




# Consistency Indices in Analytic Hierarchy Process: A Review

Sangeeta Pant <sup>1,†</sup> , Anuj Kumar <sup>1,†</sup> , Mangey Ram <sup>2,3,\*</sup>, Yury Klochkov <sup>4</sup>  and Hitesh Kumar Sharma <sup>5</sup>

<sup>1</sup> Department of Mathematics, University of Petroleum & Energy Studies, Dehradun 248007, India; pant.sangeet@gmail.com (S.P.); anuj4march@gmail.com (A.K.)

<sup>2</sup> Department of Mathematics, Graphic Era Deemed to be University, Dehradun 248002, India

<sup>3</sup> Institute of Advanced Manufacturing Technologies, Peter the Great St. Petersburg Polytechnic University, 195251 Saint Petersburg, Russia

<sup>4</sup> Academic Development Management, Peter the Great St. Petersburg Polytechnic University, 195251 Saint Petersburg, Russia; y.kloch@gmail.com

<sup>5</sup> School of Computer Science, University of Petroleum & Energy Studies, Dehradun 248007, India; hksharma@ddn.upes.ac.in

\* Correspondence: mangeyram@geu.ac.in or drmsswami@yahoo.com;

Tel.: +91-135-2642727 or +91-135-2642729 (ext. 256)

† These authors contributed equally to this work.

**Abstract:** A well-regarded as well as powerful method named the ‘analytic hierarchy process’ (AHP) uses mathematics and psychology for making and analysing complex decisions. This article aims to present a brief review of the consistency measure of the judgments in AHP. Judgments should not be random or illogical. Several researchers have developed different consistency measures to identify the rationality of judgments. This article summarises the consistency measures which have been proposed so far in the literature. Moreover, this paper describes briefly the functional relationships established in the literature among the well-known consistency indices. At last, some thoughtful research directions that can be helpful in further research to develop and improve the performance of AHP are provided as well.

**Keywords:** analytic hierarchy process (AHP); multi-criteria decision-making (MCDM); consistency measure; nature-inspired optimization technique; reliability optimization

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## 1. Introduction

Optimization [1] can be viewed as a decision-making process with some constraints wherein the task is to obtain the maximum benefit from the available resources to get the best achievable results. In literature, multicriteria decision-making (MCDM) has also been used to exploit the search space after exploring the search space with nature-inspired optimization techniques [2]. The analytic hierarchy process (AHP), one of the well-regarded MCDM tools, is attributed to Thomas Saaty [3–8]. It has been widely used in many different fields for the last forty years. In AHP the factors, which can influence the decisions, are identified and then these factors are arranged into a hierarchical structure of different levels to reduce the complexity of the decision problem. Then each factor in the corresponding level is compared pairwise. These  $\frac{n(n-1)}{2}$  comparisons are arranged above the principal diagonal of a square matrix whose diagonal entries are one. The entries below to principal diagonal are the reciprocal of the entries of the upper half of the matrix. Thus, these comparisons contribute to constructing a positive reciprocal decision matrix which is called a ‘pairwise comparison matrix’ or ‘judgement matrix’. In real life, it is always not possible for the decision-maker to make perfect judgements. Therefore, there are cases when some inconsistency may appear. Assume that there are three criteria  $x_1$ ,  $x_2$ , and  $x_3$ . The decision-maker finds that  $x_1$  is *slightly* more important than  $x_2$ , while  $x_2$  is *slightly* more important

than  $x_3$ . If the decision-maker concludes, that  $x_3$  is equally or more important than  $x_1$ , then certainly some inconsistency arises. But, if the decision-maker concludes that  $x_1$  is also slightly more important than  $x_3$ , then this decision is better than the earlier one and thus a slight inconsistency arises in this case. Hence, the second judgement is more consistent. Due to pairwise comparisons [9–14], the decision-maker always has an opportunity to estimate the irrationality of his judgements. According to Saaty [4], a pairwise comparisons matrix should be “close” to a consistent matrix. He developed an index that is known as *CI* to check the degree of inconsistency of judgements. The manuscript aims to offer a short review of consistency indices in AHP. This research article contributes to the world of decision theory as follows:

- (1) This article attempts to provide a review of consistency indices along with their limitations.
- (2) The axiomatization of consistency indices by different authors have also been summarised.
- (3) Five improvement strategies are identified under the section about potential research directions for further enhancement in the performance of consistency indices.

Our analyses are based on the papers published between 1977 and 2021 retrieved from the UPES Library, SCI-Hub, and ISI Web of Science database. We have carried out this research in three phases. In the first phase, we selected the literature which described the mathematical background of the consistency of pairwise comparison matrices. The consistency indices and their mathematical properties were studied in the next phase. In the last phase, the functional relationship and axiomatization of consistency indices were studied.

The rest of the paper is organized as follows: Section 2 presents the mathematical background behind AHP. Section 3 reviews consistency indices proposed in the literature. Section 4 demonstrates some limitations of consistency methods and the importance of the functional relationships among consistency indices. Section 5 presents some future directions of research. Finally, Section 6 concludes the overall remarks of this article.

## 2. Mathematical Background of AHP

We cannot ignore the mathematical concepts that are required for a deep understanding of the AHP. In this section, mathematical terms and definitions have been described.

