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Inverse Problem for a Time Fractional Parabolic Equation with Nonlocal Boundary Conditions

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Abstract: This article considers an inverse problem of time fractional parabolic partial differential equations with the nonlocal boundary condition. Dirichlet-measured output data are used to distinguish the unknown coefficient. A finite difference scheme is constructed and a numerical approximation is made. Examples and numerical experiments, such as man-made noise, are provided to show the stability and efficiency of this numerical method.

Keywords: fractional; differential equation; nonlocal; boundary conditions; inverse problem; numerical method; finite difference method

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1. Introduction

Numerous authors from the scientific, engineering, and mathematics fields have, in recent years, dealt with the dynamical systems described by fractional partial differential equations. This area has markedly grown worldwide.

Fractional-order partial-differential equations are the generalization of known classical-order partial-differential equations. Various methods have been formulated to solve fractional differential equations, such as the Laplace transform method, the Fourier transform method, the iteration method and the operational method. Generally speaking, nonlinear fractional differential equations do not have precise analytical solutions, which is why approximate and numerical techniques have been used. An equation in a specific region with a specific piece of data is known as a "direct problem". In contrast, determining an unknown coefficient, an unknown source function or unknown boundary condition using measured output data is termed an "inverse problem". According to unknown input, an inverse problem can be termed an inverse problem of coefficient identification, an inverse problem of source identification or an inverse problem of boundary value identification. Generally, inverse problems are ill-posed problems, since they are very sensitive to errors in measured input.

Nonlocal boundary conditions have recently received more attention in the mathematical formulation and numerical solution to inverse coefficient problems. There are physical applications in which nonlocal boundary conditions are encountered, such as chemical diffusion and heat-conduction biological processes. Inverse problems for time-fractional parabolic equations with nonlocal boundary conditions in their initial stages require exploration, as not many articles have been written on this. Furthermore, the numerical solution to these problems has still not been studied. An inverse coefficient, time-fractional parabolic partial-differential equation is studied in this paper, in the case of nonlocal boundary conditions. An analytical solution is obtained using eigenfunction expansions. An analysis of the time-dependent inverse coefficient problem is provided, with an additional measurement of the output data of Dirichlet type at the boundary point for the fractional diffusion equation, and the distinguishability of the mapping is investigated. The measured output data, the explicit form of input–output mapping, are additionally constructed. The fact that the distinguishability of input–output mapping implies the injectivity of the mapping is proved. The Fourier method is used to find a unique solution to the problem. Noisy Dirichlet measured output data are used to introduce the input–output mapping, consequently procuring an analytical representation of the mapping. Finally, a numerical approximation of the problem is constructed using the finite difference method. This paper is related to the modeling of diffusion problems, known as diffusion equation as given in (1):

$$D_t^\alpha u(x,t) = u_{xx}(x,t) - p(t)u(x,t) + F(x,t) \quad 0 < \alpha \leq 1,\ (x,t) \in \Omega_T \tag{1}$$

$$u(x,0) = g(x) \tag{2}$$

$$u_x(0,t) = u_x(1,t),\ u(0,t) = \Psi_1(t) \tag{3}$$

where $\Omega_T = (x,t) \in R^2 : 0 < x < 1,\ 0 < t \leq T$ and the fractional derivative $D_t^\alpha u(x,t)$ is defined in Caputo sense $D_t^\alpha u(x,t) = (I^{1-\alpha}u')(t)$, $0 < \alpha \leq 1$, I^α being the Riemann–Liouville fractional integral,

$$(I^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \cdot f(\tau)d\tau & 0 < \alpha \leq 1 \\ f(t) & \alpha = 0 \end{cases} \tag{4}$$

Equations (1)–(3) indicate an inverse problem with respect to the unknown function $p(t)$. $F(x,t)$ is a source function.

