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Functionally Graded Thin Circular Plates with Different Moduli in Tension and Compression: Improved Föppl–von Kármán Equations and Its Biparametric Perturbation Solution

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Abstract: The biparametric perturbation method is applied to solve the improved Föppl–von Kármán equation, in which the improvements of equations come from two different aspects: the first aspect concerns materials, and the other is from deformation. The material considered in this study has bimodular functionally graded properties in comparison with the traditional materials commonly used in classical Föppl–von Kármán equations. At the same time, the consideration for deformation deals with not only the large deflection as indicated in classical Föppl–von Kármán equations, but also the larger rotation angle, which is incorporated by adopting the precise curvature formulas but not the simple second-order derivative term of the deflection. To fully demonstrate the effectiveness of the biparametric perturbation method proposed, two sets of parameter combinations, one being a material parameter with central defection and the other being a material parameter with load, are used for the solution of the improved Föppl–von Kármán equations. Results indicate that not only the two sets of solutions from different parameter combinations are consistent, but also they may be reduced to the single-parameter perturbation solution obtained in our previous study. The successful application of the biparametric perturbation method provides new ideas for solving similar nonlinear differential equations.

Keywords: biparametric perturbation; Föppl–von Kármán equation; bimodular materials; functionally graded materials; circular plate

MSC: 34E10; 74K20

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1. Introduction

Poincaré's perturbation method [1] is one of the standard analytical methods, which is used for the solution of nonlinear problems in applied mechanics. This method consists of the development of the solution of an initial or boundary value problem in an asymptotic series of a parameter. This parameter either appears explicitly in the problem or is introduced artificially. In Poincaré's earlier work and the subsequent development of his original ideas, we may find some interesting progresses for conceivable further generalizations [2]. In the singular perturbation scheme, one of the important developments, namely, strained coordinates method [3], stems originally from Poincaré's periodic solutions of nonlinear ordinary differential equations by the straining of the independent coordinates. Another interesting generalization from Poincaré's original idea is the so-called multiparameter perturbation technique. The ordinary perturbation technique with a single parameter is extended to embrace the expansions of two or more parameters. These parameters involved may be of different characters: some, for example, describing the properties of materials, while others describing the dynamic or geometrical nature of the problem. In this study, we focus on the application of a multiparameter perturbation technique in the field of solid mechanics. More explicitly, we try to use this method to analyze a large deformation problem of thin plates with certain material properties, for example, with bimodular functionally graded properties, that is, to use a multiparameter perturbation method to solve improved Föppl–von Kármán equations.

In the perturbation problem, to conduct a linear analysis is the first step, and the perturbation occurs in the neighborhood of the solution of linearized equations. Thus, based on the known solution of a linear system, the solution of a perturbation system may be obtained. Briefly speaking, the typical practice of the perturbation method is to expand the solution in ascending powers of a known parameter, and the unknown functions in the solution are gradually determined by decomposing the governing equation and the corresponding boundary conditions; thus, the approximate solution is finally obtained according to the required computational precision.

As suggested by Poincaré, the perturbation expansion should be regarded as an asymptotic series in essence; thus, the convergence of a perturbation solution may not be discussed. Subsequent studies also showed that the perturbation solution may not rely on any small parameter [4]. Despite of this, the parameter selected plays an important role during perturbation because the appropriate selection enables us to obtain asymptotic solutions with better convergence. The early pioneer works on flexible thin plates may be traced back to studies by Vincent [5] and Chien [6]. Selecting the external load as a perturbation parameter, Vincent [5] obtained a perturbation solution of large-deflection thin plates first. Considering that the perturbation parameter either appears explicitly or is introduced artificially in the problem, Chien [6] obtained another perturbation solution by selecting the central deflection as the perturbation parameter. In Chien's solution, the important relationship of load vs. central deflection is explicit due to the appropriate selection for a parameter. For a long period of time, Chien's solution has been cited as a classical work in many subsequent studies. In addition to load and central deflection, there exist several alternatives for perturbation parameters, for example, a generalized displacement [7], a linear function of Poisson's ratio [8], and an average angular deflection [9]. For these possible perturbation parameters, Chen and Kuang [10] discussed differences of solution.

When encountering difficulties in parameter selection, usually, we will solve it via two entirely different approaches. The first is the nonparametric perturbation method suggested by Chen [11,12], in which the physical meaning of a parameter is uncertain in advance, thus naturally eliminating the empirical factors in the parameter selection. That is to say that the parameter selection is not restricted, and thus, it is "free", so it is also called a free parameter perturbation method. The basic idea of another approach is by doing the opposite; that is, if in a real problem, there are two or more parameters that may be selected as perturbation parameters, these available parameters may be included in the perturbation together. The multiparameter perturbation method, thus, appears, as mentioned at the very beginning. The earlier work may be traced back to a study by Nowinski and Ismail [13], who solved the large deflection problem of elastic anisotropic plates by selecting the anisotropy of materials and the load as two perturbation parameters. The pioneering application of the multiparameter perturbation method to beam problems was from Chien [14], who successfully solved the classical Euler–Bernoulli equation for the first time by selecting the load and the height difference of end supports as two perturbation parameters. Later, by simplifying the governing equation, He and Chen [15] derived a biparametric perturbation solution for the same problem. To generalize the application of the biparametric perturbation method to beam problems, He et al. [16] further obtained the generalized perturbation solutions under various boundary conditions. Considering the diversity of materials, He et al. [17,18] also used the biparametric perturbation method to solve the large-deflection thin plate problem, in which the classical materials are extended to modern materials with certain advanced features, for example, with bimodular functionally graded properties. Totally speaking, the superiority of a multiparameter perturbation method is that many factors that have influences on the final result can be considered into the solution through perturbation. At the same time, with the advantage of the analytical solution itself, the influence of these factors can be easily seen in the solution, which greatly strengthens the convenience of subsequent parameter analysis.

Bimodular material refers to such a material that has the different elastic moduli in tension and compression. In fact, most materials [19,20], including graphite, concrete, ceramics, rubber, and some biomedical materials, will present different tensile and compressive strains when they are subjected to tensile and compressive stresses with the same magnitude. These materials are called, by Jones [21], bimodular materials or multimodulus materials. Basically, two material models are widely used in theoretical analysis in the field of engineering. The first is the Bert model [22], which is established on the criterion of positive–negative signs in the longitudinal strain of fibers. In the analysis of orthotropic materials and laminated composites [23–25], the Bert model is widely adopted. The second is the Ambartsumyan model [26], which is established on the criterion of principal stresses' positive–negative signs, and this model is mainly applicable to isotropic materials. In structural analysis, the Ambartsumyan model is of particular significance since principal stresses' positive–negative signs determine whether a certain point in a structure is in tension or in compression. Our present study is based on the Ambartsumyan model. However, due to the fact that principal stresses are generally obtained as a final result but not as a known condition before solving, it is difficult to describe the stress state of a point in advance. Besides, experimental results are also lacking in describing elastic coefficients in the complex states of stress. In a few simple problems, analytical solutions are available, only concerning beams and plates [27–29]. In some complex problems, we have to resort to the iterative technique-based finite element method (FEM). In each iteration, the principal stress state of each element needs to be judged in order to obtain a new elastic matrix used for the next iteration. This is the direct iterative method with variable stiffness, which was widely used in earlier studies, as indicated in the reviews from Ye et al. [30] and Sun et al. [31]. Thereafter, based on the improved constitutive model for different moduli and combined with the arc-length method, Ma et al. [32] established a finite element iterative program to determine buckling critical loads of rods with different moduli. Given that the traditional iteration methods are often difficult in convergence for such kind of constitutive model, Du et al. [33] established a new computational framework. Their numerical examples showed that the computational framework proposed can be used to analyze the wrinkling of thin plane membranes and explain some unusual cell mechanosensing phenomena.

