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Article

Algebraic Representation of Topologies on a Finite Set

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Abstract: Since the 1930s, topological counting on finite sets has been an interesting work so as to enumerate the number of corresponding order relations on the sets. Starting from the semi-tensor product (STP), we give the expression of the relationship between subsets of finite sets from the perspective of algebra. Firstly, using the STP of matrices, we present the algebraic representation of the subset and complement of finite sets and corresponding structure matrices. Then, we investigate respectively the relationship between the intersection and union and intersection and minus of structure matrices. Finally, we provide an algorithm to enumerate the numbers of topologies on a finite set based on the above theorems.

Keywords: the number of topologies; semi-tensor product; structure matrices

MSC: 54-08

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1. Introduction

In the 1930s, Alexandroff pointed out in [1] that there is a one-to-one correspondence between all topologies on a finite set and all quasi-orders, and there is a one-to-one correspondence between all T_0 topologies on a finite set and all partial-orders. Therefore, enumerating the topologies on the set is equivalent to enumerating the number of corresponding order relations on the set. Since then, calculating the number of topologies on a finite set has become an interesting topic of topological research [2,3]. Sharp pointed out a topological corresponding relation matrix and tried to describe the topology by using some properties of the matrix [4]. Scholars have gradually paid attention to the calculation of the number of T_0 topologies on a finite set and obtained relevant conclusions and inferences through demonstrations [5–7], and with the deepening of research, they abandoned the traditional enumeration method and used different algorithms to calculate the topologies on a finite set from different angles [8,9]. In the study of structure matrices, similarity relations and connective relations are two special classes of binary relations, which have important applications to economics, to the theorems of Arrow, and others about social welfare functions [10].

The Cheng product, also called the semi-tensor product (STP), was firstly proposed by Prof. Cheng to study the topology of Boolean networks [11]. Since the STP can convert any finite-valued functions into algebraic forms, Prof. Cheng and his collaborators developed an algebraic state space representation method for the analysis of BNs and BCNs and obtained many essential results [12,13]. In the following years, the STP made many achievements in the study of Boolean networks and gradually formed a complete framework of Boolean network control theory [14–16]. Moreover, the STP method also shows strong vitality in many other fields, such as information security [17], finite automata [18,19], vehicle control, and networked evolutionary games [20,21].

Compared with the previous traditional enumeration method with a high time complexity, the highlights of our findings are the following characteristics:

- Using the STP of matrices, the algebraic representation of the subset and the complement of finite sets and corresponding structure matrices are obtained, respectively;
- The relationship between the intersection and union and the intersection and minus of structure matrices is investigated, and a series of inferences is given;
- The application is provided for the calculation of topologies on a finite set; in addition, the corresponding algorithms and examples can be given in order to provide more possibilities for the topology.

Our work is divided into the following sections. In Section 2, we review a few basic notations and definitions; in Section 3, the algebraic representation of the subset and the complement of finite sets are given; in Section 4, the main results are obtained; in Section 5, the corresponding algorithms and examples are given for the calculation of topologies on a finite set; finally, in Sections 6 and 7, we close with a brief conclusion and point out some possible future directions of research.

2. Preliminaries

In this section, in order to better understand the definition and conclusion of the article, we first list some notations:

(1) $\mathcal{M}_{m\times n}$ is the set of all $m \times n$ real matrices;
(2) $\mathbf{1}_n$ represents the column vector of length n all terms of which are 1, that is $\mathbf{1}_n = [1, \cdots, 1]^T \in \mathcal{V}_n$;
(3) $Col_i(M)(Row_i(M))$ is the i-th column (row) of matrix M;
(4) δ_n^i is the i-th column of n-dimensional identity matrix I_n, that is $\delta_n^i = Col_i(I_n)$;
(5) $\Delta_n := \{\delta_n^i | i = 1, 2, \cdots, n\}$;
(6) Assume $L = [\delta_n^1, \delta_n^2, \cdots, \delta_n^n]$, then its shorthand form is $L = \delta_n[1, 2, \cdots, n]$;
(7) For a finite set X, $2^X := \{S : S \subseteq X\}$.

Because the STP overcomes the limitation of traditional matrices' product on dimensions, we introduce the definitions of the Kronecker product, STP, topology, and T_0 topology.