The left boundary value function $\Psi_1(t)$ belongs to $C[0,T]$. This function $g(x)$ satisfies the following consistency conditions:

$$\begin{aligned} &(C1) \quad g(0) = \Psi_1(0) \\ &(C2) \quad g'(1) = u_x(0,0) \end{aligned} \tag{5}$$

Under $(C1)$ and $(C2)$, the initial boundary value problem (1)–(3) has the unique solution $u(x,t)$, which is defined in the domain $\bar{\Omega}_T = \{(x,t) \in R^2 : 0 \leq x \leq 1, 0 \leq t \leq T\}$ and belongs to the space

$$C(\bar{\Omega}_T) \cap W_t^1(0,T) \cap C_x^2(0,1) \tag{6}$$

where the solution u is continuous with respect to x and t and t, u_t is in L^1, u_x and u_{xx} is continuous.

2. An Analysis of the Inverse Coefficient Problem with Measured Data $H(T) = U(1,T)$

Consider the inverse problem with measured output data $h(t)$ at $x = 1$. To formulate the solution for the parabolic problem (1)–(3) by using the Fourier method of separation of variables, we first introduce an auxiliary function $v(x,t)$ as follows:

$$v(x,t) = u(x,t) - \Psi_1(t)(1-x), \quad x \in [0,1] \tag{7}$$

by which we transform problem (1)–(3) into a problem with homogeneous boundary conditions. Therefore, the initial boundary value problem (1)–(3) can be rewritten in terms of $v(x,t)$ in the given form:

$$\begin{aligned} D_t^\alpha v(x,t) - v_{xx}(x,t) = {} & D_t^\alpha \Psi_1(t)(1-x) - p(t)v(x,t) \\ & - p(t)\Psi_1(t)(1-x) + F(x,t) \end{aligned} \tag{8}$$

$$v(x,0) = g(x) - \Psi_1(0)(1-x) \tag{9}$$

$$v_x(0,t) - v_x(1,t) = 0 \tag{10}$$

$$v(0,t) = 0 \tag{11}$$

The unique solution to the initial-boundary value problem can be represented in the following form [1] :

$$\begin{aligned} v(x,t) = &\sum_{k=1}^{\infty} < \zeta(\theta), X_k(\theta) > E_{\alpha,1}(\lambda_k, t^\alpha) Y_k(x) + \\ &\sum_{k=1}^{\infty} \left(\int_0^t s^{\alpha-1} \cdot E_{\alpha,\alpha}(-\lambda_k s^\alpha) < \xi(\theta, t-s; p_j(t)), X_k(\theta) > ds \right) Y_k(x) \end{aligned} \tag{12}$$

where

$$\zeta(x) = g(x) - \Psi_1(0)(1-x) \tag{13}$$

$$\xi(x,t) = -D_t^\alpha \Psi_1(t)(1-x) - p(t)v(x,t) - p(t)\Psi_1(t)(1-x) + F(x,t) \tag{14}$$

Moreover, $< \zeta(\theta), \Phi_n(\theta) >= \int_0^1 \Phi_n(\theta)\zeta(\theta)d\theta$; $E_{\alpha,\beta}$ is the generalized Mittag–Leffler function, defined by [2]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \alpha)} \tag{15}$$

Assume that $X_k(x)$ is the solution to the following Sturm–Liouville problem [3].

$$\begin{cases} -\Phi_{xx}(x) = \lambda\Phi(x) & 0 < x < 1 \\ \Phi'(1) = \Phi'(0) & \Phi(0) = 0 \end{cases} \tag{16}$$

$Y_0(x) = x$, $Y_{2k-1}(x) = x \cdot \cos(2\pi kx)$, $Y_{2k}(x) = \sin(2\pi kx)$, $k = 1, 2, \ldots$, $X_0(x) = 2$, $X_{2k-1}(x) = 4\cos(2\pi kx)$, $X_{2k}(x) = 4(1-x)\sin(2\pi kx)$, $k = 1, 2, \ldots$. The system of functions Y_n's are biorthonormal bases, that is, $<< Y_i, X_j >>= 0$ otherwise $<< Y_i, X_j >>= 1$ if $i = j$. These are also Riesz bases in L^2.