Over the past few decades, functionally graded material (FGM) has become one of the important research topics in the engineering and technical fields, such as civil engineering, aerospace, acoustics, and microelectromechanical systems [34–37]. FGM is a new type of composite materials, composed of two or more materials whose composition usually presents continuously gradient changes, thus avoiding the interface effect. Most of the existing studies on structural elements made of FGM focus on beams and plates, but few consider the bimodular effect from FGM. However, as mentioned above, most materials can exhibit some bimodular effect, and it is just a matter of whether it is obvious or not, so FGM seems to be no exception.

Recently, more and more attention has been paid to the bimodular effect of materials in the analysis field of FGM and corresponding structures, especially bimodular FGM beams and plates. Aiming at the bimodular FGM plates, He et al. established the small deflection simplified theory based on the concept of neutral layer [38] and, thereafter, derived the equations governing the bimodular FGM thin circular plates with large deflections [39]. For a thin plate under large deformation, not only its deflection but also its rotation angle presents a relatively big value. For this purpose, Li et al. [40] used a single-parameter perturbation method to solve the Föppl–von Kármán equations without the small-rotation-angle assumption. On the other hand, some satisfactory progress has also been made in the application of the multiparameter perturbation method to similar problems. First, He et al. [17,18] solved the Föppl–von Kármán equations of thin plates with bimodular functionally graded properties by using the biparametric perturbation method. Thereafter, Yang et al. [41] solved the bending problem of piezoelectric cantilever beams by using the multiparameter perturbation method; He et al. [42] used the same method to solve the problem of functionally graded, thin, circular piezoelectric plates, in which three piezoelectric parameters were selected as perturbation parameters. These works showed that the study of the multiparameter perturbation method is continuously growing. For the improved Föppl–von Kármán equation that considers the precise curvature formulas from deformation and the bimodular functionally graded properties from materials, however, the application of this method has not been reported.

In this study, the improved Föppl–von Kármán equation was solved by using the biparametric perturbation method. The main aim of this paper is to investigate the application of this method to the improved Föppl–von Kármán equations, while the precise curvature and bimodular functionally graded effect on the mechanical performance of thin plates do not fall within the scope of this study temporarily. For this purpose, the whole paper is organized as follows: In the next Section 2, the establishment of improved Föppl–von Kármán equations is briefly described. For the effective implementation of the biparametric perturbation method, in Section 3, the relevant parameter variables in the established equations are expanded into polynomials of the material parameters, and two sets of perturbation parameter combinations are used for the perturbation. The comparison of two sets of biparametric solutions and the comparison with a single-parameter solution, as well as the regression verification, are presented in Section 4. The concluding remarks are summarized in Section 5.

2. Improved Föppl–von Kármán Equations

2.1. Establishment of Governing Equations

As shown in Figure 1, a bimodular FGM thin circular plate with thickness t and radius a is subjected to a transversely uniformly distributed load q, in which the location of an unknown neutral layer of a plate, which will be determined later, is represented by a dotted dash line at the peripheral of the plate. The neutral layer generally does not coincide with the geometrical middle plane of the plate due to the introduction of bimodular functionally graded materials. The origin o of a cylindrical coordinate system (r, ϕ, z) is placed at the plate center on the neutral layer, and r, ϕ, and z are the radial, circumferential, and transverse coordinates, respectively. Due to the axisymmetry, ϕ is ignored in Figure 1. The heights of the tensile and compressive zones are denoted by t_1 and t_2, respectively, and the corresponding modulus of the two zones is the tensile and compressive modulus, $E^+(z)$ and $E^-(z)$, which are the function of z due to the functionally graded property along the thickness direction. In order not to lose the generality, the edge constraint condition is temporarily not given.

Considering the convenience of integral and differential operations, $E^+(z)$ and $E^-(z)$ are defined as the following exponent type functions [38]:

$$E^+(z) = E_0 e^{\alpha_1 z/t},\ E^-(z) = E_0 e^{\alpha_2 z/t}, \tag{1}$$

where α_1 and α_2 are the two graded indices of tensile and compressive zones, and E_0 stands for Young's modulus of elasticity of the neutral layer. From Equation (1), it is easy to find that $E^+(z) = E^-(z) = E_0$ when $\alpha_1 = \alpha_2 = 0$ or $z = 0$. At the same time, following the general practice, Poisson's ratio is assumed as two constants, v^+ and v^-, ignoring the variation along the z direction.

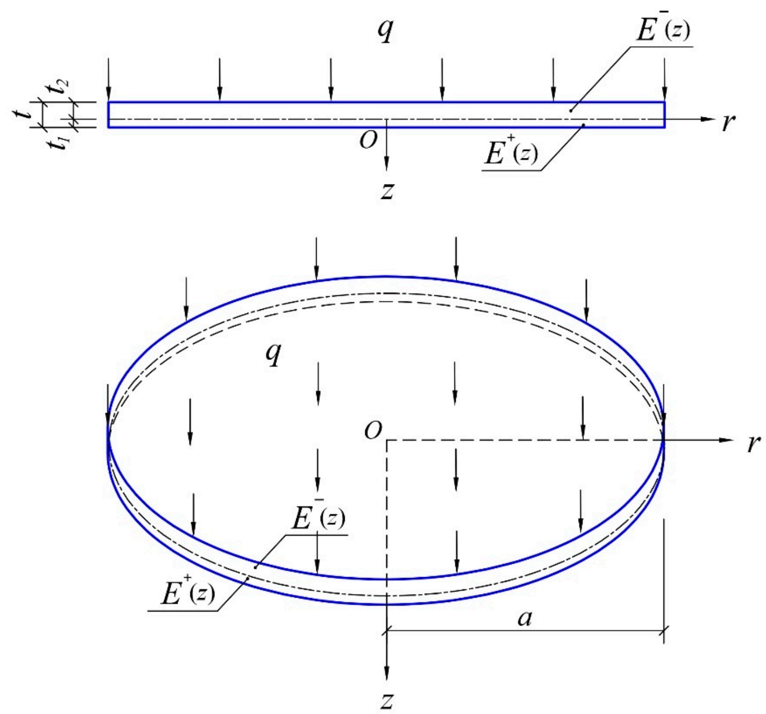

Figure 1. The bimodular FGM thin circular plate under a uniformly distributed load.

If we isolate a differential element body from the plate shown in Figure 1 and study its equilibrium conditions, the following three equations of equilibrium may be obtained:

$$\frac{d}{dr}(rN_r) - N_\theta = 0, \tag{2}$$

$$\frac{d}{dr}(rN_r \sin\beta) + \frac{d}{dr}(rQ_r) - qr = 0, \tag{3}$$

and

$$\frac{d}{dr}(rM_r) - M_\theta + rQ_r = 0, \tag{4}$$

in which N_r and N_θ are the radial and circumferential force, respectively; M_r and M_θ are the bending moment along the radial and circumferential direction, respectively; Q_r is the transverse shear force; and β denotes the rotation of the radial force.

If we let $\sigma_r^{+/-}$ and $\sigma_\theta^{+/-}$ be the radial and circumferential stresses in tensile and compressive zones, and ε_r and ε_θ be the radial and circumferential strain, respectively, the strain–stress relations give

$$\left\{ \begin{array}{l} \sigma_r^{+/-} = \frac{E^{+/-}(z)}{1-(\nu^{+/-})^2}(\varepsilon_r + \nu\varepsilon_\theta) \\ \sigma_\theta^{+/-} = \frac{E^{+/-}(z)}{1-(\nu^{+/-})^2}(\varepsilon_\theta + \nu\varepsilon_r) \end{array} \right. . \tag{5}$$

In addition, the geometrical equations are

$$\left\{ \begin{array}{l} \varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{z}{\rho_r} \\ \varepsilon_\theta = \frac{u}{r} + \frac{z}{\rho_\theta} \end{array} \right. , \tag{6}$$

in which u and w are the radial displacement and transverse displacement or deflection, respectively, and the precise curvatures for $1/\rho_r$ and $1/\rho_\theta$ are

$$\begin{cases} \frac{1}{\rho_r} = -\frac{d^2w}{dr^2}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-3/2} \\ \frac{1}{\rho_\theta} = \frac{1}{r}\sin\beta = -\frac{1}{r}\frac{dw}{dr}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-1/2} \end{cases}, \tag{7}$$

in which $\tan\beta = -dw/dr$.

