot\widehat{p}(\sigma;0,m)\cdot q\cdot\frac{\widehat{\eta}_0(\sigma)}{\widehat{\eta}_0(t)}+\frac{r_1\cdot(\varphi')^2}{4\eta_2\cdot\varphi}. \end{aligned}$$

Integrating this inequality from t_0 to t, we then obtain

$$w(t_0)\geq\int_{t_0}^{t}\left[\varphi(u)\widehat{p}(\sigma(u);0,m)q(u)\frac{\widehat{\eta}_0(\sigma(u))}{\widehat{\eta}_0(u)}-\frac{r_1(u)(\varphi'(u))^2}{4\eta_2(t_0,u)\varphi(u)}\right]\mathrm{d}u,$$

which contradicts to (11).

Hence, the proof ends. □

Corollary 1. *Suppose that* $\lim_{t\to\infty}\widehat{\eta}_0(t)=\infty$ *and* $L>1/4$, *where*

$$L:=\liminf_{t\to\infty}\left[\frac{r_1(t)\widehat{\eta}_0(t)}{\eta_2(t_0,t)}\widehat{p}(\sigma(t);0,m)q(t)\widehat{\eta}_0(\sigma(t))\right]. \tag{14}$$

Then, $S_{\uparrow}=\emptyset$.

Proof. From Theorem 1, we know that $S_{\uparrow}=\emptyset$ when condition (11) is satisfied. By choosing $\varphi(t)=\widehat{\eta}_0(t)$, condition (11) reduces to

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\widehat{p}(\sigma(u);0,m)q(u)\widehat{\eta}_0(\sigma(u))-\frac{\eta_2(t_0,u)}{4r_1(t)\widehat{\eta}_0(u)}\right]\mathrm{d}u=\infty. \tag{15}$$

Now, we will prove that (15) is necessary for the validity of $L>0$. From the definition of L, there is a $t_1\geq t_0$ such that

$$\widehat{p}(\sigma(t);0,m)q(t)\widehat{\eta}_0(\sigma(t))\geq l\frac{\eta_2(t_0,t)}{r_1(t)\widehat{\eta}_0(t)},$$

for $t \geq t_1$ and for arbitrary $l \in \left(\frac{1}{4}, L\right)$. Therefore,

$$\begin{aligned}
&\int_{t_1}^{t} \left[\widehat{p}(\sigma(u);0,m)q(u)\widehat{\eta}_0(\sigma(u)) - \frac{\eta_2(t_0,u)}{4r_1(t)\widehat{\eta}_0(u)}\right]\mathrm{d}u \\
> &\left(l - \frac{1}{4}\right)\int_{t_1}^{t}\left[\frac{\eta_2(t_0,u)}{r_1(t)\widehat{\eta}_0(u)}\right]\mathrm{d}u \\
= &\left(l - \frac{1}{4}\right)\int_{t_1}^{t}\left[\frac{\widehat{\eta}_0'(u)}{\widehat{\eta}_0(u)}\right]\mathrm{d}u \\
= &\left(l - \frac{1}{4}\right)\ln\frac{\widehat{\eta}_0(t)}{\widehat{\eta}_0(t_1)}. \qquad (16)
\end{aligned}$$

Taking $\limsup_{t\to\infty}$ on (16), we have that (15) holds.

Hence, the proof ends. $\square$

In the following results, we improve the monotonic properties of the solutions in category $S_{\uparrow}$, and then obtain better criteria confirming that $S_{\uparrow} = \emptyset$. For that, we define the functional sequences (μ_ℓ), (ν_ℓ), and $(\widehat{\eta}_\ell)$ as

$$\mu_\ell(t) := \eta_2(t_0,t) + \int_{t_0}^{t}\eta_2(t_0,u)q(u)\widehat{p}(\sigma(u);\ell,m)\widehat{\eta}_0(\sigma(u))\mathrm{d}u,$$

$$\nu_\ell(t) := \exp\left[\int_{t_0}^{t}\frac{\mathrm{d}u}{\mu_\ell(u)r_2(u)}\right]$$

and

$$\widehat{\eta}_{\ell+1}(t) := \int_{t_0}^{t}\frac{\nu_\ell(u)}{r_1(u)}\mathrm{d}u,$$

for $\ell = 0, 1, \ldots$.