The Dirichlet type of the measured output data at the boundary $x = 1$ in terms of $v(x,t)$ can be written in the following form [4–15]:

$$h(t) = u(1,t) = v(1,t) \tag{17}$$

To simplify (12), define the following:

$$z_k(t) = \sum_{k=1}^{\infty} < \zeta(\theta), X_k(\theta) > E_{\alpha,1}(\lambda_k t^\alpha) \tag{18}$$

$$w_k(t) = \sum_{k=1}^{\infty} \left(\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha) < \xi(\theta, t-s; p_j(t)), X_k(\theta) > ds \right) \tag{19}$$

By using $z_k(t)$ and $w_k(t)$, we can write the solution as follows:

$$v(x,t) = \sum_{k=1}^{\infty} z_k(t) Y_k(x) + \sum_{k=1}^{\infty} w_k(t) Y_k(x) \tag{20}$$

The analytical solution to the problem [16–30] in series form is given in (20). Therefore, by substituting $x = 1$,

$$h(t) = v(1,t) = \sum_{k=1}^{\infty} z_k(t) Y_k(1) + \sum_{k=1}^{\infty} w_k(t) Y_k(1) \tag{21}$$

is obtained.

As a result, $h(t)$ is analytically determined as a series representation. The right-hand side of (21) defines the input–output mapping $\Psi[p]$:

$$\Psi[p] := \sum_{k=1}^{\infty} z_k(t)Y_k(1) + \sum_{k=1}^{\infty} w_k(t)Y_k(1) \tag{22}$$

The relationship between the functions $p_1(t), p_2(t) \in K$ at $x = 1$ and the corresponding outputs $h_j(t) = u(1, t; p_j),\ j = 1, 2$ are given in the following lemma.

Lemma 1. *Let $v_1(x,t) = v(x,t;p_1)$ and $v_2(x,t) = v(x,t;p_2)$ be the solutions to the direct problem (8)–(11) corresponding to the admissible coefficients $p_1(t), p_2(t) \in K$. If $h_j(t) = u(1,t;p_j)$ $j = 1,2$ are the corresponding outputs [31,32] $h_j(t)$ $j = 1,2$ satisfy the following series identity.*

$$\Delta h(t) = \sum_{k=1}^{\infty} \Delta w_k(t) Y_k(1) \tag{23}$$

for each $t \in (0, T]$ where $\Delta h(t) = h_1(t) - h_2(t)$, $\Delta w_k(t) = w_k^1(t) - w_k^2(t)$.

Proof. By using identity (21), the measured output data $h_j(t) = v(1,t)$ $j = 1,2$ can be written as:

$$h_1(t) = \sum_{k=1}^{\infty} z_k^1(t)Y_k(1) + \sum_{k=1}^{\infty} w_k^1(t)Y_k(1) \tag{24}$$

$$h_2(t) = \sum_{k=1}^{\infty} z_k^2(t)Y_k(1) + \sum_{k=1}^{\infty} z_k^2(t)Y_k(1) \tag{25}$$

Respectively, since, $z_k^1 = z_k^2(t)$ from the definition of $z_k(t)$. The difference of these formulas implies the desired result. □

The lemma and the definitions enable us to reach the following conclusion.

Corollary 1. *Let the conditions of Lemma 1 hold. If, in addition, $< \xi(x,t;p_1(t)) - \xi(x,t;p_2(t)), X_k(x) >= 0\ \forall\ t \in (0,T]$ hold, then $h_1(t) = h_2(t)$ $\forall t \in (0,T]$.*

Since $Y_k(x)$ $\forall k = 0,1,2,\ldots$ forms a basis for the space and $Y_k(1) \neq 0$ $\forall k = 0,1,2,\ldots$, then $< \xi_1(x,t;p_1(t)) - \xi_2(x,t;p_2(t)), X_k(x) >\neq 0$, at least for some $k \in \mathbb{N}$. Hence, through the lemma, we can conclude that $h_1(t) \neq h_2(t)$, which leads us to the following consequence: $\Psi[p_1] \neq \Psi[p_2]$ implies that $p_1(t) \neq p_2(t)$.