Substituting Equation (6) into Equation (5) yields

$$\begin{cases} \sigma_r^{+/-} = \frac{E^{+/-}(z)}{1-(\nu^{+/-})^2}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu^{+/-}\frac{u}{r}+z\left(\frac{1}{\rho_r}+\nu^{+/-}\frac{1}{\rho_\theta}\right)\right] \\ \sigma_\theta^{+/-} = \frac{E^{+/-}(z)}{1-(\nu^{+/-})^2}\left[\frac{u}{r}+\nu^{+/-}\frac{du}{dr}+\frac{\nu^{+/-}}{2}\left(\frac{dw}{dr}\right)^2+z\left(\frac{1}{\rho_\theta}+\nu^{+/-}\frac{1}{\rho_r}\right)\right] \end{cases}. \tag{8}$$

The radial and circumferential force, N_r and N_θ, are the sum of integrals in tensile zone and compressive areas, that is,

$$\begin{cases} N_r = \int_0^{t_1}\sigma_r^+dz + \int_{-t_2}^0\sigma_r^-dz \\ N_\theta = \int_0^{t_1}\sigma_\theta^+dz + \int_{-t_2}^0\sigma_\theta^-dz \end{cases}. \tag{9}$$

After substituting Equation (8) into Equation (9), N_r and N_θ may be computed as

$$\begin{cases} N_r = A_0\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu^+\frac{u}{r}\right] \\ N_\theta = A_0\left[\frac{u}{r}+\nu^+\frac{du}{dr}+\nu^+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right] \end{cases}, \tag{10}$$

where

$$A_0 = \int_{-t_2}^{t_1}E^+(z)dz = \int_{-t_2}^{t_1}E_0e^{\alpha_1 z/t}dz = \frac{E_0t}{\alpha_1}\left(\frac{e^{\alpha_1}-1}{e^{\alpha_1 t_2/t}}\right). \tag{11}$$

It should be pointed out that any point of the plate is stretched along the radial and circumferential directions when the plate is under large deflection. Thus, all integrals along the z direction should be calculated, based only on the tensile components $E^+(z)$. This is the reason that the only integrand is $E^+(z)$. Besides, the integrals of the items containing z are also equal to zero in Equation (9).

According to a bimodular theory, M_r and M_θ may be computed as, in the form of a subarea integral:

$$\begin{cases} M_r = \int_0^{t_1}\sigma_r^+zdz + \int_{-t_2}^0\sigma_r^-zdz \\ M_\theta = \int_0^{t_1}\sigma_\theta^+zdz + \int_{-t_2}^0\sigma_\theta^-zdz \end{cases}. \tag{12}$$

Substituting Equation (8) into Equation (12), also considering Equation (1), will yield

$$\begin{cases} M_r = A_1^+\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu^+\frac{u}{r}\right]+A_2^+\left(\frac{1}{\rho_r}+\frac{\nu^+}{\rho_\theta}\right) \\ \qquad +A_1^-\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu^-\frac{u}{r}\right]+A_2^-\left(\frac{1}{\rho_r}+\frac{\nu^-}{\rho_\theta}\right) \\ M_\theta = A_1^+\left[\frac{u}{r}+\nu^+\frac{du}{dr}+\nu^+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]+A_2^+\left(\frac{1}{\rho_\theta}+\nu^+\frac{1}{\rho_r}\right) \\ \qquad +A_1^-\left[\frac{u}{r}+\nu^+\frac{du}{dr}+\nu^-\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]+A_2^-\left(\frac{1}{\rho_\theta}+\nu^-\frac{1}{\rho_r}\right) \end{cases}, \tag{13}$$

where

$$\begin{cases} A_1^+ = \frac{\int_0^{t_1} zE_0 e^{\alpha_1 z/t}dz}{1-(\nu^+)^2} = \frac{1}{1-(\nu^+)^2}\left[\frac{E_0 t^2}{\alpha_1^2} + \left(\frac{t_1 t}{\alpha_1} - \frac{t^2}{\alpha_1^2}\right)E_0 e^{\alpha_1 t_1/t}\right] \\ A_1^- = \frac{\int_{-t_2}^{0} zE_0 e^{\alpha_2 z/t}dz}{1-(\nu^-)^2} = \frac{1}{1-(\nu^-)^2}\left[-\frac{E_0 t^2}{\alpha_2^2} + \left(\frac{t_2 t}{\alpha_2} + \frac{t^2}{\alpha_2^2}\right)E_0 e^{-\alpha_2 t_2/t}\right] \\ A_2^+ = \frac{\int_0^{t_1} z^2 E_0 e^{\alpha_1 z/t}dz}{1-(\nu^+)^2} = \frac{1}{1-(\nu^+)^2}\left[\left(\frac{2t^3}{\alpha_1^3} + \frac{t_1^2 t}{\alpha_1} - \frac{2t^2 t_1}{\alpha_1^2}\right)E_0 e^{\alpha_1 t_1/t} - \frac{2E_0 t^3}{\alpha_1^3}\right] \\ A_2^- = \frac{\int_{-t_2}^{0} z^2 E_0 e^{\alpha_2 z/t}dz}{1-(\nu^-)^2} = \frac{1}{1-(\nu^-)^2}\left[-\left(\frac{2t^3}{\alpha_2^3} + \frac{t_2^2 t}{\alpha_2} + \frac{2t^2 t_2}{\alpha_2^2}\right)E_0 e^{-\alpha_2 t_2/t} + \frac{2E_0 t^3}{\alpha_2^3}\right] \end{cases}. \tag{14}$$

Note that the integral of the items containing z in Equation (12) has been determined as zero, that is, $A_1^+ + A_1^- = 0$, since it is exactly the condition used for determining the unknown neutral layer, according to our previous study [38]. Thus, M_r and M_θ can be further rewritten as

$$\begin{cases} M_r = A_2^+\left(\frac{1}{\rho_r} + \frac{\nu^+}{\rho_\theta}\right) + A_2^-\left(\frac{1}{\rho_r} + \frac{\nu^-}{\rho_\theta}\right) \\ M_\theta = A_2^+\left(\frac{1}{\rho_\theta} + \frac{\nu^+}{\rho_r}\right) + A_2^-\left(\frac{1}{\rho_\theta} + \frac{\nu^-}{\rho_r}\right) \end{cases}. \tag{15}$$

Now, Equations (2)–(4), (10), and (15), seven in total, may be used for u, w, N_r, N_θ, M_r, M_θ, and Q_r, and the problem is solvable.

First, eliminating the term Q_r via Equations (3) and (4), and then substituting Equations (10) and (15) into the derived equation and also considering the precise curvature formulas shown in Equation (7), we may obtain the equilibrium equation of the improved Föppl–von Kármán equations as follows:

$$\begin{aligned} &D^* \frac{d^2}{dr^2}\left\{r\frac{d^2 w}{dr^2}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-3/2} + \nu^{+/-}\frac{dw}{dr}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-1/2}\right\} \\ &-D^*\frac{d}{dr}\left\{\frac{1}{r}\frac{dw}{dr}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-1/2} + \nu^{+/-}\frac{d^2 w}{dr^2}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-3/2}\right\} \\ &+\frac{d}{dr}\left\{-rN_r\frac{dw}{dr}\left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-1/2}\right\} - qr = 0 \end{aligned} \tag{16}$$

where

$$D^* = A_2^+ + A_2^-. \tag{17}$$

Note if we neglect the bending stiffness D^*, Equation (16) will be reduced to the out-plane equation of the equilibrium of a circular membrane problem without the small-rotation-angle assumption, as shown in Equation (3) in our previous study [43,44].