Lemma 4. *Suppose that $x \in S_{\uparrow}$. Then, eventually,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r_1(t)z'(t)}{\nu_{\ell-1}(t)}\right) \leq 0, \qquad (17)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z(t)}{\widehat{\eta}_\ell(t)}\right) \qquad (18)$$

and

$$\left(r_2(t)\left(r_1(t)z'(t)\right)'\right)' + q(t)\widehat{p}(\sigma(t);\ell,m)z(\sigma(t)) \leq 0, \qquad (19)$$

for $\ell = 1, 2, \ldots$.

Proof. Suppose that $x \in S_{\uparrow}$. From (9), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[r_1(t)z'(t) - \eta_2(t_0,t)r_2(t)\left[r_1(t)z'(t)\right]'\right] &= -\eta_2(t_0,t)\left(r_2(t)\left[r_1(t)z'(t)\right]'\right)' \\
&\geq \eta_2(t_0,t)q(t)\widehat{p}(\sigma(t);0,m)z(\sigma(t)). \qquad (20)
\end{aligned}$$

Integrating (20) from t_0 to t and using (7), we obtain

$$\begin{aligned}
r_1(t)z'(t) \geq\ &\eta_2(t_0,t)r_2(t)\left[r_1(t)z'(t)\right]' \\
&+ \int_{t_0}^{t}\eta_2(t_0,u)q(u)\widehat{p}(\sigma(u);0,m)z(\sigma(u))\mathrm{d}u. \qquad (21)
\end{aligned}$$

Now, integrating (7) from t_0 to t, we find

$$\begin{aligned} z(t) &\geq r_2(t)\left[r_1(t)z'(t)\right]' \int_{t_0}^{t} \frac{\eta_2(t_0,u)}{r_1(u)}\mathrm{d}u \\ &= \widehat{\eta}_0(t) r_2(t)\left[r_1(t)z'(t)\right]', \end{aligned} \tag{22}$$

and so

$$\begin{aligned} z(\sigma(t)) &\geq \widehat{\eta}_0(\sigma(t)) r_2(\sigma(t))\left[r_1(\sigma(t))z'(\sigma(t))\right]' \\ &\geq \widehat{\eta}_0(\sigma(t)) r_2(t)\left[r_1(t)z'(t)\right]'. \end{aligned} \tag{23}$$

Combining (21) and (23), we get

$$\begin{aligned} & r_1(t)z'(t) \\ \geq\; & r_2(t)\left[r_1(t)z'(t)\right]'\left[\eta_2(t_0,t) + \int_{t_0}^{t} \eta_2(t_0,u) q(u) \widehat{p}(\sigma(u);0,m) \widehat{\eta}_0(\sigma(u))\mathrm{d}u\right] \\ =\; & \mu_0(t) r_2(t)\left[r_1(t)z'(t)\right]'. \end{aligned} \tag{24}$$

Multiplying this inequality by

$$\exp\left[-\int_{t_0}^{t} \frac{\mathrm{d}u}{\mu_0(u) r_2(u)}\right],$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r_1(t)z'(t)}{\nu_0(t)}\right) \leq 0.$$

Using this fact, we obtain

$$z(t) \geq \int_{t_0}^{t} \frac{r_1(u)z'(u)}{\nu_0(t)} \frac{\nu_0(t)}{r_1(u)}\mathrm{d}u \geq \widehat{\eta}_1(t)\frac{r_1(u)z'(u)}{\nu_0(t)},$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z}{\widehat{\eta}_1}\right) = \frac{1}{\widehat{\eta}_1^2}\left[\widehat{\eta}_1 z' - \frac{\nu_0}{r_1} z\right] \leq 0. \tag{25}$$

Now, it follows from (25) that the relationship (4) becomes $x(t) > \widehat{p}(\sigma(u);1,m)z(t)$. Moreover, Equation (1) turns into (19) at $\ell = 1$.

Next, Using (19) at $\ell = 1$ instead of (9), and completing the proof with the same previous approach, we get (19) at $\ell = 2$.

Similarly, we can validate the relations (17), (18), and (19) for $\ell = 3, 4, \ldots$.