Theorem 1. *Let conditions (C1) and (C2) hold. Assume that $\Psi[\cdot] : K \longrightarrow C[0,T]$ is the input–output mapping defined by ((22)) and corresponding to the measured output $h(t) = u(1,t)$. In this case, the mapping $\Psi[p]$ has the distinguishability property in the class of admissible parameter K, i.e., $\Psi[p_1] \neq \Psi[p_2]$ $\forall\ p_1, p_2 \in K$ implies $p_1(t) \neq p_2(t)$.*

Proof. From the above explanation, the proof is clear. □

3. Numerical Method

This section considers the inverse problem given by (1)–(3) and (17). We use the finite difference method to discretize this problem. The domain $[0,1] \times [0,T]$ is divided into an $M \times N$ mesh with the spatial step size $k = 1/M$ in x direction and the time step size $\tau = T/N$, respectively.

The grid points x_i, t_n are defined by
$x_i = ik; i = 0;1;2;\ldots;M;$
$t_j = j\tau; j = 0;1;2;\ldots;N;$
in which M and N, are integers. The notations u_i^j, F_i^j, p^j, g_i, ψ_1^j and h^j finite difference approximations of $u(x_i,t_j)$, $F(x_i,t_j)$, $p(t_j)$, $g(x_i)$, $\psi_1(t_j)$ and $h(t_j)$, respectively.

The finite-difference approximation for discretizing problem (1)–(3) and (17) is:

$$\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{j+1}\frac{\Gamma(j-m-\alpha+1)}{(j-m)!}\left(\frac{u_i^m-u_i^{m-1}}{\tau^\alpha}\right)=\frac{1}{h^2}\left(u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}\right) - p^j u_i^{j+1}+F_i^j \tag{26}$$

$$u_i^0=g_i \tag{27}$$

$$u_0^j=\psi_1^j \tag{28}$$

$$u_{M+1}^j=\left(u_M^j+u_1^j-\psi_1^j\right), \tag{29}$$

where $1 \le i \le M$ and $0 \le j \le N$.

Now, let us construct the predicting-correcting mechanism. Firstly, if we use the measured output data-$u(1,t)=h(t)$, we obtain

$$p(t)=\frac{D_t^\alpha h(t)-u_{xx}(1,t)-F(1,t)}{h(t)}. \tag{30}$$

The finite difference approximation of $p(t)$ is

$$p^j=\frac{\left[H^j-\frac{1}{k^2}\left(u_{M+1}^{j+1}-2u_M^{j+1}+u_{M-1}^{j+1}\right)-F_M^j\right]}{h^j}, \tag{31}$$

where $H^j=D_t^\alpha h(t_j), j=0,1,\dots,N$.

In numerical computation, since the time step is very small, we can take $p^{j(0)}=p^{j-1}$, $u_i^{j(0)}=u_i^{j-1}, j=0,1,2,\dots,N, i=1,2,\dots,M$. At each s-th iteration step, we first determine $p^{j(s)}$ from the formula.

$$p^{j(s)}=\frac{\left[H^j-\frac{1}{k^2}\left(u_{M+1}^{j+1(s)}-2u_M^{j+1(s)}+u_{M-1}^{j+1(s)}\right)-F_M^j\right]}{h^j}. \tag{32}$$