Definition 1 (Reference [22]). *Let $A = (a_{ij}) \in \mathcal{M}_{m\times n}, B = (b_{ij}) \in \mathcal{M}_{p\times q}$. Then, the Kronecker product of matrices is:*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Definition 2 (Reference [23]). *Let $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times q}, t = lcm(n, p)$. Then, the STP of A and B is:*

$$A \ltimes B := (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $lcm(n, p)$ represents the least common multiple of n and p.

An example of the STP of two matrices denoted as A and B is given below.

Example 1. *Assume $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, then:*

$$
\begin{aligned}
A \ltimes B &= (A \otimes I_2)(B \otimes I_3) \\
&= \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 2 \\ 4 & 1 & 2 & 2 & 1 & 1 \\ 1 & 4 & 3 & 1 & 2 & 3 \\ 6 & 1 & 4 & 3 & 1 & 2 \\ 1 & 4 & 1 & 1 & 2 & 1 \\ 2 & 1 & 4 & 1 & 1 & 2 \end{pmatrix}
\end{aligned}
$$

Definition 3 (Reference [24]). *A topological space on a set X is a collection τ of subsets of X, called the open sets, satisfying:*

(1) Any union of elements of τ belongs to τ;
(2) Any finite intersection of elements of τ belongs to τ;
(3) $\emptyset$ and X belong to τ.

Remark 1. *For a finite set X, if the union or intersection of any two elements of τ belongs to τ, then any union or intersection of elements of τ also belongs to τ.*

A topological space τ on a set X is also called a topology on X and often denoted by (X, τ).

Definition 4 (Reference [24]). *A topological space (X, τ) is a T_0 space iff whenever x and y are distinct points in X, there is an open set containing one and not the other.*

Note that a T_0 space is also called a T_0 topology.

Example 2. *Assume set $X = \{x_1, x_2, x_3\}$:*

(1) Take a subset $S = \{x_1, x_2\}$, and assume a family of subset $\tau = \{X, S, \emptyset\}$. When $x_1 \in S$, $x_2 \in S$ certainly; the same can be proven when $x_1 \in X$, $x_2 \in X$ certainly. Therefore, (X, τ) is not a T_0 topology;
(2) Take subsets $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2, x_3\}$, and $S_3 = \{x_2\}$, and assume a family of subset $\tau = \{X, S_1, S_2, S_3, \emptyset\}$. For x_1 and x_2, there exists an open set named S_3 meeting the above definition; similarly, the combination containing x_2 and x_3 also meets the definition; for x_1 and x_3, there exists an open set named $S_1(S_2)$ meeting the above definition. Therefore, (X, τ) is a T_0 topology;
(3) It is easy to prove that if $\tau = \{X, \emptyset\}$, then (X, τ) is not T_0. This result also holds for all finite sets X containing more than one element.

3. Algebraic Representation

In this part, we present the algebraic representation of the subset and the complement of finite sets.

Proposition 1 (Algebraic representation of the subset of a finite set). *Given a finite set $X = \{x_1, x_2, \cdots, x_n\}$ and a subset S of X, then denote $S \sim \delta_2^{i_1} \ltimes \delta_2^{i_2} \ltimes \cdots \ltimes \delta_2^{i_n}$, where for all $k, k = 1, 2, \cdots, n$, $i_k = 1$ if $x_k \in S$, otherwise, $i_k = 2$. In particular, $X \sim \delta_{2^n}^1$, and $\emptyset \sim \delta_{2^n}^{2^n}$.*

Example 3. *Assume $X = \{x_1, x_2, x_3\}$, $S = \{x_1, x_2\}$, and $\tau = \{X, S, \emptyset\}$ of X, then:*

(1) $X \sim \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^1 = \delta_8^1$, $\emptyset \sim \delta_2^2 \ltimes \delta_2^2 \ltimes \delta_2^2 = \delta_8^8$;

(2) $2^X \sim \Delta_8$;

(3) $S \sim \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^2 = \delta_8^2$.

In algebraic form, the family τ *can be represented by* $\tau = \{\delta_8^1, \delta_8^2, \delta_8^8\}$.

As is known to all, given a finite set $X = \{x_1, x_2, \cdots, x_n\}$ and a subset S of X, the complement of S is defined as $S' = \{x_i : x_i \in X, x_i \notin S\}$.