At the same time, from Equations (2) and (10), we have

$$\frac{u}{r} = \frac{1}{1-(\nu^+)^2}\frac{1}{A_0}(N_\theta - \nu^+ N_r) = \frac{1}{1-(\nu^+)^2}\frac{1}{A_0}\left[\frac{d}{dr}(rN_r) - \nu^+ N_r\right]. \tag{18}$$

Substituting u in Equation (18) into the first formula of Equation (10) yields

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2 N_r\right)\right] + \frac{A_0}{2}\left(\frac{dw}{dr}\right)^2 = 0. \tag{19}$$

which is the compatible equation of the improved Föppl–von Kármán equations.

Lastly, Equations (16) and (19) constitute the improved Föppl–von Kármán equations, which consider the precise curvature formulas from deformation and the bimodular functionally graded effect from materials. Specifically, when the bimodular functionally graded properties of the materials disappear, A_0 in Equation (19) is reduced to $E_0 t$, and D^* in Equation (16) is reduced to the familiar expression, $D = E_0 t^3/[12(1-\nu^2)]$. At the same time, the term 1 + $(-dw/dr)^2$ in Equation (16) is approximated as 1 under the small-rotation-angle assumption. Eventually, Equations (16) and (19) are reduced to the classical Föppl–von Kármán equations as follows:

$$\begin{cases} D\left(\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right)-\frac{1}{r}\frac{d}{dr}\left(rN_r\frac{dw}{dr}\right)=q \\ r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2N_r\right)\right]+\frac{E_0t}{2}\left(\frac{dw}{dr}\right)^2=0 \end{cases}, \tag{20}$$

which verifies the correctness of the equations established.

2.2. Equation Simplification and Boundary Conditions

Note that in Equation (16), there are many nonlinear items concerning 1 + $(-dw/dr)^2$, and their presence will hinder solving the equations; therefore, it is necessary to make a moderate simplification. To this end, the negative exponential terms in Equation (16) are expanded in the form of the power series of $(-dw/dr)^2$, that is,

$$\begin{cases} \left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-3/2}=1-\frac{3}{2}\left(-\frac{dw}{dr}\right)^2+\ldots \\ \left[1+\left(-\frac{dw}{dr}\right)^2\right]^{-1/2}=1-\frac{1}{2}\left(-\frac{dw}{dr}\right)^2+\ldots \end{cases} \tag{21}$$

Substituting them into Equation (16), we have

$$\begin{aligned} & D^*\frac{d^2}{dr^2}\left\{r\frac{d^2w}{dr^2}\left[1-\frac{3}{2}\left(-\frac{dw}{dr}\right)^2\right]+\nu^{+/-}\frac{dw}{dr}\left[1-\frac{1}{2}\left(-\frac{dw}{dr}\right)^2\right]\right\} \\ & -D^*\frac{d}{dr}\left\{\frac{1}{r}\frac{dw}{dr}\left[1-\frac{1}{2}\left(-\frac{dw}{dr}\right)^2\right]+\nu^{+/-}\frac{d^2w}{dr^2}\left[1-\frac{3}{2}\left(-\frac{dw}{dr}\right)^2\right]\right\} \\ & +\frac{d}{dr}\left\{-rN_r\frac{dw}{dr}\left[1-\frac{1}{2}\left(-\frac{dw}{dr}\right)^2\right]\right\}-qr=0 \end{aligned} \tag{22}$$

Integrating Equation (22) and also considering the symmetry conditions, $dw/dr = 0$ and $N_r = 0$ at $r = 0$, will yield

$$\begin{aligned} & D^*\left[r\frac{d^3w}{dr^3}+\frac{d^2w}{dr^2}-\frac{1}{r}\frac{dw}{dr}-3r\left(\frac{d^2w}{dr^2}\right)^2\frac{dw}{dr}-\frac{3}{2}r\frac{d^3w}{dr^3}\left(\frac{dw}{dr}\right)^2-\frac{3}{2}\frac{d^2w}{dr^2}\left(\frac{dw}{dr}\right)^2+\frac{1}{2r}\left(\frac{dw}{dr}\right)^3\right] \\ & =\frac{1}{2}qr^2+rN_r\frac{dw}{dr}-\frac{1}{2}rN_r\left(\frac{dw}{dr}\right)^3 \end{aligned} \tag{23}$$

while the counterpart in the classical Föppl–von Kármán equation is

$$D\left(r\frac{d^3w}{dr^3}+\frac{d^2w}{dr^2}-\frac{1}{r}\frac{dw}{dr}\right)-rN_r\frac{dw}{dr}=\frac{1}{2}qr^2. \tag{24}$$

Obviously, there are many nonlinear items generated in the equation due to the introduction of precise curvature formulas.

We consider the following four edge constraints, that is, at $r = a$:

(i) Rigidly clamped,

$$w=0,\ \frac{dw}{dr}=0,\ u=0 \tag{25a}$$

(ii) Movably clamped,

$$w=0,\ \frac{dw}{dr}=0,\ N_r=0 \tag{25b}$$

(iii) Simply hinged,

$$w = 0,\ M_r = 0,\ u = 0 \tag{25c}$$

(iv) Simply supported,

$$w = 0,\ M_r = 0,\ N_r = 0. \tag{25d}$$

Via Equation (18), the condition of $u = 0$ may be expressed in terms of N_r, that is,

$$r\frac{dN_r}{dr} + (1-\nu^+)N_r = 0. \tag{26}$$

Via the first formula of Equation (15), the condition of $M_r = 0$ may be expressed in terms of w, that is,

$$A_2^+\left(\frac{d^2w}{dr^2} + \frac{\nu^+}{r}\frac{dw}{dr}\right) + A_2^-\left(\frac{d^2w}{dr^2} + \frac{\nu^-}{r}\frac{dw}{dr}\right) = 0, \tag{27}$$

in which $1 + (-dw/dr)^2 \approx 1$ is adopted to simplify the derivation.

3. Application of Biparametric Perturbation Method

3.1. Nondimensionalization and Perturbation Preparation

We first introduce the dimensionless quantities as follows:

$$P = \frac{qa^4}{E_0t^4},\ \eta = 1 - \frac{r^2}{a^2},\ W = \frac{w}{t},\ T = \frac{t}{a},\ S = \frac{N_ra^2}{E_0t^3}, \tag{28}$$

and

$$K = \frac{D^*}{E_0t^3} = \frac{K^+ + K^-}{E_0t^3} = \frac{A_2^+ + A_2^-}{E_0t^3},\ V = \frac{[1-(\nu^+)^2]A_0}{E_0t}. \tag{29}$$

Equations (19) and (23) are transformed into

$$\frac{d^2}{d\eta^2}[(1-\eta)S] + \frac{V}{2}\left(\frac{dW}{d\eta}\right)^2 = 0 \tag{30}$$

and

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW}{d\eta} - 2T^2(1-\eta)^2\left(\frac{dW}{d\eta}\right)^3\right] = -\frac{P}{16K} + \frac{S}{4K}\frac{dW}{d\eta} - \frac{T^2S}{2K}\left(\frac{dW}{d\eta}\right)^3. \tag{31}$$

Considering Equations (10) and (27)–(29), the third of the boundary conditions, that is, Equation (25c), may be changed as

$$W = 0,\ \lambda_1\frac{d^2W}{d\eta^2} - \frac{dW}{d\eta} = 0,\ \lambda_2\frac{dS}{d\eta} - S = 0\ \ at\ \ \eta = 0, \tag{32}$$

where λ_1 and λ_2 are two parameters newly introduced, and they are [29]

$$\lambda_1 = \frac{2K}{K^+(1+\nu^+) + K^-(1+\nu^-)},\ \lambda_2 = \frac{2}{1-\nu^+}. \tag{33}$$

According to our previous study [29], other three boundary conditions (25a,b,d) may be obtained simply by prescribing λ_1 and λ_2 as zero, for example, for rigidly clamped, $\lambda_1 = 0$; for movably clamped, $\lambda_1 = \lambda_2 = 0$; and for simply supported, $\lambda_2 = 0$. Thus, in the next solving, what we need to do is to seek the solution of Equations (30) and (31) under the general boundary conditions, that is, Equation (32).