Then, from (26)–(29), we obtain:

$$\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{j+1}\frac{\Gamma(j-m-\alpha+1)}{(j-m)!}\left(\frac{u_i^{m(s+1)}-u_i^{m-1(s)}}{\tau^\alpha}\right)= \frac{1}{h^2}\left(u_{i+1}^{j+1(s+1)}-2u_i^{j+1(s+1)}+u_{i-1}^{j+1(s+1)}\right) -p^{j(s)}u_i^{j+1(s+1)}+F_i^j \tag{33}$$

$$u_0^{j(s)}=\psi_1^j \tag{34}$$

$$u_{M+1}^{j(s)}=\left(u_M^{j(s)}+u_1^{j(s)}-\psi_1^j\right), \tag{35}$$

The system of Equations (27) and (33)–(35) can be solved by the Gauss elimination method and $u_i^{j+1(s+1)}$ is determined. If the difference in values between the two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $p^{j(s)}$, $u_i^{j+1(s+1)}(i=1,2,\dots,N_x)$ as p^j, $u_i^{j+1}(i=1,2,\dots,N_x)$, on the ($j$)-th time step, respectively. By virtue of this iteration, we can move from level j to level $j+1$.

Example 1. *Consider the following problem for* $\alpha=1/2$*:*

$$F(x,t) = \left(\frac{16t^2}{5\sqrt{\pi}}\sqrt{t} + t^5 - t^3(2\pi)^2\left(-\sin(2\pi x) + \cos^2(2\pi x)\right)\right)\exp(\sin(2\pi x)),$$
$$\varphi(x) = 0, \quad \Psi_1(t) = t^3, \textit{ and the measured output data is } h(t) = t^3,$$

It is easy to check that the exact solution is:

$$\{p(t), u(x,t)\} = \left\{t^2, t^3\exp(\sin(2\pi x))\right\}.$$

Let us apply the scheme above for the step sizes $k = 0.05$, $\tau = 0.05$. Figures 1 and 2 show the exact and the numerical solutions of $\{p(t), u(x,t)\}$ when $T = 1$.

We can see from these figures that the agreement between the numerical and exact solutions for $p(t)$ and $u(x,T)$ is excellent.

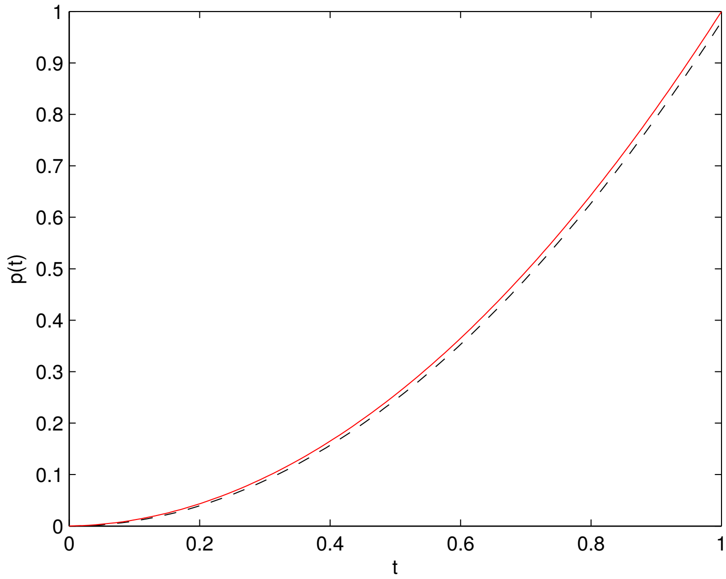

Figure 1. The exact and numerical solutions of $p(t)$. The exact solution is shown with dashes line.