Hence, the proof ends. □

Using (18) and (19) instead of (6) and (9), respectively, we directly get the following theorem:

Theorem 2. *Suppose that there is a $\varphi \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t}\left[\varphi(u)\widehat{p}(\sigma(u);\ell,m)q(u)\frac{\widehat{\eta}_\ell(\sigma(u))}{\widehat{\eta}_\ell(u)} - \frac{r_1(u)(\varphi'(u))^2}{4\eta_2(t_0,u)\varphi(u)}\right]\mathrm{d}u = \infty, \tag{26}$$

for any $\ell, m \geq 0$. Then, $S_\uparrow = \emptyset$.

Theorem 3. *Suppose that*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(u)\widehat{p}(\sigma(u);\ell,m)\widehat{\mu}_\ell(\sigma(u))\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{27}$$

for any $\ell, m \geq 0$, *where*

$$\widehat{\mu}_\ell(t) = \int_{t_0}^{t} \frac{\mu_\ell(u)}{r_1(u)} \mathrm{d}u.$$

Then, $S_\uparrow = \emptyset$.

Proof. Suppose the contrary that $x \in S_\uparrow$. As in the proof of Lemma 4, we arrive at

$$r_1(t)z'(t) \geq \mu_\ell(t)r_2(t)\left[r_1(t)z'(t)\right]'.$$

Integrating this inequality from t_0 to t, we find

$$z(t) \geq \widehat{\mu}_\ell(t)r_2(t)\left[r_1(t)z'(t)\right]'. \tag{28}$$

Substituting from (28) into (19), we conclude that

$$\begin{aligned} &\left(r_2(t)\left(r_1(t)z'(t)\right)'\right)' \\ \leq\ & -q(t)\widehat{p}(\sigma(t);\ell,m)\widehat{\mu}_\ell(\sigma(t))r_2(\sigma(t))\left[r_1(\sigma(t))z'(\sigma(t))\right]'. \end{aligned}$$

Setting $\mathcal{G} := r_2(r_1z')' > 0$, we have that $\mathcal{G}$ is a positive solution of the inequality

$$\mathcal{G}'(t) + q(t)\widehat{p}(\sigma(t);\ell,m)\widehat{\mu}_\ell(\sigma(t))\mathcal{G}(\sigma(t)) \leq 0. \tag{29}$$

However, from Theorem 2.1.1 in [34], condition (27) confirms the oscillation of all solutions to (29), a contradiction.

Hence, the proof ends. □

For the following result, we assume that $r_1(t) = 1$. In Corollary 1 in [24], by replacing the inequality

$$\left(r_2(t)z''(t)\right)' + (1 - p(\sigma(t)))q(t)z(\sigma(t)) \leq 0$$

by (19), we obtain the following theorem:

Theorem 4. *Suppose that* $r_1(t) = 1$, *and*

$$\liminf_{t\to\infty} \frac{t}{r_2(t)} \int_t^\infty q(\ell)\frac{\sigma^2(\ell)}{\ell}\mathrm{d}\ell > \frac{1}{2\widehat{p}(t;\ell,m)},$$

for any $\ell, m \geq 0$. *Then,* $S_\uparrow = \emptyset$.

Example 1. *Consider the NDE of Euler type*

$$(x(t) + p_0x(\alpha t))''' + \frac{q_0}{t^3}x(\beta t) = 0, \tag{30}$$

where $t > 0$, $p_0 \geq 0$, $q_0 > 0$, *and* $\alpha, \beta \in (0,1)$. *Now, we define the sequences* (A_ℓ) *and* (B_ℓ) *as* $B_0 = 1$,

$$A_\ell := \begin{cases} 1 & \text{for } p_0 = 0, \\ [1 - p_0]\sum_{k=0}^{m} p_0^{2k}\alpha^{2(1+B_\ell)k} & \text{for } p_0 < 1, \\ \left[p_0 - \alpha^{-1-B_\ell}\right]\sum_{k=1}^{n} p_0^{-2k} & \text{for } p_0 > 1/\alpha^2, \end{cases}$$

and

$$B_{\ell+1} := \frac{1}{1 + \frac{1}{2}A_\ell\beta^2 q_0},$$

for $\ell = 0, 1, \ldots$. *It is easy to verify that* $\widehat{p}(\sigma(u);\ell,m) = A_\ell$,