**Definition 1.** *Positive Reciprocal Matrix.*

A square matrix  $A = [a_{ij}]$  of order  $n$  having only positive elements and satisfying the property  $a_{ij} = \frac{1}{a_{ji}} \forall i, j$  is called a positive reciprocal matrix.

Let  $P$  be a matrix of order  $n$  with each element equal to 1. We can generate nontrivial positive reciprocal matrices of the same order with the help of the matrix. Here, by using a nontrivial reciprocal matrix, meaning a positive reciprocal matrix whose entries are not all necessarily 1. Let  $D = \text{diag}(d_1, d_2, \dots, d_n)$  be a diagonal matrix (which is not an identity or a null matrix for the nontrivial case) of order  $n$  with the positive diagonal entries. Then the matrix  $A = DPD^{-1}$  is a positive reciprocal matrix. Another way to generate a reciprocal matrix  $A = [a_{ij}]$  of order  $n$  is by taking  $a_{ij} = w_i/w_j$ , where  $w_i, w_j$  are the elements of a finite set  $W = \{w_1, w_2, \dots, w_n : w_i \in \mathbb{R}, i = 1, 2, \dots, n\}$ . The structure of a pairwise comparison matrix of order  $n$  is as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where  $a_{ij} > 0$  and  $a_{ji} = \frac{1}{a_{ij}} \forall i, j$ .

According to Saaty [3], if  $w = \{w_1, w_2, \dots, w_n : w_i \in \mathbb{R}, i = 1, 2, \dots, n\}$  is the weight vector (priority vector), then the elements of the above matrix can be approximated as

$a_{ij} \approx \frac{w_i}{w_j}$ . Thus, the matrix  $A = [a_{ij}]$  can be expressed in terms of the ratios of weights  $A = \left[ \frac{w_i}{w_j} \right]$  as follows:

$$A = \begin{bmatrix} 1 & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & 1 \end{bmatrix}$$

**Definition 2.** *Spectrum and Spectral Radius of a Square Matrix.*

Spectrum  $\sigma(A)$  of a square matrix  $A$  is a collection of all of its eigenvalues in which the eigenvalues are repeated according to their algebraic multiplicity.

The multiplicity of an eigenvalue in spectrum is equal to the dimension of generalized eigenspace. The spectral radius  $\rho(A)$  of  $A$  is the maximum value of the modulus of its eigenvalues i.e.,

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$$

**Definition 3.** *Primitive Matrix.*

If all the elements  $a_{ij}$  of a square matrix  $A$  are nonnegative (i.e.,  $a_{ij} \geq 0$ ) then such a matrix is known as the non-negative matrix. A primitive matrix is a special type of nonnegative matrix. A nonnegative matrix  $A$  is called primitive if there exist a natural number  $k$  such that  $a_{ij}^{(k)} > 0$ ,  $\forall (i, j)$ , where  $a_{ij}^{(k)}$  is the element of  $A^k$  at  $i$ th row and  $j$ th column. Thus, every positive reciprocal matrix is a primitive matrix.

The Perron–Frobenius theorem [11] is a well-known theorem for identifying the primitive matrix. According to this theorem, if  $A$  is a primitive matrix with spectral radius  $\rho(A)$ , then there exists a unique largest eigenvalue  $\lambda_{max}$  such that:

- (1)  $\rho(A) = |\lambda_{max}|$ , i.e.,
- (2) The algebraic multiplicity of  $\lambda_{max}$  must be one, and hence, the geometric multiplicity of  $\lambda_{max}$  is one.
- (3) The eigenvectors corresponding to  $\lambda_{max}$  are strictly positive.

For example,

$\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$  is a primitive matrix with eigenvalues 3 and  $-2$ .

$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$  is not a primitive matrix with eigenvalues 2 and  $-2$ .

$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$  is not a primitive matrix with repeated eigenvalues 2.

**Definition 4.** *Consistency of Reciprocal Matrix.*

Let  $A$  be a positive reciprocal matrix of order  $n$ . If  $\lambda_{max}$  is the eigenvalue of  $A$  such that  $\rho(A) = |\lambda_{max}|$ , then  $\lambda_{max}$  is called the principal eigenvalue or Perron value. The value of  $\lambda_{max}$  can never be less than  $n$ , i.e.,  $\lambda_{max} \geq n$ . If  $\lambda_{max}$  is equal to  $n$ , then the matrix  $A$  satisfies the consistency property, which is also known as transitive relation  $a_{ij}a_{jk} = a_{ik}$ , where  $i, j, k = 1, 2, 3 \dots n$ . If  $A$  is a consistent reciprocal matrix, then it will satisfy following properties:

- (1) A positive reciprocal matrix  $A$  of order  $n$  has  $\lambda_{max} = n$ , if and only if  $A$  is consistent.
- (2) A positive reciprocal matrix  $A$  of order  $n$  is consistent if and only if its characteristic polynomial  $P_A(\lambda)$  is of the form  $P_A(\lambda) = \lambda^n - n\lambda^{n-1}$ .