Besides, at the plate center, the axisymmetric conditions give

$$\frac{dW}{d\eta} = 0(\neq \ \infty) \ and \ S \neq \ \infty, \ at \ \eta = 1. \tag{34}$$

Note that there are two important physical quantities, V and K, in Equations (30) and (31). Among the parameters that constitute V and K, there could be parameters that we prefer to select, for example, the graded index in the tensile zone, α_1. Therefore, according to our perturbation experiences, V and K need to be expanded as, with respect to α_1, [18]

$$\begin{cases} \frac{1}{K} = \frac{E_0 t^3}{D^+ + D^-} = l\left(1 + m_1\alpha_1 + m_2\alpha_1^2 + m_3\alpha_1^3 + \cdots\right) \\ V = \frac{e^{\alpha_1} - 1}{\alpha_1 e^{\alpha_1 T_2}} = 1 + n_1\alpha_1 + n_2\alpha_1^2 + n_3\alpha_1^3 + \cdots \end{cases}, \tag{35}$$

where

$$\begin{cases} l = \frac{3\left[1-(\mu^+)^2\right]}{T_1{}^2}, m_1 = \frac{-T_1{}^2 + kT_2{}^2}{2}, \\ m_2 = \frac{\left(-T_1{}^2 + kT_2{}^2\right)^2}{4} - \frac{T_1{}^3 + k^2T_2{}^3}{2}, \ m_3 = \frac{(T_1{}^2 - kT_2{}^2)(T_1{}^3 + k^2T_2{}^3)}{2} - \frac{(T_1{}^2 - kT_2{}^2)^3}{8}, \\ n_1 = \frac{1}{2} - T_2, n_2 = \frac{1}{6} - \frac{T_2}{2} + \frac{T_2^2}{2}, n_3 = \frac{1}{24} - \frac{T_2}{6} + \frac{T_2^2}{4} - \frac{T_2^3}{6} \end{cases} \tag{36}$$

where $k = \alpha_1/\alpha_2$, and $k \neq 1$. Now, Equations (30) and (31) are further modified as

$$\frac{d^2}{d\eta^2}[(1-\eta)S] + \frac{1}{2}\left(1 + n_1\alpha_1 + n_2\alpha_1{}^2 + n_3\alpha_1{}^3 + \cdots\right)\left(\frac{dW}{d\eta}\right)^2 = 0 \tag{37}$$

and

$$\begin{gathered} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW}{d\eta} - 2T^2(1-\eta)^2\left(\frac{dW}{d\eta}\right)^3\right] \\ = l\left(1 + m_1\alpha_1 + m_2\alpha_1{}^2 + m_3\alpha_1{}^3 + \cdots\right)\left[\frac{S}{4}\frac{dW}{d\eta} - \frac{P}{16} - \frac{T^2 S}{2}\left(\frac{dW}{d\eta}\right)^3\right] \end{gathered} \tag{38}$$

3.2. Biparametric Perturbation on α_1 with W_m

The two parameters, α_1 and W_m, are selected as the perturbation parameters, in which α_1 is the graded index in the tensile zone and W_m is the dimensionless central deflection, that is,

$$W_m = (W)_{\eta=1} = \left(\frac{w}{t}\right)_{r=0} = \frac{w_0}{t}. \tag{39}$$

P, S, and W in Equations (30) and (31) are expressed in the perturbation parameter,

$$P = P(\alpha_1, W_m), \ W = W(\alpha_1, W_m, \eta), \ S = S(\alpha_1, W_m, \eta). \tag{40}$$

Thus, P, S, and W are expanded in the power series of α_1 and W_m,

$$\begin{aligned} \frac{P}{16} &= P_1\alpha_1 + P_2W_m + P_3\alpha_1{}^2 + P_4\alpha_1W_m + P_5W_m{}^2 + P_6\alpha_1{}^3 + P_7\alpha_1{}^2W_m + P_8\alpha_1W_m{}^2 + P_9W_m{}^3 \\ &+P_{10}\alpha_1{}^4 + P_{11}\alpha_1{}^3W_m + P_{12}\alpha_1{}^2W_m{}^2 + P_{13}\alpha_1W_m{}^3 + P_{14}W_m{}^4 + P_{15}\alpha_1{}^5 + P_{16}\alpha_1{}^4W_m \\ &+P_{17}\alpha_1{}^3W_m{}^2 + P_{18}\alpha_1{}^2W_m{}^3 + P_{19}\alpha_1W_m{}^4 + P_{20}W_m{}^5 + \cdots \end{aligned} \tag{41}$$

$$\begin{aligned} W = \ & W_1\alpha_1 + W_2W_m + W_3\alpha_1{}^2 + W_4\alpha_1W_m + W_5W_m{}^2 + W_6\alpha_1{}^3 + W_7\alpha_1{}^2W_m + W_8\alpha_1W_m{}^2 \\ &+W_9W_m{}^3 + W_{10}\alpha_1{}^4 + W_{11}\alpha_1{}^3W_m + W_{12}\alpha_1{}^2W_m{}^2 + W_{13}\alpha_1W_m{}^3 + W_{14}W_m{}^4 + W_{15}\alpha_1{}^5 \\ &+W_{16}\alpha_1{}^4W_m + W_{17}\alpha_1{}^3W_m{}^2 + W_{18}\alpha_1{}^2W_m{}^3 + W_{19}\alpha_1W_m{}^4 + W_{20}W_m{}^5 + \cdots \end{aligned} \tag{42}$$

and

$$\begin{aligned} S = \ & S_1\alpha_1 + S_2W_m + S_3\alpha_1{}^2 + S_4\alpha_1W_m + S_5W_m{}^2 + S_6\alpha_1{}^3 + S_7\alpha_1{}^2W_m + S_8\alpha_1W_m{}^2 \\ &+S_9W_m{}^3 + S_{10}\alpha_1{}^4 + S_{11}\alpha_1{}^3W_m + S_{12}\alpha_1{}^2W_m{}^2 + S_{13}\alpha_1W_m{}^3 + S_{14}W_m{}^4 + S_{15}\alpha_1{}^5 \\ &+S_{16}\alpha_1{}^4W_m + S_{17}\alpha_1{}^3W_m{}^2 + S_{18}\alpha_1{}^2W_m{}^3 + S_{19}\alpha_1W_m{}^4 + S_{20}W_m{}^5 + \cdots \end{aligned} \tag{43}$$

where P_i (i = 1, 2, 3 ...) are undetermined constants, and $W_i(\eta)$ and $S_i(\eta)$ (i = 1, 2, 3 ...) are unknown functions with respect to η. In the expansion of P, the introduction of 1/16 can make the next calculation easier. Substituting Equations (41)–(43) into Equations (37) and (38), and also into Equations (32) and (34), a series of decomposed differential equations and the corresponding conditions used for solving P_i, $W_i(\eta)$, and $S_i(\eta)$ may be obtained.