$$\widehat{\eta}_\ell(t) = \frac{1}{1 + B_\ell}t^{1+B_\ell},$$

$$\mu_\ell(t) = \frac{1}{B_{\ell+1}} t$$

and

$$\nu_\ell(t) = t^{B_{\ell+1}},$$

for $\ell = 0, 1, \ldots$ *. Using Theorems 2 and choosing* $\varphi(t) = t^2$*, we have that* $S_\uparrow = \emptyset$ *if*

$$A_\ell \beta^{1+B_{\ell+1}} q_0 > 1. \tag{31}$$

While Theorem 3 confirms that $S_\uparrow = \emptyset$ *if*

$$q_0 A_\ell \beta^2 \frac{1}{2B_{\ell+1}} \ln \frac{1}{\beta} > \frac{1}{\mathrm{e}}. \tag{32}$$

Moreover, Theorem 4 confirms that $S_\uparrow = \emptyset$ *if*

$$q_0 > \frac{1}{2\beta^2 A_\ell}. \tag{33}$$

Remark 1. *In [24–26,28], several conditions are presented that guarantee that* $S_\uparrow = \emptyset$*. By applying these results to Equation (30), we get the following, see Table 1:*

1. *Corollary 1 in [24] guarantees that* $S_\uparrow = \emptyset$ *if*
$$q_0 > \frac{2}{4(1-p_0)\beta^2}; \tag{34}$$
2. *Theorem 2.7 in [25] guarantees that* $S_\uparrow = \emptyset$ *if*
$$\frac{1}{2} q_0 (1-p_0) \beta^2 \ln \frac{1}{\beta} > \frac{1}{\mathrm{e}}; \tag{35}$$
3. *Theorem 1 in [26] guarantees that* $S_\uparrow = \emptyset$ *if*
$$q_0 > \frac{1}{\beta^2}\left(1 + \frac{p_0}{\alpha}\right); \tag{36}$$
4. *Theorem 2.8 in [28] guarantees that* $S_\uparrow = \emptyset$ *if* $\alpha \geq \beta$ *and*
$$q_0 p_* \left(\frac{\beta}{\alpha}\right)^2 > 2, \tag{37}$$
where
$$p_* = \frac{1}{p_0}\left(1 - \frac{1}{p_0 \alpha^2}\right).$$

Table 1. Lower bounds for the values of q_0 in conditions (31)–(37).

Criterion p_0	α	β	**(31)**	**(32)**	**(33)**	**(34)**	**(35)**	**(36)**	**(37)**
0.5	0.5	0.5	6.4399	6.0415	3.9375	4.0000	8.4918	8.0000	fail
3.0	0.7	0.5	27.282	25.594	16.681	fail	fail	21.143	36.780
0.7	0.9	0.1	133.74	63.388	113.09	166.67	106.51	177.78	fail

2.1.2. Category $S_{\downarrow}$

For convenience, we define

$$\widetilde{\eta}_0(h,k) := \int_h^k \frac{\eta_2(u,k)}{r_1(u)} \mathrm{d}u$$

and

$$\widetilde{q}(t) = \min\{q(t), q(\tau(t))\}.$$

In this section, in addition to conditions (A1)–(A4), we also need the following assumption:

(A5) $\tau \circ \sigma = \sigma \circ \tau$ and $\tau'(t) \geq \tau_0 > 0$,where $(\tau \circ \sigma)(t) = \tau(\sigma(t))$.

Lemma 5. *Suppose that $x \in S_{\downarrow}$ and there is a positive function $\rho \in \mathbf{C}([t_0,\infty))$ such that $\sigma(t) < \rho(t) < \tau(t)$. Then*

$$z(h) \geq \widetilde{\eta}_n(h,k) r_2(k) \left[r_1(k) z'(k)\right]', \ \ n = 0,1,..., \tag{38}$$

for $h \leq \tau(k)$, where $\widetilde{q}(t) := \min\{q(\tau(t)), q(t)\}$, and

$$\widetilde{\eta}_{n+1}(h,k) := \int_h^k \frac{1}{r_1(v)} \int_v^k \frac{1}{r_2(s)} \exp\left[\frac{\tau_0}{\tau_0 + p_0} \int_{\tau^{-1}(s)}^k \widetilde{q}(u) \widetilde{\eta}_n(\sigma(u), \rho(u)) \mathrm{d}u\right] \mathrm{d}s \mathrm{d}v.$$