- (3) The column vectors of  $A$  are proportional and hence the rank of a consistent positive reciprocal matrix is always one. Thus, if a matrix is less consistent then its columns will be less proportional.

The main objective of any multi-criteria decision-making method is to decide the weight for each criterion. In AHP, as its name suggests, the process of decision-making starts with breaking down the multi-criteria decision-making problem into a hierarchy modal, and then by using mathematical calculation, basically based on linear algebra, one can find the weights. These weights can be generated with the help of the pairwise comparisons of two alternatives under the given criterion. The decision maker judges the weak, strong, very weak or very strong preference under the particular criterion. In the discrete case these pairwise comparisons lead to a matrix and in the continuous case to kernels of Fredholm operators [8,12].

Total  $n(n-1)/2$  pairwise comparisons contribute to form a pairwise comparison matrix  $A = [a_{ij}]$  (PCM) of order  $n$ . The diagonal entries of PCM equal to 1 and the remaining entries are simply the reciprocals of these  $n(n-1)/2$  comparison. If  $a_{ij}$  denotes the preference of  $i$ th alternative over the  $j$ th alternative, where  $i, j = 1, 2, \dots, n$  then

$$A = [a_{ij}], \text{ where } a_{ij} = \begin{cases} 1 & i = j \\ \frac{1}{a_{ji}} & i < j \end{cases}$$

This matrix  $A$  is always positive reciprocal in nature which may or may not be consistent. Fechner [13] was the one who introduced the pairwise comparison method in 1860. Further, Thurstone [14] developed this method in 1927. Saaty used this pairwise comparison method to develop analytic hierarchy process (AHP) as a method for multi-criteria decision-making. Pairwise comparison between the two criteria is measured by using a numerical scale from 1 to 9, which was proposed by Saaty [3]. This scale establishes one-to-one correspondence between the set of alternatives and a discrete set  $\{9, 8, 7, 6, 5, 4, 3, 2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$ . Other scales have also been proposed by others [15,16]. As discussed earlier in the mathematical working of AHP this matrix  $A$  is consistent if and only if  $a_{ij}a_{jk} = a_{ik}$ . In other words, if  $A$  is consistent then its characteristic polynomial is of the form  $\lambda^n - n\lambda^{n-1} = 0$ . The priority weights derived from a PCM have been used to judge the importance of criteria in AHP. The AHP uses a principal eigenvalue method (EM) to derive priority vectors [4,5]. Several other prioritization methods have also been introduced such as the eigenvector method (EVM), the arithmetic mean method (AMM), the Row geometric mean method (RGMM), the logarithmic least squares method, and singular value decomposition [17–23].

### 3. Consistency Indices in the Analytic Hierarchy Process

In real-world problems, it is not possible to obtain a perfectly consistent judgmental matrix after pairwise comparison, so the goal is to acquire a positive reciprocal matrix which is near to some consistent positive reciprocal matrix. The consistency index is a number, which tells us how far we are from the consistent matrix. Mathematically, one can define the consistency index as a function from the set of the judgmental matrices to the set of the real numbers. The first consistency index was proposed by Kendall and Smith in 1940 [24]. Since then, several consistency indices have been suggested in the literature.

Aupetit and Genest [25] have shown that there is a direct effect on  $CI$  if we change an element of the matrix. If any upper triangular entry of the matrix increases, then  $CI$  must be always increasing, always decreasing or decreasing to a minimum and then increasing. Thus, there should be a unique local minimum in  $CI$  functions. If the consistency measure exceeds the threshold value, then the earlier judgements must be changed. The idea of a consistency measure is meaningless without the thresholds associated to it. However, many consistency indices have been proposed in literature without telling the thresholds associated with them.

To measure inconsistency, Saaty [3] introduced the consistency index:

$$CI = \frac{(\lambda_{max} - n)}{(n - 1)}$$

This inconsistency measure is the negative of the average of the other eigenvalues of the positive reciprocal matrix  $A$ :

$$\begin{aligned} \because \text{Trace of } A &= \lambda_{max} + \sum_{i=1}^{n-1} \lambda_i = n \\ \Rightarrow \frac{(\lambda_{max} - n)}{(n - 1)} &= -\frac{\sum_{i=1}^{n-1} \lambda_i}{n - 1} \end{aligned}$$

If  $A$  is consistent, then the average of other eigenvalues must be 0, and hence,  $CI = 0$ . Saaty calculated the  $CI$  of a large number of matrices of the same order. The random consistency index ( $RI$ ) is the average of these  $CI$  of the matrices of same order. Saaty introduced a consistency ratio which is the rescaled version of  $CI$  and defined as

$$CR = \frac{CI}{RI}$$

Saaty decided the threshold of 0.10. If  $CR$  is greater than this threshold, then it questions the credibility of judgements. These judgements are revised by the decision-maker until he/she achieves a  $CR$  smaller than 0.10 [5]. Saaty [4] further suggested that for the matrices of order three and four the thresholds can be taken as 0.5 and 0.8, respectively.