(I) First-order approximation

(i) The differential equation used for the solution of $S_i(\eta)$ (i = 1, 2) can be obtained from the coefficient of α_1 and W_m in Equation (37):

$$\frac{d^2}{d\eta^2}[(1-\eta)S_i] = 0, \tag{44}$$

which should satisfy the boundary conditions (i = 1, 2)

$$\begin{cases} \lambda_2 \frac{dS_i}{d\eta} - S_i = 0 \ \ at\ \eta = 0 \\ S_i \neq \infty \ at\ \eta = 1 \end{cases}; \tag{45}$$

Thus, the solution give

$$S_1 = S_2 = 0. \tag{46}$$

(ii) The differential equation used for the solution of P_i and $W_i(\eta)$ (i = 1, 2) can be obtained from the coefficient of α_1 and W_m in Equation (38):

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_i}{d\eta}\right] = -lP_i \tag{47}$$

which should be solved under (i = 1, 2)

$$\begin{cases} W_i = 0,\ \lambda_1 \frac{d^2W_i}{d\eta^2} - \frac{dW_i}{d\eta} = 0 \ at\ \eta = 0 \\ W_1 = 0, W_2 = 1, \frac{dW_i}{d\eta} \neq \infty \ at\ \eta = 1 \end{cases}; \tag{48}$$

Thus, we obtain

$$\begin{cases} P_1 = 0, P_2 = \frac{4}{l(2\lambda_1+1)} \\ W_1 = 0, W_2 = \frac{\eta^2+2\lambda_1\eta}{2\lambda_1+1} \end{cases}. \tag{49}$$

(II) Second-order approximation

(i) The differential equation used for the solution of $S_i(\eta)$ (i = 3, 4, 5) can be obtained from the coefficient of $\alpha_1^2, \alpha_1 W_m, W_m^2$ in Equation (37):

$$\begin{cases} \frac{d^2}{d\eta^2}[(1-\eta)S_3] + \frac{1}{2}\left(\frac{dW_1}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_4] + \frac{dW_1}{d\eta}\frac{dW_2}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_5] + \frac{1}{2}\left(\frac{dW_2}{d\eta}\right)^2 = 0 \end{cases}, \tag{50}$$

which should satisfy the boundary conditions, that is, Equation (45), in which (i = 3, 4, 5); thus, the solution gives

$$\begin{cases} S_3 = S_4 = 0 \\ S_5 = \frac{1}{6(2\lambda_1+1)^2}[\eta^3 + (4\lambda_1+1)\eta^2 + (6\lambda_1{}^2 + 4\lambda_1 + 1)\eta + \lambda_2(6\lambda_1{}^2 + 4\lambda_1 + 1)] \end{cases}. \tag{51}$$

(ii) The differential equation used for the solution of P_i and $W_i(\eta)$ (i = 3, 4, 5) can be obtained from the coefficient of $\alpha_1^2, \alpha_1 W_m, W_m^2$ in Equation (38):

$$\left\{\begin{array}{l} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_3}{d\eta}\right] = l\frac{S_1}{4}\frac{dW_1}{d\eta} - l(P_3 + P_1 m_1) \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_4}{d\eta}\right] = l\left(\frac{S_1}{4}\frac{dW_2}{d\eta} + \frac{S_2}{4}\frac{dW_1}{d\eta}\right) - l(P_4 + P_2 m_1) \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_5}{d\eta}\right] = l\frac{S_2}{4}\frac{dW_2}{d\eta} - lP_5 \end{array}\right. \tag{52}$$

which should be solved under (i = 3, 4, 5)

$$\left\{\begin{array}{l} W_i = 0,\ \lambda_1\frac{d^2W_i}{d\eta^2} - \frac{dW_i}{d\eta} = 0\ at\ \eta = 0 \\ W_i = 0, \frac{dW_i}{d\eta} \neq \infty\ at\ \eta = 1 \end{array}\right. ; \tag{53}$$

Thus, we obtain

$$\left\{\begin{array}{l} P_3 = P_5 = 0, P_4 = \frac{-4m_1}{l(2\lambda_1+1)} \\ W_3 = W_4 = W_5 = 0 \end{array}\right. . \tag{54}$$

(III) Third-order approximation

(i) The differential equation used for the solution of $S_i(\eta)$ (i = 6, 7, 8, 9) can be obtained from the coefficient of $\alpha_1^3, \alpha_1^2 W_m, \alpha_1 W_m^2, W_m^3$ in Equation (37):

$$\left\{\begin{array}{l} \frac{d^2}{d\eta^2}[(1-\eta)S_6] + \frac{1}{2}n_1\left(\frac{dW_1}{d\eta}\right)^2 + \frac{dW_1}{d\eta}\frac{dW_3}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_7] + \frac{dW_1}{d\eta}\frac{dW_4}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_2}{d\eta} + n_1\frac{dW_1}{d\eta}\frac{dW_2}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_8] + \frac{dW_1}{d\eta}\frac{dW_5}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_4}{d\eta} + \frac{1}{2}n_1\left(\frac{dW_2}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_9] + \frac{dW_2}{d\eta}\frac{dW_5}{d\eta} = 0 \end{array}\right. , \tag{55}$$

which should satisfy the boundary conditions, that is, Equation (45), in which (i = 6, 7, 8, 9); thus, the solution gives

$$\left\{\begin{array}{l} S_6 = S_7 = S_9 = 0 \\ S_8 = \frac{n_1}{6(2\lambda_1+1)^2}[\eta^3 + (4\lambda_1+1)\eta^2 + (6\lambda_1{}^2 + 4\lambda_1 + 1)\eta + \lambda_2(6\lambda_1{}^2 + 4\lambda_1 + 1)] \end{array}\right. . \tag{56}$$

(ii) The differential equation used for the solution of P_i and $W_i(\eta)$ (i = 6, 7, 8, 9) can be obtained from the coefficient of $\alpha_1^3, \alpha_1^2 W_m, \alpha_1 W_m^2, W_m^3$ in Equation (38):

$$\left\{\begin{array}{l} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_6}{d\eta}\right] = l\left(\frac{S_1}{4}\frac{dW_3}{d\eta} + \frac{S_3}{4}\frac{dW_1}{d\eta} + m_1\frac{S_1}{4}\frac{dW_1}{d\eta}\right) - l(P_6 + P_3 m_1 + P_1 m_2) \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_7}{d\eta} - 6T^2(1-\eta)^2\left(\frac{dW_1}{d\eta}\right)^2\frac{dW_2}{d\eta}\right] \\ \quad = l\left(\frac{S_1}{4}\frac{dW_4}{d\eta} + \frac{S_2}{4}\frac{dW_3}{d\eta} + m_1\frac{S_2}{4}\frac{dW_1}{d\eta}\right) - l(P_7 + P_4 m_1 + P_2 m_2) \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_8}{d\eta} - 6T^2(1-\eta)^2\frac{dW_1}{d\eta}\left(\frac{dW_2}{d\eta}\right)^2\right] \\ \quad = l\left(\frac{S_1}{4}\frac{dW_5}{d\eta} + \frac{S_4}{4}\frac{dW_2}{d\eta} + m_1\frac{S_2}{4}\frac{dW_2}{d\eta}\right) - l(P_8 + P_5 m_1) \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_9}{d\eta} - 2T^2(1-\eta)^2\left(\frac{dW_2}{d\eta}\right)^3\right] = l\left(\frac{S_2}{4}\frac{dW_5}{d\eta} + m_1\frac{S_5}{4}\frac{dW_2}{d\eta}\right) - lP_9 \end{array}\right. \tag{57}$$

which should be solved under Equation (53), in which (i = 6, 7, 8, 9); thus, we obtain

$$\begin{cases} P_6 = P_8 = 0, P_7 = \frac{4(m_1^2 - m_2)}{l(2\lambda_1+1)} \\ P_9 = \frac{1}{1080l(2\lambda_1+1)^4} \begin{bmatrix} (69120T^2 + 1080l\lambda_2 + 360l){\lambda_1}^4 + \\ (-172800T^2 + 1620l\lambda_2 + 840l){\lambda_1}^3 + \\ (-34560T^2 + 1080l\lambda_2 + 825l){\lambda_1}^2 + \\ (-17280T^2 + 350l\lambda_2 + 388l)\lambda_1 - 3456T^2 + 50l\lambda_2 + 73l \end{bmatrix} \end{cases}. \tag{58a}$$