Proof. Assume that $x \in S_{\downarrow}$. Since $r_1 \cdot [r_1 \cdot z']'$ is nonincreasing, we find, for all $h \leq k$,

$$-r_1(h) z'(h) \geq \int_h^k \frac{r_2(u)[r_1(u) z'(u)]'}{r_2(u)} \mathrm{d}u \geq \eta_2(h,k) r_2(k) \left[r_1(k) z'(k)\right]',$$

and so

$$-z'(h) \geq r_2(k) \left[r_1(k) z'(k)\right]' \frac{\eta_2(h,k)}{r_1(h)}, \text{ for } h \leq k.$$

Moreover,

$$z(h) \geq r_2(k) \left[r_1(k) z'(k)\right]' \int_h^k \frac{\eta_2(u,k)}{r_1(u)} \mathrm{d}u = \widetilde{\eta}_0(h,k) \left[r_2(k) \left(r_1(k) z'(k)\right)'\right] \text{ for } h \leq k.$$

Next, by using induction, we will demonstrate that (38) holds at $n+1$ depending on the assumption that

$$z(h) \geq \widetilde{\eta}_n(h,k) \left[r_1(k) z'(k)\right]'. \tag{39}$$

It follows from (A5) and (1) that

$$\begin{aligned} p_0(q \circ \tau) \cdot (x \circ \tau \circ \sigma) &= p_0(q \circ \tau) \cdot (x \circ \sigma \circ \tau) \\ &= -\frac{p_0}{\tau'(t)} \left((r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'\right)' \\ &\leq -\frac{p_0}{\tau_0} \left((r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'\right)'. \end{aligned}$$

Combining this inequality and (1), we arrive at

$$\begin{aligned} \widetilde{q} \cdot (z \circ \sigma) &\leq q \cdot (x \circ \sigma) + p_0(q \circ \tau) \cdot (x \circ \tau \circ \sigma) \\ &\leq -\left(r_2 \cdot (r_1 \cdot (z'))'\right)' - \frac{p_0}{\tau_0} \left((r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'\right)'. \end{aligned} \tag{40}$$

Using (39) with $h = \sigma$ and $k = \rho$, we get $z \circ \sigma \geq \widetilde{\eta}_n(\sigma, \rho) \cdot [(r_1 \circ \rho) \cdot (z' \circ \rho)]'$, which with (40) gives

$$\left[r_2 \cdot (r_1 \cdot (z'))' + \frac{p_0}{\tau_0}(r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'\right]' \leq -\widetilde{q} \cdot \widetilde{\eta}_n(\sigma, \rho) \cdot [(r_1 \circ \rho) \cdot (z' \circ \rho)]'. \tag{41}$$

Now, we define the function

$$\phi := r_2 \cdot (r_1 \cdot (z'))' + \frac{p_0}{\tau_0}(r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'.$$

From (P1), we get

$$r_2 \cdot (r_1 \cdot (z'))' \leq (r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))'.$$

Thus,

$$\left[1 + \frac{p_0}{\tau_0}\right] r_2 \cdot (r_1 \cdot (z'))' \leq \phi \leq \left[1 + \frac{p_0}{\tau_0}\right](r_2 \circ \tau) \cdot ((r_1 \circ \tau) \cdot (z' \circ \tau))', \tag{42}$$

or

$$\frac{\tau_0}{\tau_0 + p_0}\left(\phi \circ \tau^{-1}\right) \leq r_2 \cdot (r_1 \cdot z')',$$

which with (41) gives

$$\phi' + \frac{\tau_0}{\tau_0 + p_0}\widetilde{q} \cdot \widetilde{\eta}_n(\sigma, \rho) \cdot \left(\phi \circ \tau^{-1} \circ \rho\right) \leq 0. \tag{43}$$