Crawford [23] introduced another consistency index which is known as the 'geometric index'  $GCI$ . This index was further reformulated by Aguaron and Moreno-Jimenez [18]. The  $i$ th element  $w_i$  of priority vector  $w$  (normalized priority vector) is evaluated by using geometric mean of the elements of the  $i$ th row of the pairwise comparison matrix  $A = [a_{ij}]$ , i.e.,

$$w_i = \left( \prod_{j=1}^n a_{ij} \right)^{1/n} \div \sum_{i=1}^n \left( \prod_{j=1}^n a_{ij} \right)^{1/n}$$

The error term  $e_{ij}$  associated with each entry  $a_{ij}$  of the matrix  $A$  is given by

$$e_{ij} = a_{ij} \frac{w_j}{w_i}$$

If the matrix is consistent then it is obvious that  $a_{ij} = \frac{w_i}{w_j}$ , and hence, for a consistent matrix,  $e_{ij} = 1$ .

The consistency index  $GCI$  is found by evaluating the distance from a specific consistent matrix by using the following formula:

$$GCI = \frac{2}{(n-2)(n-1)} \sum_{i < j} (\ln e_{ij})^2$$

They added the squared deviations of the log of the elements of a matrix from the log of the matrix elements generated by the row geometric mean solution. They proved that for an arbitrary judgment matrix  $A$ , the geometric mean vector gives rise to the m-closest consistent matrix to  $A$ . The normalized geometric mean scale is similar to the normalized eigenvector scale for a consistent matrix  $A$ . If the dimension is not more than three, then two scales are always the same even for the inconsistent matrices.

Several similarity measures have been developed in literature [26] like the Dice similarity measure, overlap similarity measure, Jaccard similarity measure, and cosine similarity measure, etc. The cosine similarity measure is the building block behind the development

of the cosine consistency index. The cosine similarity identifies the similarity between two vectors. Let  $u$  and  $v$  be two vectors in an inner product space  $V$ ; then, the cosine similarity measure is the modulus of the cosine of the angle between  $u$  and  $v$ , i.e.,

$$\text{cosine similarity measure between } u \text{ and } v = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

If the two vectors have the same orientation, then their cosine similarity measure is equal to one. If two vectors are orthogonal, then they have a 0 similarity measure. Thus, the cosine similarity measure is a function from  $V \times V$  to the closed interval  $[0, 1]$ . A consistent positive reciprocal matrix has rank one and columns of  $A$  are linearly dependent of each other (collinear). Thus, if we want to find the near consistent matrix corresponding to an inconsistent matrix, we can use cosine similarity measure. Cosine similarity measure has also been widely used to derive the priority vector in AHP [27–29]. The sum of the cosine of the angle between the priority vector and each column vector of the judgment matrix is maximized by Kuo and Lin [27]. They modelled the optimization problem

$$\text{Max } C = \sum_{j=1}^n c_j = \frac{\sum_{j=1}^n \sum_{i=1}^n w_i a_{ij}}{\sqrt{\sum_{k=1}^n w_k^2 \sum_{k=1}^n a_{kj}^2}}$$

subject to

$$\sum_{i=1}^n w_i = 1$$

$$w_i \geq 0, i = 1, 2, \dots, n.$$

They further proposed a cosine consistency index  $CCI = C^*/n$ , where  $C^*$  is the optimal value of the above optimization model. For the perfectly consistent matrix  $C^* = n$ , otherwise,  $0 < C^* < n$ . In other words, for the perfectly consistent matrix  $CCI$  must be 1 otherwise  $CCI \in (0, 1)$ .  $CCI$  must be greater than or equal to 90% for accepting the approximation. Cosine maximization method was used by Khatwani and Kar [29] to revise the entries of judgement matrix.

Salo-Hamalainen index (CMSH) was introduced in 1997 [30]. This index is different from others as it doesn't require any prioritization method unlike  $CI$  and  $GCI$ . Unfortunately, this index could not catch that much attention because the thresholds associate to this measure was not described. Later, in 2019 Amanta et al. [31] introduced the threshold associated to this consistency index.

If the preferences are represented by additive approach, then the geometric consistency index ( $GCI$ ) of Crawford [23] corresponds to the Euclidean norm. Recently, Fedrizzi, Civolani and Critch [32] proposed a new measure to evaluate inconsistency which can be considered as the generalization of geometric consistency index provided by Crawford [23]. They introduced an inconsistency of the pairwise comparison matrix  $A$  with index  $I_d(A)$ , which is a normed based distance of a matrix  $A$  from the nearest consistent matrix in linear subspace  $L^*$  of consistent matrices:

$$I_d(A) = d(A, L^*) = \min_{B \in L^*} d(A, B)$$

A very interesting result was found by Shiraishi, Obata, and Daigo [33–35]. They found that the inconsistency of a matrix  $A$  of order  $n \geq 3$  is related to the coefficient  $c_3$  of  $\lambda^{n-3}$  of the characteristic polynomial of  $A$ . From the Perron–Frobenius theorem the characteristic polynomial of any consistent positive reciprocal has the form:

$$P_A(\lambda) = \lambda^n - n\lambda^{n-1}$$

Inclusion of any other term in this formulation will certainly make the matrix inconsistent. Shiraishi, Obata, and Daigo [35] further proved that for a positive reciprocal matrix of



order  $n \geq 3$ ,  $c_3$  must be either negative or zero, and the matrix is consistent if  $c_3 = 0$ , and for better consistency the value of  $c_3$  must tend towards 0. In other words, maximization of the  $c_3$  is expected to obtain a consistent matrix. Thus, measure of inconsistency index is the value of  $c_3$

$$c_3 = \sum_{i < j < k} 2 - \left( \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} \right) = 2 \binom{n}{3} - \sum_{i < j < k} \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}}$$

as the algebraic mean is always greater than or equal to geometric mean, i.e.,

$$\frac{1}{2} \left( \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} \right) \geq \sqrt{\left( \frac{a_{ik}}{a_{ij}a_{jk}} \frac{a_{ij}a_{jk}}{a_{ik}} \right)} \Rightarrow \left( \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} \right) \geq 2$$