$$\begin{cases} W_6 = W_7 = W_8 = 0 \\ W_9 = -\frac{1}{4320(2\lambda_1+1)^4} \begin{bmatrix} (4\lambda_1 l + 2l)\eta^6 + \\ \left(27648T^2\lambda_1 + 36{\lambda_1}^2 l + 13824T^2 + 30\lambda_1 l + 6l\right)\eta^5 + \\ \begin{pmatrix} 103680T^2{\lambda_1}^2 + 150{\lambda_1}^3 l + 17280T^2\lambda_1 \\ +195{\lambda_1}^2 l - 17280T^2 + 90\lambda_1 l + 15l \end{pmatrix}\eta^4 + \\ \begin{pmatrix} 138240T^2{\lambda_1}^3 + 240l{\lambda_1}^4 + 240{\lambda_1}^3 l\lambda_2 - 69120T^2{\lambda_1}^2 \\ +480{\lambda_1}^3 l + 280{\lambda_1}^2 l\lambda_2 - 69120T^2\lambda_1 + 380{\lambda_1}^2 l \\ +120\lambda_1 l\lambda_2 + 140\lambda_1 l + 20l\lambda_2 + 20l \end{pmatrix}\eta^3 + \\ \begin{pmatrix} -69120T^2{\lambda_1}^2 - 120{\lambda_1}^3 l - 120{\lambda_1}^2 l\lambda_2 + 17280T^2\lambda_1 \\ -255{\lambda_1}^2 l - 80\lambda_1 l\lambda_2 + 3456T^2 - 178\lambda_1 l - 20l\lambda_2 - 43l \end{pmatrix}\eta^2 + \\ \begin{pmatrix} -138240T^2{\lambda_1}^3 - 240l{\lambda_1}^4 - 240{\lambda_1}^3 l\lambda_2 + 34560T^2{\lambda_1}^2 \\ -510{\lambda_1}^3 l - 160{\lambda_1}^2 l\lambda_2 + 6912T^2\lambda_1 - 356{\lambda_1}^2 l - 40\lambda_1 l\lambda_2 - 86\lambda_1 l \end{pmatrix}\eta \end{bmatrix} \end{cases}. \tag{58b}$$

(IV) Fourth-order approximation

(i) The differential equation used for the solution of $S_i(\eta)$ (i = 10, 11, 12, 13, 14) can be obtained from the coefficient of $\alpha_1^4, \alpha_1^3 W_m, \alpha_1^2 W_m^2, \alpha_1 W_m^3, W_m^4$ in Equation (37):

$$\begin{cases} \frac{d^2}{d\eta^2}[(1-\eta)S_{10}] & +\frac{dW_1}{d\eta}\frac{dW_6}{d\eta} + \frac{1}{2}\left(\frac{dW_3}{d\eta}\right)^2 + n_1\frac{dW_1}{d\eta}\frac{dW_3}{d\eta} + \frac{1}{2}n_2\left(\frac{dW_1}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_{11}] & +\frac{dW_1}{d\eta}\frac{dW_7}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_4}{d\eta} + \frac{dW_6}{d\eta}\frac{dW_2}{d\eta} \\ & +n_1\left(\frac{dW_1}{d\eta}\frac{dW_4}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_2}{d\eta}\right) + n_2\frac{dW_1}{d\eta}\frac{dW_2}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_{12}] & +\frac{1}{2}\left(\frac{dW_4}{d\eta}\right)^2 + \frac{dW_3}{d\eta}\frac{dW_5}{d\eta} + \frac{dW_1}{d\eta}\frac{dW_8}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_7}{d\eta} \\ & +n_1\left(\frac{dW_1}{d\eta}\frac{dW_5}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_2}{d\eta}\right) + \frac{1}{2}n_2\left(\frac{dW_2}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_{13}] & +\frac{dW_1}{d\eta}\frac{dW_9}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_5}{d\eta} + \frac{dW_8}{d\eta}\frac{dW_2}{d\eta} + n_1\frac{dW_2}{d\eta}\frac{dW_5}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}[(1-\eta)S_{14}] & +\frac{dW_2}{d\eta}\frac{dW_9}{d\eta} + \frac{1}{2}\left(\frac{dW_5}{d\eta}\right)^2 = 0 \end{cases} \tag{59}$$

which should satisfy the boundary conditions, that is, Equation (45), in which (i = 10, 11, 12, 13, 14); thus, the solution gives

$$\begin{cases} S_{10} = S_{11} = S_{13} = 0 \\ S_{12} = \frac{n_2}{6(2\lambda_1+1)^2}[\eta^3 + (4\lambda_1 + 1)\eta^2 + (6{\lambda_1}^2 + 4\lambda_1 + 1)\eta + \lambda_2(6{\lambda_1}^2 + 4\lambda_1 + 1)] \end{cases}. \tag{60a}$$

$$S_{14} = -\frac{1}{90720(2\lambda_1+1)^5} \times \left[\begin{array}{l} (18l\lambda_1+9l)\eta^7 + \left(138240T^2\lambda_1 + 204l{\lambda_1}^2 + 69120T^2 + 180l\lambda_1 + 39l\right)\eta^6 + \\ \left(774144T^2{\lambda_1}^2 + 1092l{\lambda_1}^3 + 331776T^2\lambda_1 + 1506l{\lambda_1}^2 - 27648T^2 + 726l\lambda_1 + 123l\right)\eta^5 + \\ \left(\begin{array}{l} 1741824T^2{\lambda_1}^3 + 2772l{\lambda_1}^4 + 1512l{\lambda_1}^3\lambda_2 + 483840T^2{\lambda_1}^2 + 5754l{\lambda_1}^3 + 1764l{\lambda_1}^2\lambda_2 \\ -248832T^2\lambda_1 + 4656l{\lambda_1}^2 + 756l\lambda_1\lambda_2 - 27648T^2 + 1734l\lambda_1 + 126l\lambda_2 + 249l \end{array} \right)\eta^4 + \\ \left(\begin{array}{l} 1451520T^2{\lambda_1}^4 + 2520l{\lambda_1}^5 + 2520l{\lambda_1}^4\lambda_2 + 1016064T^2{\lambda_1}^3 + 7812l{\lambda_1}^4 \\ +4452l{\lambda_1}^3\lambda_2 - 725760T^2{\lambda_1}^2 + 8904l{\lambda_1}^3 + 2184l{\lambda_1}^2\lambda_2 - 127872T^2\lambda_1 + 4341l{\lambda_1}^2 \\ +406l\lambda_1\lambda_2 - 3456T^2 + 698l\lambda_1 - 14l\lambda_2 - 52l \end{array} \right)\eta^3 + \\ \left(\begin{array}{l} 1451520T^2{\lambda_1}^4 + 2520l{\lambda_1}^5 + 2520l{\lambda_1}^4\lambda_2 - 919296T^2{\lambda_1}^3 + 4452l{\lambda_1}^4 + 1092l{\lambda_1}^3\lambda_2 \\ -241920T^2{\lambda_1}^2 + 1764l{\lambda_1}^3 - 56l{\lambda_1}^2\lambda_2 - 31104T^2\lambda_1 - 643l{\lambda_1}^2 \\ -154l\lambda_1\lambda_2 - 3456T^2 - 506l\lambda_1 - 14l\lambda_2 - 52l \end{array} \right)\eta^2 + \\ \left(\begin{array}{l} -1451520T^2{\lambda_1}^4 - 2520l{\lambda_1}^5 - 2520l{\lambda_1}^4\lambda_2 - 193536T^2{\lambda_1}^3 - 6258l{\lambda_1}^4 \\ -2268l{\lambda_1}^3\lambda_2 - 96768T^2{\lambda_1}^2 - 5712l{\lambda_1}^3 - 896l{\lambda_1}^2\lambda_2 - 31104T^2\lambda_1 - 2449l{\lambda_1}^2 \\ -154l\lambda_1\lambda_2 - 3456T^2 - 506l\lambda_1 - 14l\lambda_2 - 52l \end{array} \right)(\eta+\lambda_2) \end{array} \right] \tag{60b}$$