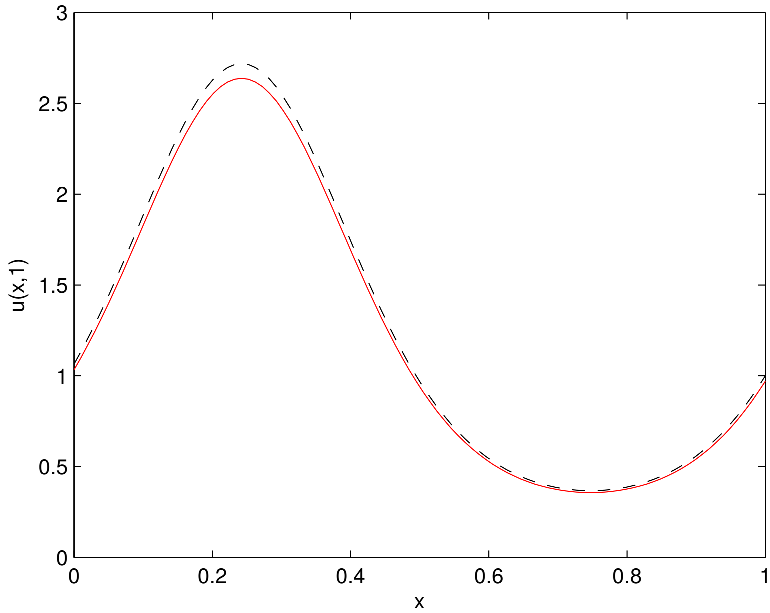

Figure 2. The exact and numerical solutions of $u(x,1)$. The exact solution is shown with dashes line.

4. Conclusions

The distinguishability property of the input–output mapping $\Psi[\cdot] : K \longrightarrow C[0,T]$ was investigated using measured output data $x = 1$. The measured output data $h(t)$ were obtained analytically as a series representation. This also leads to the input–output mapping $\Psi[\cdot]$ in an explicit form. In future studies, the authors plan to consider various fractional inverse coefficients or inverse source problems.

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References

1. Wei, T.; Zhang, Z.Q. Reconstruction of a time-dependent source term in a time-fractional diffusion equation. *Eng. Anal. Bound. Elem.* **2013**, *37*, 23–31. [CrossRef]
2. Caepinteri, A.; Mainardi, F. *Fractals and Fractional Calculus in Continuum Mechanics*; Springer: Berlin/Heidelberg, Germany, 1997.
3. Li, Z.; Yamamoto, M. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl. Anal.* **2015**, *94*, 570–579. [CrossRef]
4. Djrbashian, M.M. Differential operators of fractional order and boundary value problems in the complex domain. In *The Gohberg Anniversary Collection*; Springer: Berlin/Heidelberg, Germany, 1989; pp. 153–172.
5. Mittag-Leffler, G.M. Sur la nouvelle fonction eα (x). *CR Acad. Sci. Paris* **1903**, *137*, 554–558.
6. Humbert, P. Quelques résultats relatifs à la fonction de mittag-leffler. *C. R. Hebd. Des Seances Acad. Des Sci.* **1953**, *236*, 1467–1468.
7. Cannon, J.R.; Lin, Y. Determination of source parameter in parabolic equations. *Mechanica* **1992**, *27*, 85–94. [CrossRef]
8. Dehghan, M. Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement. *Numer. Methods Partial. Differ. Equ.* **2004**, *21*, 621–622. [CrossRef]
9. Luchko, Y. Initial boundary value problems for the one dimensional time-fractional diffusion equation. *Frac. Calc. Appl. Anal.* **2012**, *15*, 141–160. [CrossRef]
10. Ozbilge, E. Identification of the unknown diffusion coefficient in a quasi-linear parabolic equation by semigroup approach with mixed boundary conditions. *Math. Meth. Appl.Sci.* **2008**, *31*, 1333–1344. [CrossRef]
11. Renardy, M.; Rogers, R.C. *An Introduction to Partial Differential Equations*; Springer: New York, NY, USA, 2004.
12. Showalter, R.E. *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*; American Mathematical Society: Waltham, MA, USA, 1997.
13. Wei, H.; Chen, W.; Sun, H.; Li, X. A coupled method for inverse source problem of spatial fractional anomalous diffusion equations. *Inverse Probl. Sci. Eng.* **2010**, *18*, 945–956. [CrossRef]
14. Chen, W.; Fu, Z.J. Boundary particle method for inverse cauchy problems of inhomogeneous helmholtz equations. *J. Mar. Sci. Technol.* **2009**, *17*, 157–163. [CrossRef]
15. Zhang, Y.; Xiang, X. Inverse source problem for a fractional diffusion equation. *Inverse Probl.* **2011**, *27*, 035010. [CrossRef]
16. Francesco, M.; Luchko, Y.; Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *arXiv* **2007**, arXiv:cond-mat/0702419.
17. Gorenflo, R.; Mainardi, F.; Moretti, D.; Paradisi, P. Time fractional diffusion: A discrete random walk approach. *Nonlinear Dyn.* **2002**, *29*, 129–143. [CrossRef]
18. Plociniczak, L. Analytical studies of a time-fractional porous medium equation. Derivation, approximation and applications. *Commun. Nonlinear Sci. Num. Simul.* **2015**, *24*, 169–183. [CrossRef]
19. Baglan, I. Determination of a coefficient in a quasilinear parabolic equation with periodic boundary condition. *Inverse Probl. Sci. Eng.* **2015**, *23*, 884–900. [CrossRef]
20. Ozbilge, E.; Demir, A. Identification of unknown coefficient in time fractional parabolic equation with mixed boundary conditions via semigroup approach. *Dynam. Syst. Appl.* **2015**, *24*, 341–348.
21. Ozbilge, E.; Demir, A. Inverse problem for a time-fractional parabolic equation. *J. Inequal. Appl.* **2015**, *2015*, 1–9. [CrossRef]
22. Ozbilge, E.; Demir, A. Semigroup approach for identification of the unknown diffusion coefficient in a linear parabolic equation with mixed output data. *Bound. Value Probl.* **2013**, *2013*, 1–12. [CrossRef]
23. Özkum, G.; Demir, A.; Erman, S.; Korkmaz, E.; Özgür, B. On the inverse problem of the fractional heat-like partial differential equations: Determination of the source function. *Adv. Math. Phys.* **2013**, *2013*, 476154. [CrossRef]