Proposition 2 (Algebraic representation of the complement of a finite set)**.** *Given a finite set* $X = \{x_1, x_2, \cdots, x_n\}$ *and a subset S of X, then* $S' = \delta_2^{i_1} \ltimes \delta_2^{i_2} \ltimes \cdots \ltimes \delta_2^{i_n}$*, where for each* $k \in \{1, 2, \cdots, n\}$*,* $i_k = 2$ *if* $x_k \in S$*, otherwise,* $i_k = 1$*. In particular,* $X' = \emptyset = \delta_{2^n}^{2^n}$*, and* $\emptyset' = X = \delta_{2^n}^1$*.*

4. Structure Matrices

In this part, we explain the structure matrices of the union, intersection, and minus of subsets. On this basis, some inferences are given to illustrate the relationships between them.

Proposition 3 (Matrix representation of the complement)**.** *Given a finite set* $X = \{x_1, x_2, \cdots, x_n\}$*, let* $S = \delta_{2^n}^a$ *be a subset of X. Then the complement* S' *of S satisfies* $S' = \delta_{2^n}^{2^n - a + 1}$*. Moreover,* $S' = M_n \ltimes S$*, where* $M_n = \delta_{2^n}[2^n, 2^n - 1, \cdots, 1]$*.*

Proof of Proposition 3. Suppose

$$S = \delta_2^{i_1} \ltimes \delta_2^{i_2} \ltimes \cdots \ltimes \delta_2^{i_n} = \delta_{2^n}^{\sum_{l=1}^{n-1}(i_l - 1)2^{n-l} + i_n}.$$

Then we have that

$$S' = \delta_2^{\overline{i_1}} \ltimes \delta_2^{\overline{i_2}} \ltimes \cdots \ltimes \delta_2^{\overline{i_n}} = \delta_{2^n}^{\sum_{l=1}^{n-1}(\overline{i_l} - 1)2^{n-l} + \overline{i_n}};$$

where $\{i_l, \overline{i_l}\} = \{1, 2\}$ for each $l \in \{1, 2, \cdots, n-1\}$.

Then, summing over them:

$$\begin{aligned}
&\sum_{l=1}^{n-1}(i_l - 1)2^{n-l} + i_n + \sum_{l=1}^{n-1}(\overline{i_l} - 1)2^{n-l} + \overline{i_n} \\
=&\sum_{l=1}^{n-1}(i_l + \overline{i_l} - 1)2^{n-l} + i_n + \overline{i_n} \\
=&\sum_{l=1}^{n-1}2^{n-l} + 3 \\
=&2^n - 2 + 3 \\
=&2^n + 1.
\end{aligned}$$

Therefore, we have that $S' = \delta_{2^n}^{2^n - a + 1}$. Moreover, for $M_n = \delta_{2^n}[2^n, 2^n - 1, \cdots, 1]$, it is not difficult to check that

$$M_n \ltimes S = M_n \ltimes \delta_{2^n}^a = \delta_{2^n}^{2^n - a + 1},$$

and hence $S' = M_n \ltimes S$. □

Example 4. *Assume set* $X = \{x_1, x_2, x_3\}$ *and the subset* $S = \{x_1, x_2\}$*, then* $S' = \{x_3\}$*, and:*

(1) *From the above propositions, it is not difficult for us to find:* $S = \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^2 = \delta_8^2$ *and* $S' = \delta_2^2 \ltimes \delta_2^2 \ltimes \delta_2^1 = \delta_8^7$;

(2) *Because* $n = 3$*, it is easy to obtain* $M_3 = \delta_8[8, 7, \cdots, 1]$;

(3) *We use the formula to find:*

$$
\begin{aligned}
S' &= M_3 \ltimes S \\
&= \delta_8[8,7,\cdots,1] \ltimes \delta_8^2 \\
&= \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix} \\
&= \begin{pmatrix} 0&0&0&0&0&0&1&0 \end{pmatrix}^\top = \delta_8^7.
\end{aligned}
$$

Example 5. *Take the set $X = \{x_1, x_2\}$ for example. The family of subsets $\tau = \{X, \{x_1\}, \{x_2\}, \varnothing\}$. All subsets of X are treated as rows and columns of the table, respectively:*

(1) *Example of the union:*
The calculated results are shown in the form of a set (Table 1) and the form of a vector (Table 2), respectively.

Table 1. Union in set form.

$A \cup B$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\varnothing$
$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$
$\{x_1\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_1, x_2\}$	$\{x_1\}$
$\{x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_2\}$	$\{x_2\}$
$\varnothing$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\varnothing$

Table 2. Union in vector form.