Thus,  $c_3$  is always negative or zero. They suggested to skip an entry while making the pairwise comparison matrix. Let this entry be  $x$ . Now the objective is to find an appropriate value of  $x$  such that  $c_3(x)$  becomes maximum.

Lamata and Peláez [21,36] proposed a consistency index  $CI^*$  which is based upon the determinant of pairwise comparison matrix of order 3. This index is developed by using the fact that for three alternatives  $x_i, x_j$  and  $x_k$  if the judgement matrix  $M$  is

$$M = \begin{bmatrix} 1 & a_{ij} & a_{ik} \\ \frac{1}{a_{ij}} & 1 & a_{jk} \\ \frac{1}{a_{ik}} & \frac{1}{a_{jk}} & 1 \end{bmatrix}$$

then the judgements are perfect if and only if

(i) the entries of  $M$  are transitive and (ii)  $M$  is a singular matrix.

If  $M$  is non-singular, i.e.,

$$\det(M) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 > 0$$

then judgements are inconsistent. For a judgmental matrix  $A$  of order  $n > 3$ , the consistency index  $CI^*$  was taken as the mean of the determinants of all sub matrices of order 3 of the matrix  $A$ . The total number of submatrices of order 3 of a matrix of order  $n$  are  $\binom{n}{3} = \frac{n!}{3!(n-3)!}$ . Hence, the mathematical formula for the consistency index  $CI^*$  becomes

$$CI^* = \begin{cases} 0 & n < 3 \\ \det(M) & n = 3 \\ \frac{\sum_{i < j < k} \left( \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right)}{\binom{n}{3}} & n > 3 \end{cases}$$

It is easy to identify [37,38] that for  $n \geq 3$  the consistency index  $CI^*$  proposed by Lamata is related to the consistency index  $c_3$  proposed by Shiraish as:

$$c_3 = - \binom{n}{3} CI^*$$

In [39], the further scope of improvement in  $CI^*$  was found and this improved version of  $CI^*$  was denoted by  $CI^+$ . By considering Saaty's scale the minimum value of  $\left| \frac{a_{13}}{a_{12}a_{23}} \right|$

should be  $9^{-3}$  and maximum value of  $\left| \frac{a_{13}}{a_{12}a_{23}} \right|$  should be  $9^3$ . Using these values, they defined the consistency index of a matrix of order 3 which is bounded in  $[0, 1]$  as

$$CI^+ = \frac{9^3 + 9^{-3} - \left( \left| \frac{a_{13}}{a_{12}a_{23}} \right| + \left| \frac{a_{12}a_{23}}{a_{13}} \right| \right)}{9^3 + 9^{-3} - 2} = 1 + \frac{2 - \left( \left| \frac{a_{13}}{a_{12}a_{23}} \right| + \left| \frac{a_{12}a_{23}}{a_{13}} \right| \right)}{9^3 + 9^{-3} - 2}$$

Thus, for a consistent matrix  $CI^+_{max} = 1$ , because for a consistent matrix of order three  $\frac{a_{13}}{a_{12}a_{23}} = 1 = \frac{a_{12}a_{23}}{a_{13}}$ .

The value of  $CI^+_{min} = 0$ , when  $\left| \frac{a_{13}}{a_{12}a_{23}} \right|$  takes its maximum or minimum value, i.e.,  $9^3$  or  $9^{-3}$ , respectively. For any other value of  $\left| \frac{a_{13}}{a_{12}a_{23}} \right|$ ,  $CI^+$  falls in the interval  $[0, 1]$ . They formulated  $CI^+$  for any matrix  $A$  of order  $n$  as the mean value of the  $CI^+$  of all the submatrices of order three of  $A$  i.e.,

$$CI^+ = \begin{cases} 0 & n < 3 \\ CI^+(A_{3 \times 3}) & n = 3 \\ \frac{\sum_{i=1}^n CI^+_i}{\binom{n}{3}} & n > 3 \end{cases}$$

where  $CI^+_i$  is the consistency of  $i$ th submatrix order three of  $A$ .