24. Ozbilge, E.; Demir, A. Identification of the unknown coefficient in a quasi-linear parabolic equation by a semigroup approach. *J. Inequal. Appl.* **2013**, *2013*, 1–7. [CrossRef]
25. Ozbilge, E.; Demir, A. Distinguishability of a source function in a time fractional inhomogeneous parabolic equation with robin boundary condition. *Hacet. J. Math. Stat.* **2018**, *47*, 1503–1511. [CrossRef]
26. Bayrak, M.A.; Demir, A.; Ozbilge, E. An improved version of residual power series method for space-time fractional problems. *Adv. Math. Phys.* **2022**, *2022*, 6174688. [CrossRef]
27. Bayrak, M.A.; Demir, A.; Ozbilge, E. A novel approach for the solution of fractional diffusion problems with conformable derivative. *Numer. Methods Partial. Differ. Equ.* **2021**, 1–18. [CrossRef]
28. Demir, A.; Bayrak, M.A.; Ozbilge, E. A new approach for the approximate analytical solution of space-time fractional differential equations by the homotopy analysis method. *Adv. Math. Phys.* **2019**, *2019*, 5602565. [CrossRef]
29. Cetinkaya, S.; Demir, A. Diffusion equation including a local fractional derivative and weighted inner product. *J. Appl. Math. Comput. Mech.* **2022**, *21*, 19–27. [CrossRef]
30. Bulut, A.; Hacioglu, I. The energy of all connected cubic circulant graphs. *Linear Multilinear Algebra* **2020**, *68*, 679–685. [CrossRef]
31. Bulut, A.; Hacioglu, I. Asymptotic energy of connected cubic circulant graphs. *Akce Int. J. Graphs Comb.* **2021**, *18*, 25–28. [CrossRef]
32. Hacioglu, I.; Bulut, A.; Kaskaloglu, K. The minimal and maximal energies of all cubic circulant graphs. *Akce Int. J. Graphs Comb.* **2021**, *18*, 148–153. [CrossRef]