$A \cup B$	δ_4^1	δ_4^2	δ_4^3	δ_4^4
δ_4^1	δ_4^1	δ_4^1	δ_4^1	δ_4^1
δ_4^2	δ_4^1	δ_4^2	δ_4^1	δ_4^2
δ_4^3	δ_4^1	δ_4^1	δ_4^3	δ_4^3
δ_4^4	δ_4^1	δ_4^2	δ_4^3	δ_4^4

Thus, the structure matrix of the union is derived:

$$
M_\cup = \delta_4[1,1,1,1,|1,2,1,2,|1,1,3,3,|1,2,3,4];
$$

(2) *Example of the intersection:*
The calculated results are shown in the form of a set (Table 3) and the form of a vector (Table 4), respectively.

Table 3. Intersection in set form.

$A \cap B$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\varnothing$
$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_1\}$	$\{x_2\}$	$\varnothing$
$\{x_1\}$	$\{x_1\}$	$\{x_1\}$	$\varnothing$	$\varnothing$
$\{x_2\}$	$\{x_2\}$	$\varnothing$	$\{x_2\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Table 4. Intersection in vector form.

$A \cap B$	δ_4^1	δ_4^2	δ_4^3	δ_4^4
δ_4^1	δ_4^1	δ_4^2	δ_4^3	δ_4^4
δ_4^2	δ_4^2	δ_4^2	δ_4^4	δ_4^4
δ_4^3	δ_4^3	δ_4^4	δ_4^3	δ_4^4
δ_4^4	δ_4^4	δ_4^4	δ_4^4	δ_4^4

Thus, the structure matrix of the intersection is derived:

$$M_\cap = \delta_4[1,2,3,4,|2,2,4,4,|3,4,3,4,|4,4,4,4].$$

Following the example above, compare the structure matrices of the union and intersection.

$$M_\cup = \delta_4[1,1,1,1,|1,2,1,2,|1,1,3,3,|1,2,3,4],$$
$$M_\cap = \delta_4[1,2,3,4,|2,2,4,4,|3,4,3,4,|4,4,4,4].$$

Via computation, we have:

$$M_\cap M_n^{16} = \delta_4[4,4,4,4,|4,3,4,3,|4,4,2,2,|4,3,2,1],$$

where $M_n^{16} = \delta_{16}[16,15,\cdots,1]$. Denote:

$$Col_i(M_\cup) := \delta_{2^n}^{\alpha_i},$$

$$Col_i(M_\cap M_n^{16}) := \delta_{2^n}^{\beta_i}.$$

It is obvious that:

$$\alpha_i + \beta_i = 5, i = 1,2,\cdots,16.$$

Proposition 4. *Denote:*

$$Col_j(M_\cap) := \delta_{2^n}^{\alpha_i},$$

$$Col_i(M_\cup M_n^{2^{2n}}) := \delta_{2^n}^{\beta_i}.$$

Then:

$$\alpha_i + \beta_i = 2^n + 1, i = 1,2,\cdots,2^{2n}.$$

Proposition 5. *Denote:*

$$Col_j Block_i(M_\cap) := \delta_{2^n}^{\alpha_{ij}},$$

$$Col_{2^n-j+1} Block_{2^n-i+1}(M_\cup) := \delta_{2^n}^{\beta_{ij}}.$$

Then:

$$\alpha_{ij} + \beta_{ij} = 2^n + 1, i,j = 1,2,\cdots,2^n.$$

Proposition 6. *From $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$, it is derived that:*

$$M_\cap = M_n M_\cup M_n (I_{2^n} \otimes M_n),$$

$$M_\cup = M_n M_\cap M_n (I_{2^n} \otimes M_n).$$

Example 6. *Take the set $X = \{x_1, x_2, x_3\}$ for example. Denote 2^X by Δ_8, then:*

(1) We can obtain the matrix of the union (Table 5).

Table 5. Matrix of the union of 2^X.