Benítez et al. [39,40] proposed linearization technique to obtain the nearest consistent matrix corresponding to a given inconsistent matrix. Orthogonal projection in linear space is used to obtain the nearest consistent matrix. Let  $F^{m \times n}$  be the set of all  $m \times n$  real matrices and let  $F^{m \times n}_+$  be the set of all positive matrices. Then it is obvious that  $F^{m \times n}_+ \subseteq F^{m \times n}$ . They defined a nonlinear bijective map  $L : F^{n \times n}_+ \rightarrow F^{n \times n}$  as

$$[L(X)]_{ij} = \log([X]_{ij})$$

Thus,  $L$  maps a positive reciprocal matrix  $B$  to a skew Hermitian matrix  $L(B)$ . They further defined a subspace  $L_n$  consisting of the images  $L(A)$  of all consistent matrices  $A$  in  $F^{n \times n}_+$ . The dimension of  $L_n$  is of course  $n - 1$ . The objective is to find the nearest consistent matrix  $L(A)$  in subspace  $L_n$  to  $L(B)$ . A linear map  $f$  from  $\mathbb{R}^n$  to vector space  $F^{n \times n}$  of all  $n \times n$  matrices is defined as

$$[f(x)]_{i,j} = x_i - x_j, \quad x = (x_1, x_2, \dots, x_n)^T$$

This function maps any vector of  $\mathbb{R}^n$  to the skew Hermitian matrix of order  $n$ . Thus  $L_n$  coincides with  $Im(f)$ . If  $W$  is the one-dimensional subspace spanned by vector  $(1, 1, \dots, 1)^T$ , then this subspace is, of course, the null space of  $f$ . Let  $\{y_1, y_2, \dots, y_{n-1}\}$  be the orthogonal basis of the orthogonal complement of  $W$  then they proved that  $\{f(y_1), f(y_2), \dots, f(y_{n-1})\}$  is the orthogonal basis of  $L_n$ . If  $(v_1, v_2, \dots, v_n)$  is an orthogonal basis of  $\mathbb{R}^n$ , then any vector  $v \in \mathbb{R}^n$  can be expressed as the linear combination of the vectors  $v_1, v_2, \dots, v_n$  as  $v = \sum_{i=1}^n \frac{\langle v^T, v_i \rangle}{\langle v_i, v_i \rangle} v_i$ . The nearest consistent matrix to  $L(B)$  in  $L_n$  is the orthogonal projection  $X_B$  of  $L(B)$  on  $L_n$ ,

$$X_B = \frac{1}{2n} \sum_{i=1}^{n-1} \frac{\langle L(B), f(y_i) \rangle}{\langle y_i, y_i \rangle} f(y_i)$$

where, the inner product  $\langle \rangle$  on the  $n^2$  dimensional vector space  $F^{n \times n}$  is defined as

$$\langle A, B \rangle = \text{trace}(A^T B)$$



As  $L$  is a bijective mapping hence the inverse mapping of  $L$  is  $E$  which is defined as  $[E(X)]_{ij} = \exp[X]_{i,j}$ . Thus if  $B$  is a positive reciprocal matrix in  $F_+^{n \times n}$  then  $E(X_B)$  is the nearest consistent matrix to  $B$  in the sense of the distance defined in  $F_+^{n \times n}$  as

$$d(A, B) = \|L(A) - L(B)\|_F.$$

This distance is developed from the Frobenius norm  $\|\cdot\|_F$  i.e.,

$$\|X\|^2 = \text{Trace}(X^T X)$$

Benítez et al. [41] proposed the same formula in a much simpler form. The nearest consistent matrix to  $L(B)$  in  $L_n$  is the orthogonal projection  $X_B$  of  $L(B)$  on  $L_n$ ,

$$X_B = \frac{1}{n} [BU_n - (BU_n)^T]$$

where  $U_n$  is a  $n \times n$  singular matrix of rank one whose elements are all 1. Then  $BU_n$  is a matrix such that the elements in  $i$ th row of  $BU_n$  are the same and equal to the sum of the elements of  $i$ th row of  $B$ . The resultant matrix  $X_B$  is, of course, a skew Hermitian matrix, whose inverse image  $E$  will give the nearest consistent matrix corresponding to  $B$ .

Koczkodaj [42,43] introduced to the research community a new definition of consistency denoted by CM which was based on a triad of any pair-wise comparison matrix. Triad is a vector  $(a_{ij}, a_{ik}, a_{jk})$  of  $\mathbb{R}^3$  where  $1 \leq i < j < k \leq n$  such that  $a_{ij}a_{jk} = a_{ik}$ . For any pair-wise comparison matrix of order three, there is only one triad  $(a_{12}, a_{13}, a_{23})$ . If  $(a, b, c)$  is the triad of any pair-wise comparison matrix  $A$  of order three, then they defined consistency measure as

$$\begin{aligned} CM(a, b, c) &= \min \left\{ \left| \frac{1}{a} \left| a - \frac{b}{c} \right|, \frac{1}{b} \left| b - ac \right|, \frac{1}{c} \left| c - \frac{b}{a} \right| \right\} \\ &= \min \left\{ \left| 1 - \frac{b}{ac} \right|, \left| 1 - \frac{ac}{b} \right| \right\} \end{aligned}$$

Thus,

$$CM(a_{12}, a_{13}, a_{23}) = \min \left\{ \left| 1 - \frac{a_{13}}{a_{12}a_{23}} \right|, \left| 1 - \frac{a_{12}a_{23}}{a_{13}} \right| \right\}.$$

The total number of triads of any pair-wise comparison matrix of order  $n$  are  $\frac{n(n-1)(n-2)}{6}$ . Thus, CM corresponding to each triad  $(a_{ij}, a_{ik}, a_{jk})$  can be evaluated with the help of the formula  $CM(a_{ij}, a_{ik}, a_{jk}) = \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\}$ . They generalized the consistency measure of any PCM of order  $n$  as the maximum value of  $CM(a_{ij}, a_{ik}, a_{jk})$ ,  $1 \leq i < j < k \leq n$ , among the  $\frac{n(n-1)(n-2)}{6}$  CM corresponding to each triad.