Hence, ϕ is nonincreasing, and so $\phi \leq \left(\phi \circ \tau^{-1} \circ \rho\right)$. Now, Equation (43) reduces to

$$\phi' + \frac{\tau_0}{\tau_0 + p_0}\widetilde{q} \cdot \widetilde{\eta}_n(\sigma, \rho) \cdot \phi \leq 0.$$

By separating the variables and integrating from h to k, we get

$$\phi(h) \geq \phi(k) \exp\left[\frac{\tau_0}{\tau_0 + p_0}\int_h^k \widetilde{q}(u)\widetilde{\eta}_n(\sigma(u), \rho(u))\mathrm{d}u\right],$$

which with (42) yields

$$\phi\left(\tau^{-1}(h)\right) \geq \left[1 + \frac{p_0}{\tau_0}\right] r_2(k)(r_1(k)z'(k))' \exp\left[\frac{\tau_0}{\tau_0 + p_0}\int_{\tau^{-1}(h)}^k \widetilde{q}(u)\widetilde{\eta}_n(\sigma(u), \rho(u))\mathrm{d}u\right],$$

and then

$$(r_1(s)z'(s))' \geq r_2(k)(r_1(k)z'(k))' \frac{1}{r_2(s)} \exp\left[\frac{\tau_0}{\tau_0 + p_0}\int_{\tau^{-1}(s)}^k \widetilde{q}(u)\widetilde{\eta}_n(\sigma(u), \rho(u))\mathrm{d}u\right].$$

Integrating this inequality twice from h to k, we get

$$-z'(v) \geq r_2(k)(r_1(k)z'(k))' \frac{1}{r_1(v)}\int_v^k \frac{1}{r_2(s)} \exp\left[\frac{\tau_0}{\tau_0 + p_0}\int_{\tau^{-1}(s)}^k \widetilde{q}(u)\widetilde{\eta}_n(\sigma(u), \rho(u))\mathrm{d}u\right]\mathrm{d}s,$$

and

$$z(h) \geq \widetilde{\eta}_{n+1}(h, k)\left[r_2(k)(r_1(k)z'(k))'\right].$$

Hence, the proof ends. $\square$

Theorem 5. *Suppose that there is a positive function $\rho \in \mathbf{C}([t_0, \infty))$ such that $\sigma(t) < \rho(t) < \tau(t)$ and $\sigma(t) \le \tau(\rho(t))$. If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\rho(t))}^{t} \widetilde{q}(u)\widetilde{\eta}_n(\sigma(u), \rho(u))\mathrm{d}u > \frac{\tau_0 + p_0}{\mathrm{e}\tau_0}, \tag{44}$$

then $S_\downarrow = \emptyset$.

Proof. Assume that $x \in S_\downarrow$. As in the proof of Lemma 5, we obtain that (41) holds. Next, we suppose that

$$w := r_2 \cdot \left(r_1 \cdot (z')\right)' + \frac{p_0}{\tau_0}(r_2 \circ \tau) \cdot \left((r_1 \circ \tau) \cdot (z' \circ \tau)\right)' > 0.$$

It is follows from (P1) and (41) that

$$w' + \frac{\tau_0}{\tau_0 + p_0}\widetilde{q} \cdot \widetilde{\eta}_n(\sigma, \rho) \cdot \left(w \circ \tau^{-1} \circ \rho\right) \le 0. \tag{45}$$

Then, w is a positive solution of (45). However, from Theorem 2.1.1 in [34], condition (44) confirms the oscillation of all solutions to (45), a contradiction.

Hence, the proof ends. □

Example 2. *Consider the NDE (30), where $\beta \in \left(0, \frac{\alpha^2}{2-\alpha}\right)$. By choosing $\rho(t) = \frac{\alpha+\beta}{2}t := \gamma t$, we note that $\sigma(t) < \rho(t) < \tau(t)$ and $\sigma(t) \le \tau(\rho(t))$. Now, we define the sequences (a_i) and (b_i) as $a_0 := (\gamma - \beta)^2/2$,*

$$b_i := \frac{q_0\alpha}{\alpha + p}a_i$$

and

$$a_{i+1} = \frac{\alpha^{b_i}}{b_i - 1}\gamma^{b_i}\left(\frac{1}{b_i - 2}\beta^{2-b_i} - \frac{b_i - 1}{b_i - 2}\gamma^{2-b_i} + \beta\gamma^{1-b_i}\right),$$

for $i = 0, 1, \ldots$. Then, we have $\widetilde{\eta}_0(h, k) = \frac{1}{2}(k - h)^2$and

$$\widetilde{\eta}_i(h, k) = \frac{1}{1 - b_i}\alpha^{b_i}k^{b_i}\left(\frac{1 - b_i}{2 - b_i}k^{2-b_i} - \left(hk^{1-b_i} - \frac{1}{2 - b_i}k^{2-b_i}\right)\right).$$