$A \cup B$	δ_8^1	δ_8^2	δ_8^3	δ_8^4	δ_8^5	δ_8^6	δ_8^7	δ_8^8
δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^1
δ_8^2	δ_8^1	δ_8^2	δ_8^1	δ_8^2	δ_8^1	δ_8^2	δ_8^1	δ_8^2
δ_8^3	δ_8^1	δ_8^1	δ_8^3	δ_8^3	δ_8^1	δ_8^1	δ_8^3	δ_8^3
δ_8^4	δ_8^1	δ_8^2	δ_8^3	δ_8^4	δ_8^1	δ_8^2	δ_8^3	δ_8^4
δ_8^5	δ_8^1	δ_8^1	δ_8^1	δ_8^1	δ_8^5	δ_8^5	δ_8^5	δ_8^5
δ_8^6	δ_8^1	δ_8^2	δ_8^1	δ_8^2	δ_8^5	δ_8^6	δ_8^5	δ_8^6
δ_8^7	δ_8^1	δ_8^1	δ_8^3	δ_8^3	δ_8^5	δ_8^5	δ_8^7	δ_8^7
δ_8^8	δ_8^1	δ_8^2	δ_8^3	δ_8^4	δ_8^5	δ_8^6	δ_8^7	δ_8^8

Thus, the structure matrix of the intersection is derived:

$$\begin{aligned} M_{\cup} = \delta_8\, [&1,1,1,1,1,1,1,1,\\ &1,2,1,2,1,2,1,2,\\ &1,1,3,3,1,1,3,3,\\ &1,2,3,4,1,2,3,4,\\ &1,1,1,1,5,5,5,5,\\ &1,2,1,2,5,6,5,6,\\ &1,1,3,3,5,5,7,7,\\ &1,2,3,4,5,6,7,8]; \end{aligned}$$

(2) *We can obtain the matrix of the intersection (Table 6).*

Table 6. Matrix of the intersection of 2^X.

$A \cap B$	δ_8^1	δ_8^2	δ_8^3	δ_8^4	δ_8^5	δ_8^6	δ_8^7	δ_8^8
δ_8^1	δ_8^1	δ_8^2	δ_8^3	δ_8^4	δ_8^5	δ_8^6	δ_8^7	δ_8^8
δ_8^2	δ_8^2	δ_8^2	δ_8^4	δ_8^4	δ_8^6	δ_8^6	δ_8^8	δ_8^8
δ_8^3	δ_8^3	δ_8^4	δ_8^3	δ_8^4	δ_8^7	δ_8^8	δ_8^7	δ_8^8
δ_8^4	δ_8^4	δ_8^4	δ_8^4	δ_8^4	δ_8^8	δ_8^8	δ_8^8	δ_8^8
δ_8^5	δ_8^5	δ_8^6	δ_8^7	δ_8^8	δ_8^5	δ_8^6	δ_8^7	δ_8^8
δ_8^6	δ_8^6	δ_8^6	δ_8^8	δ_8^8	δ_8^6	δ_8^6	δ_8^8	δ_8^8
δ_8^7	δ_8^7	δ_8^8	δ_8^7	δ_8^8	δ_8^7	δ_8^8	δ_8^7	δ_8^8
δ_8^8	δ_8^8	δ_8^8	δ_8^8	δ_8^8	δ_8^8	δ_8^8	δ_8^8	δ_8^8

Thus, the structure matrix of the intersection is derived:

$$\begin{aligned} M_{\cap} = \delta_8\, [&1,2,3,4,5,6,7,8,\\ &2,2,4,4,6,6,8,8,\\ &3,4,3,4,7,8,7,8,\\ &4,4,4,4,8,8,8,8,\\ &5,6,7,8,5,6,7,8,\\ &6,6,8,8,6,6,8,8,\\ &7,8,7,8,7,8,7,8,\\ &8,8,8,8,8,8,8,8]. \end{aligned}$$

Proposition 7. *Given a finite set X and its subset family $\tau = \{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \cdots, \delta_{2^n}^{i_r}\}$, then τ is a topology of X, if and only if:*

(1) $\delta_{2^n}^1 \in \tau$, $\delta_{2^n}^{2^n} \in \tau$;

(2) *For any two elements $\delta_{2^n}^i$ and $\delta_{2^n}^j$ of τ, both $Col_i Block_j(M_\cup) \in \tau$ and $Col_i Block_j(M_\cap) \in \tau$ hold.*

Example 7 (Example of the minus)**.** *For the set $X = \{x_1, x_2\}$, for example, it is easy to have the table of its minus (Table 7).*

Table 7. Minus of 2^X.