Szybowski et al. [44] proposed Manhattan-index, and K-index for the incomplete pairwise comparisons matrices. Mazurek [45] presented row inconsistency index (RIC). Metaheuristics [46–49] have also been used to reduce the inconsistency in pairwise comparison matrices. Several iterative algorithms are also available in the literature for the reduction of the inconsistency in pairwise comparison matrices. Recently, Mazurek [50] have done a numerical comparison of such iterative methods.

#### 4. Functional Relationship and Axiomatization of Consistency Indices

Several studies agree that the consistency indices are meaningless if the associated threshold is not present. If the consistency index is less than the threshold, then the judgements performed by the decision-maker are accepted. Otherwise, the decision-maker has to revise the judgements. In literature, the threshold is defined for a few consistency indices such as CI, GCI and CM. There are several other consistency indices that are not associated with a threshold [51–57]. In addition, if the number of elements to be compared

increases, then the consistency ratio defined by Saaty falls above 0.10. Due to this reason Saaty's consistency index was criticised in literature. According to Murphy [58], the 9-point scale proposed by Saaty is responsible for this behaviour. On the other hand, in [59] it was suggested that the small value of the standard deviation of  $CI$  of randomly generated matrices by using the 9-point scale is the reason behind the restrictive threshold. In [30], the dependency of  $CR$  threshold on the granularity of the scale was presented. Bozók and Rapcsák [60] compared Saaty's and Koczkodaj's consistency indices and arose valid questions on these consistency indices. The effect of increasing the number of objects to be compared on the inconsistency indices was experimentally studied by [61]. Determination of the strength of the consistency test is still a meaningful and significant topic of research. In recent years, work on establishing the functional relationships between different consistency indices has also been done. The functional dependency of two consistency indices has the following meanings:

- (1) Both indices satisfy the same set of properties
- (2) Both indices bring out the same results, which means that the one which is easy to compute can be used.
- (3) Functional dependency unifies the two different indices which have been developed independently.

Brunelli [37] investigated the linear relationship between  $CI^*$ ,  $c_3$  and  $GCI$ ,  $\rho$ . Brunelli [62] further studied ten consistency indices numerically to identify similarity among them. Brunelli [63] has again functionally related the two different consistency indices that arise in two different frameworks, i.e., (i) fuzzy preference relations and (ii) multiplicative preference relations. In [64] functional the dependency of nine different  $CI$  on each other has been investigated, and for  $n = 3$ , all consistency indices were found functionally dependent except  $RE$  and  $CCI$ . In [65,66], a comparison between different indices on the basis of statistical parameters has been performed.

In the last few years, the main focus of research has been shifted to the axiomatic properties of the consistency index. Axiomatic properties are a set of mathematical properties to be satisfied by any consistency index which makes consistency indices more reliable to evaluate the deviation of PCM from the consistent matrices. First detailed study on axiomatization was done by Koczkodaj and Szwarc in 2014 [67] and was revised by Koczkodaj et al. [68]. Further, Brunelli and Fedrizzi [69] suggested five axiomatic properties to characterize consistency indices. Some consistency measures [55,70–72] were not able to satisfy these axioms suggested by Brunelli and Fedrizzi [69] while others do [3,22,23,36,73]. This axiomatic framework, consisting of five axioms, was expanded further, and one more axiom was added by Brunelli in 2017 [74]. These six axioms are as follows:

**Axiom 1.** *A is consistent if there exists a unique real number  $r \in \mathbb{R}$  which represents the situation of full consistency. That is, there must be a unique minimum value of the consistency index at which it is fully consistent.*

**Axiom 2.** *Consistency indices should be invariant under the order of alternatives.*

**Axiom 3.** *Intensified preferences should not decrease the value of consistency indices.*

**Axiom 4.** *Monotonicity of consistent indices should be maintained for the single comparison as well.*

**Axiom 5.** *Consistency indices should be continuous function of the  $\binom{n}{2}$  variables  $a_{ij}$  ( $i > j$ ) which are the entries of the pairwise comparison matrix  $A = [a_{ij}]$ .*

**Axiom 6.** *The consistency indices of  $A$  must be the same as the consistency indices of  $A^T$ . Thus, the consistency index must be invariant under the inversion of preferences.*

Another set of six properties for consistency indices was proposed by Csató [75] and the consistency index suggested by Koczkodaj [35,36] was characterized. Csató [76] added two more axioms in the axiomatic framework proposed by Brunelli and Fedrizzi [74]. Recently, Mazurek and Ramík [77] introduced row inconsistency indices  $RIC$  and added

one more axiom in [69]. He further found that only Koczkodaj's consistency index  $K$  was able to satisfy all six axioms. Brunelli and Cavallo [78] have recently developed a new categorization of consistency indices. The behavior of some consistency indices on different sets of properties is listed in Table 1. Here, in the Table 1,  $A_i$  stands for the  $i$ th axiom in the proposed set of axioms given by the author.