Therefore, condition (44) reduce to

$$q_0 a_n \ln\frac{\alpha}{\gamma} > \frac{\alpha + p_0}{\alpha \mathrm{e}}. \tag{46}$$

Using Theorem 5, we have that $S_\downarrow = \emptyset$ if (46) for some $n \in \mathbb{N}$.

2.2. Oscillation Criteria

By combining the criteria that ensure that $S_\uparrow = \emptyset$ and $S_\downarrow = \emptyset$, we obtain oscillation criteria for solutions of Equation (1).

Theorem 6. *Suppose that there are $\varphi \in \mathbf{C}^1([t_0, \infty), (0, \infty))$ and $\rho \in \mathbf{C}([t_0, \infty), \mathbb{R})$ such that $\sigma(t) < \rho(t) < \tau(t)$, $\sigma(t) \le \tau(\rho(t))$, and (11) and (44) hold. Then, Equation (1) is oscillatory.*

Theorem 7. *Suppose that $\lim_{t\to\infty} \widehat{\eta}_0(t_0, t) = \infty$, $L > 1/4$, and there is a $\rho \in \mathbf{C}([t_0, \infty), \mathbb{R})$ such that $\sigma(t) < \rho(t) < \tau(t)$, $\sigma(t) \le \tau(\rho(t))$, and (44) holds, where L is defined as in (14). Then, Equation (1) is oscillatory.*

Theorem 8. *Suppose that there are $\varphi \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ and $\rho \in \mathbf{C}([t_0,\infty),\mathbb{R})$ such that $\sigma(t) < \rho(t) < \tau(t)$, $\sigma(t) \leq \tau(\rho(t))$, and (44) hold. If that there is a $\varphi \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ such that (26) holds for some $\ell, m \geq 0$, then Equation (1) is oscillatory.*

Theorem 9. *Suppose that there are $\varphi \in \mathbf{C}^1([t_0,\infty),(0,\infty))$ and $\rho \in \mathbf{C}([t_0,\infty),\mathbb{R})$ such that $\sigma(t) < \rho(t) < \tau(t)$, $\sigma(t) \leq \tau(\rho(t))$, and (44) hold. If (27) holds for some $\ell, m \geq 0$, then Equation (1) is oscillatory.*

Example 3. *Consider the NDE*

$$\left(x(t) + \frac{1}{2}x\left(\frac{t}{2}\right)\right)''' + \frac{q_0}{t^3}x\left(\frac{t}{7}\right) = 0, \tag{47}$$

where $q_0 > 0$. Using Theorem 6, Equation (47) is oscillatory if $q_0 > 104.44$.

3. Conclusions

The idea of obtaining oscillation criteria for differential equations is often based on obtaining conditions that exclude each case of the derivatives of the positive solution. In this work, in the canonical case, the oscillatory behavior of third-order NDEs is investigated. In the oscillation theory of NDEs, the relationships between the solution and its corresponding function are crucial. So, using the modified monotonic properties of positive solutions, we enhance these relationships. The lack of solutions in Categories $S_\uparrow$ and $S_\downarrow$ was then confirmed by the conditions we obtained using these relationships. Afterward, we applied several techniques to infer a set of oscillation criteria utilizing the new relationships and features. Also, we provided examples that highlight the significance of the findings and contrast them with comparable findings in the literature. Extending the findings to half-linear higher-order neutral DDEs will be an interesting suggestion for the future.

Author Contributions: Conceptualization, O.M. and Y.A.; methodology, O.M. and Y.A.; investigation, O.M. and Y.A.; writing—original draft preparation, O.M. and Y.A.; writing—review and editing, O.M. and Y.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

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