$A \backslash B$	δ_4^1	δ_4^2	δ_4^3	δ_4^4
δ_4^1	δ_4^4	δ_4^3	δ_4^2	δ_4^1
δ_4^2	δ_4^4	δ_4^4	δ_4^2	δ_4^2
δ_4^3	δ_4^4	δ_4^3	δ_4^4	δ_4^3
δ_4^4	δ_4^4	δ_4^4	δ_4^4	δ_4^4

Thus, the structure matrix of the minus is derived:

$$M_\backslash = \delta_4[4,4,4,4,|3,4,3,4,|2,2,4,4,|1,2,3,4].$$

Following the example above, compare the structure matrices of the minus and intersection,

$$M_\backslash = \delta_4[4,4,4,4,|3,4,3,4,|2,2,4,4,|1,2,3,4],$$
$$M_\cap = \delta_4[1,2,3,4,|2,2,4,4,|3,4,3,4,|4,4,4,4].$$

We have:

$$Block_1(M_\cap) = Block_4(M_\backslash), Block_2(M_\cap) = Block_3(M_\backslash),$$
$$Block_3(M_\cap) = Block_2(M_\backslash), Block_4(M_\cap) = Block_1(M_\backslash).$$

Proposition 8. *Given a finite set X, assume $|X| = n$. Separate $M_\backslash$ and $M_\cap$ into equal n blocks, then:*

$$Block_i(M_\cap) = Block_{2^n-i+1}(M_\backslash).$$

5. Application

Given a finite set $X = \{x_1, x_2, \cdots, x_n\}$, $n \geq 3$, the STP structure matrix $M_\cup$ corresponding to the union of any two sets A and B in all subsets and the STP structure matrix $M_\cap$ corresponding to the intersection of any two sets in all subsets are obtained and shown separately as follows (Tables 8 and 9).

Table 8. Union in vector form.

$A \cup B$	$\delta_{2^n}^1$	$\cdots$	$\delta_{2^n}^{2^n}$
$\delta_{2^n}^1$	$\delta_{2^n}^1$	$\cdots$	$\delta_{2^n}^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\delta_{2^n}^{2^n}$	$\delta_{2^n}^1$	$\cdots$	$\delta_{2^n}^{2^n}$

Table 9. Intersection in vector form.

$A \cap B$	$\delta_{2^n}^1$	$\cdots$	$\delta_{2^n}^{2^n}$
$\delta_{2^n}^1$	$\delta_{2^n}^1$	$\cdots$	$\delta_{2^n}^{2^n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\delta_{2^n}^{2^n}$	$\delta_{2^n}^{2^n}$	$\cdots$	$\delta_{2^n}^{2^n}$

Each element in $M_\cup$ and $M_\cap$ retains only the number k in the upper right corner to obtain two simplified structure matrices, denoted as G_1 and G_2, respectively. According to Remark 1, the following conclusions can be made.

Proposition 9. *The k-order principal minors consisting of* $1, i_1, \cdots, i_{k-2}, 2^n$ *rows and columns in* G_1 *and* G_2 *are denoted as* $D_{1,k}$ *and* $D_{2,k}$*, respectively, where* $k \geq 3$*. The family of subsets corresponding to* $\{\delta_{2^n}^1, \delta_{2^n}^{i_1}, \cdots, \delta_{2^n}^{i_{k-2}}, \delta_{2^n}^{2^n}\}$ *constitutes a topology on X iff all elements of* $D_{1,k}$ *and* $D_{2,k}$ *belong to* $\{1, i_1, \cdots, i_{k-2}, 2^n\}$*. In addition, the family of subsets is clearly a topology when the form* $\tau = \{\delta_{2^n}^1, \delta_{2^n}^{2^n}\}$ *or* $\tau = \{\delta_{2^n}^1, \delta_{2^n}^{i}, \delta_{2^n}^{2^n}\}$*, where* $i = 2, \cdots, 2^n - 1$.

Therefore, it is sufficient to find how many pairs $(D_{1,k}, D_{2,k})$ can satisfy the above Proposition 9, to prove the number of all topologies on X. The following Algorithm 1 is given.

Algorithm 1: Application of the STP structure matrix.