**Table 1.** Axiomatic properties satisfied by different indices.

S. No.	Indices	Brunelli & Fedrizzi [69]	Brunelli [74]	Csató [75]	Csató [76]	Koczkodaj [68]	Mazurek & Ramík [77]
1	$CI$ [3]	Satisfies all axioms.	Satisfies all axioms	-	-	Dissatisfies- $A_2$ and $A_4$	Satisfies- $A_1$ to $A_5$ Dissatisfies- $A_6$
2	$CI^*$ [22,36]	Satisfies all axioms.	Satisfies all axioms	-	-	-	Satisfies- $A_1$ to $A_5$ Dissatisfies- $A_6$
3	$GCI$ [23]	Satisfies all axioms.	Satisfies all axioms	-	-	-	Satisfies- $A_1$ to $A_5$ Dissatisfies- $A_6$
4	$CM$ [42,43]	Satisfies all five axioms.	Satisfies all six axioms	Satisfies all six axioms	Satisfies all eight axioms	-	Satisfies all six axioms
5	$RE$ [70]	Satisfies- $A_1, A_2, A_3$ Dissatisfies- $A_4, A_5$	Satisfies- $A_1, A_2, A_3$ Dissatisfies- $A_4, A_5, A_6$	-	-	-	-
6	$HCI$ [71]	Satisfies- $A_1, A_2, A_3$ and $A_5$ Dissatisfies- $A_4$	Satisfies- $A_1, A_2, A_3, A_5$ and $A_6$ Dissatisfies- $A_4$	-	-	-	-
7	$GW$ [53]	Satisfies- $A_1, A_2$ and $A_5$ Dissatisfies- $A_3$	Satisfies- $A_1, A_2, A_5$ and $A_6$ Dissatisfies $A_3$	-	-	-	Satisfies- $A_1, A_2$ and $A_5, A_6$ Dissatisfies- $A_3$
8	$NI_n^r$ [72]	Satisfies- $A_1, A_2$ and $A_5$ Dissatisfies- $A_4$	Satisfies- $A_1, A_2, A_5$ and $A_6$ Dissatisfies $A_4$	-	-	-	-
9	$CMSH$ [30]	Satisfies- $A_1, A_2, A_4$ and $A_5$ Dissatisfies- $A_3$	Satisfies- $A_1, A_2, A_4, A_5$ and $A_6$ Dissatisfies- $A_3$	-	-	-	-
10	$CI_H$ [73]	Satisfies all five axioms.	Satisfies-all six axioms.	-	-	-	-
11	$CCI$ [27]	Satisfies- $A_1, A_2$ and $A_5$ Dissatisfies- $A_3$	Satisfies- $A_1, A_2, A_5$ and $A_6$ Dissatisfies $A_3$	-	-	-	-
12	$RIC$ [77]	Satisfies- $A_1, A_2, A_4$ and $A_5$ Dissatisfies- $A_3$	-	-	-	-	Satisfies- $A_1, A_2, A_4, A_5$ and $A_6$ Dissatisfies- $A_3$

## 5. Research Gaps and Potential Research Direction

Extensive research has been done in the field of consistency in AHP, but there is still a scope to improve the existing consistency indices and develop new consistency indices. We have listed some potential future directions on the basis of the existent research gaps in the field of the consistency indices in AHP as follows:

- (1) While the AHP method was developed in early seventies, there is still there a bright scope to perform mathematical analysis of AHP especially, in the area of evaluation of consistency index.

- (2) Intensive work can be done to determine the threshold of existing consistency indices. Many other indices have been developed by the researchers so far, but some of them are not that meaningful because they do not provide the thresholds associated with the indices.
- (3) The linear scale (Saaty's scale) has been criticized in literature as it is not large enough to handle the ambiguity in real-life problems, and hence gives rise to the absurdity in consistency index.
- (4) As discussed in Section 4, there is a strong need to unify consistency indices with the help of axiomatic properties. In recent years, the main focus of research has been shifted to the axiomatic properties of consistency index. Axiomatization to unify the existing consistency indices is another promising research direction.
- (5) The weak consistency of preference relations with triangular numbers, interval numbers, and trapezoidal fuzzy numbers is not well studied yet.

## 6. Conclusions

AHP is one of the most popular tools in multi-criteria decision-making (MCDM). The main disadvantage of AHP is a large number of pairwise comparisons, which can certainly cause errors to arise. Extensive research has been performed to identify and minimize these errors by developing consistency indices. This article starts by explaining the mathematical concepts of AHP. Then, it reviews the different consistency indices with their proper mathematical structure. This article also includes the limitation of consistency indices on the basis of their functional relationship and the satisfaction level of different axiomatizations. In a nutshell, axiomatization is the need of the hour to unify consistency indices on the same platform. This article also covers some potential research directions as there is still room for improvement in the field of consistency indices. These directions can help researchers to think about unexplored areas in this field.

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