Input :
The finite set $X = \{x_1, x_2, \cdots, x_n\}, n \geq 3$;

Output:
$\mathcal{T} = \{\delta_{2^n}^1, \delta_{2^n}^{i_1}, \cdots, \delta_{2^n}^{i_{k-2}}, \delta_{2^n}^{2^n}\}$;
The number of topologies on X is N;

obtain the STP structure matrix $M_\cup$ and $M_\cap$;
Simplify $M_\cup$ and $M_\cap$ into G_1 and G_2;
initialize the number of subsets in the subset family $k \leftarrow 2$ ($\mathcal{T} = \{\delta_{2^n}^1, \delta_{2^n}^{2^n}\}$);
initialize $N \leftarrow 1$ ($\mathcal{T} = \{\delta_{2^n}^1, \delta_{2^n}^{2^n}\}$ is a topology);
repeat
  $k \leftarrow k + 1$;
  for $i_1, \cdots, i_{k-2} = 1, \cdots, 2^n - 1$, *and* $i_1 < \cdots < i_{k-2}$ **do**
    Let $D_{1,k}$ and $D_{2,k}$ be the k-order principal minors consisting of $1, i_1, \cdots, i_{k-2}, 2^n$ rows and columns in G_1 and G_2, respectively;
    $S := \{1, i_1, \cdots, i_{k-2}, 2^n\}$;
    Let x_{ij} and y_{ij} be the elements of $D_{1,k}$ and $D_{2,k}$, respectively;
    if $x_{ij} \in S$ *and* $y_{ij} \in S$ **then**
      $N \leftarrow N + 1$;
      $\mathcal{T} = \{\delta_{2^n}^1, \delta_{2^n}^{i_1}, \cdots, \delta_{2^n}^{i_{k-2}}, \delta_{2^n}^{2^n}\}$;
  end
until $k > 2^n + 1$;

Example 8. *Continuing with Example 6 above, we obtain two simplified structure matrices of the union and intersection* G_1 *and* G_2.

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 \\ 1 & 2 & 1 & 2 & 5 & 6 & 5 & 6 \\ 1 & 1 & 3 & 3 & 5 & 5 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 4 & 4 & 6 & 6 & 8 & 8 \\ 3 & 4 & 3 & 4 & 7 & 8 & 7 & 8 \\ 4 & 4 & 4 & 4 & 8 & 8 & 8 & 8 \\ 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 \\ 6 & 6 & 8 & 8 & 6 & 6 & 8 & 8 \\ 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \end{bmatrix}.$$

First, when $\tau = \{\delta_8^1, \delta_8^8\}$, it obviously is a topology on X. When there is one nonempty proper subset in τ, for example $\tau = \{\delta_8^1, \delta_8^2, \delta_8^8\}$, the third-order principal minors composed of the first, second, and eighth rows and columns of G_1 and G_2 are obtained and denoted by $D_{1,3}$ and $D_{2,3}$.

$$D_{1,3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 8 \end{bmatrix}, \quad D_{2,3} = \begin{bmatrix} 1 & 2 & 8 \\ 2 & 2 & 8 \\ 8 & 8 & 8 \end{bmatrix}.$$

Each element in $D_{1,3}$ and $D_{2,3}$ belongs to $\{1, 2, 8\}$. Therefore, such a family τ is a topology on X. In this way and by Algorithm 1, we can obtain six, nine, six, six, zero, and one topology with one, two, three, four, five, and six nonempty proper subsets of X, respectively. Therefore, there are twenty-nine topologies on X in total.

6. Problems

The method proposed above is helpful to enumerate all finite topologies on a finite set and puts forward some new ideas for scholars to study this field. However, several problems have not been solved.

Problem 1. *Given an arbitrary integer $n > 3$, how can we obtain the structure matrices $M_\cup$ and $M_\cap$ on a set X with n elements quickly?*

The solution of Problem 1 will help enumerate the topologies on a finite set by Algorithm 1. Moreover, we hope to find a similar method to enumerate the T_0 topologies as we have shown in Section 5 for general topologies.

Problem 2. *Is it possible to enumerate all the T_0 topologies with the help of the structure matrices $M_\cup$ and $M_\cap$ on a finite set X?*

7. Conclusions

In this paper, enumerating topologies on a finite set was investigated from an algebraic point of view. Through the STP of matrices, the algebraic representation of the subset and complement of a finite set and the corresponding structure matrices were obtained, respectively. In addition, the relationship between the intersection and union and intersection and minus of structure matrices was investigated, and a series of inferences was given. Finally, the corresponding algorithms and examples were given for the calculation of the number of topologies on a finite set based on the intersection and union of structure matrices. It is noted that all the calculations involved are matrix operations, which can be easily proven and applied by mathematical software. The method proposed in this paper is helpful to enumerate all finite topologies on a set, which gets rid of the limitation that the traditional enumeration method makes the computation of topologies on a set of n elements difficult to achieve because of the high time complexity and lays a good foundation for future topology research.

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