(ii) The differential equation used for the solution of P_i and $W_i(\eta)$ (i = 10, 11, 12, 13, 14) can be obtained from the coefficient of $\alpha_1^4, \alpha_1^3 W_m, \alpha_1^2 W_m^2, \alpha_1 W_m^3, W_m^4$ in Equation (38):

$$\begin{array}{l} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_{10}}{d\eta} - 6T^2(1-\eta)^2\frac{dW_3}{d\eta}\left(\frac{dW_1}{d\eta}\right)^2\right] \\ = l\left[\begin{array}{l} \frac{S_1}{4}\frac{dW_6}{d\eta} + \frac{S_3}{4}\frac{dW_3}{d\eta} + \frac{S_6}{4}\frac{dW_1}{d\eta} + \\ m_1\left(\frac{S_1}{4}\frac{dW_3}{d\eta} + \frac{S_3}{4}\frac{dW_1}{d\eta}\right) + m_2\frac{S_1}{4}\frac{dW_1}{d\eta} \end{array} \right] - \frac{T^2 l}{2}S_1\left(\frac{dW_1}{d\eta}\right)^3 - l(P_{10} + P_6 m_1 + P_3 m_2 + P_1 m_3) \end{array} \tag{61a}$$

$$\begin{array}{l} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_{11}}{d\eta} - 6T^2(1-\eta)^2\left(\frac{dW_1}{d\eta}\right)^2\frac{dW_4}{d\eta}\right] \\ = l\left[\begin{array}{l} \frac{S_1}{4}\frac{dW_7}{d\eta} + \frac{S_3}{4}\frac{dW_4}{d\eta} + \frac{S_6}{4}\frac{dW_2}{d\eta} + \frac{S_4}{4}\frac{dW_3}{d\eta} + \frac{S_7}{4}\frac{dW_1}{d\eta} + \frac{S_2}{4}\frac{dW_6}{d\eta} + \\ m_1\left(\frac{S_1}{4}\frac{dW_4}{d\eta} + \frac{S_2}{4}\frac{dW_3}{d\eta} + \frac{S_3}{4}\frac{dW_2}{d\eta} + \frac{S_4}{4}\frac{dW_1}{d\eta}\right) + m_2\left(\frac{S_1}{4}\frac{dW_2}{d\eta} + \frac{S_2}{4}\frac{dW_1}{d\eta}\right) \end{array} \right] \\ -\frac{T^2 l}{2}\left[S_2\left(\frac{dW_1}{d\eta}\right)^3 + 3S_1\frac{dW_2}{d\eta}\left(\frac{dW_1}{d\eta}\right)^2\right] - l(P_{11} + P_7 m_1 + P_4 m_2 + P_2 m_3) \end{array} \tag{61b}$$

$$\begin{array}{l} \frac{d^2}{d\eta^2}\left\{(1-\eta)\frac{dW_{12}}{d\eta} - 6T^2(1-\eta)^2\left[\frac{dW_3}{d\eta}\left(\frac{dW_2}{d\eta}\right)^2 + \frac{dW_5}{d\eta}\left(\frac{dW_1}{d\eta}\right)^2\right]\right\} \\ = l\left[\begin{array}{l} \frac{S_1}{4}\frac{dW_8}{d\eta} + \frac{S_2}{4}\frac{dW_7}{d\eta} + \frac{S_3}{4}\frac{dW_5}{d\eta} + \frac{S_4}{4}\frac{dW_4}{d\eta} + \frac{S_5}{4}\frac{dW_3}{d\eta} + \frac{S_7}{4}\frac{dW_2}{d\eta} + \frac{S_8}{4}\frac{dW_1}{d\eta} + \\ m_1\left(\frac{S_1}{4}\frac{dW_5}{d\eta} + \frac{S_4}{4}\frac{dW_2}{d\eta} + \frac{S_2}{4}\frac{dW_4}{d\eta} + \frac{S_5}{4}\frac{dW_1}{d\eta}\right) + m_2\frac{S_2}{4}\frac{dW_2}{d\eta} \end{array} \right] \\ -\frac{3T^2 l}{2}\left[S_1\frac{dW_1}{d\eta}\left(\frac{dW_2}{d\eta}\right)^2 + S_2\frac{dW_2}{d\eta}\left(\frac{dW_1}{d\eta}\right)^2\right] - l(P_{12} + P_8 m_1 + P_5 m_2) \end{array} \tag{61c}$$

$$\begin{array}{l} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_{13}}{d\eta} - 6T^2(1-\eta)^2\left(\frac{dW_2}{d\eta}\right)^2\frac{dW_4}{d\eta}\right] \\ = l\left[\begin{array}{l} \frac{S_1}{4}\frac{dW_9}{d\eta} + \frac{S_4}{4}\frac{dW_5}{d\eta} + \frac{S_8}{4}\frac{dW_2}{d\eta} + \frac{S_2}{4}\frac{dW_8}{d\eta} + \\ \frac{S_5}{4}\frac{dW_4}{d\eta} + \frac{S_9}{4}\frac{dW_1}{d\eta} + m_1\left(\frac{S_2}{4}\frac{dW_5}{d\eta} + \frac{S_5}{4}\frac{dW_2}{d\eta}\right) \end{array} \right] - \frac{T^2 l}{2}S_1\left(\frac{dW_2}{d\eta}\right)^3 - l(P_{13} + P_9 m_1) \end{array} \tag{61d}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW_{14}}{d\eta}\right] = l\left(\frac{S_2}{4}\frac{dW_9}{d\eta} + \frac{S_5}{4}\frac{dW_5}{d\eta} + \frac{S_9}{4}\frac{dW_2}{d\eta}\right) - \frac{T^2 l}{2}S_2\left(\frac{dW_2}{d\eta}\right)^3 - lP_{14} \tag{61e}$$

which should be solved under Equation (53), in which (i = 10, 11, 12, 13, 14); thus, we obtain

$$\begin{cases} P_{10} = P_{12} = P_{14} = 0, P_{11} = -\frac{4(m_1^3 - 2m_2 m_1 + m_3)}{l(2\lambda_1+1)} \\ P_{13} = \frac{1}{1080 l (2\lambda_1+1)^4} \begin{bmatrix} \left(-69120 m_1 T^2 + (1080\lambda_2 + 360) n_1 l\right)\lambda_1{}^4 + \\ \left(172800 m_1 T^2 + (1620\lambda_2 + 840) n_1 l\right)\lambda_1{}^3 + \\ \left(34560 m_1 T^2 + (1080\lambda_2 + 825) n_1 l\right)\lambda_1{}^2 + \\ \left(17280 m_1 T^2 + (350\lambda_2 + 388) n_1 l\right)\lambda_1 + \\ 3456 m_1 T^2 + (50\lambda_2 + 73) n_1 l \end{bmatrix} \end{cases}. \tag{62a}$$

$$\begin{cases} W_{10} = W_{11} = W_{12} = W_{14} = 0 \\ W_{13} = -\frac{(n_1+m_1)}{4320(2\lambda_1+1)^4} \times \begin{bmatrix} (4\lambda_1 l + 2l)\eta^6 + \\ \left(27648 T^2\lambda_1 + 36\lambda_1{}^2 l + 13824 T^2 + 30\lambda_1 l + 6l\right)\eta^5 + \\ \begin{pmatrix} 103680 T^2\lambda_1{}^2 + 150\lambda_1{}^3 l + 17280 T^2\lambda_1 + \\ 195\lambda_1{}^2 l - 17280 T^2 + 90\lambda_1 l + 15l \end{pmatrix}\eta^4 + \\ \begin{pmatrix} 138240 T^2\lambda_1{}^3 + 240 l\lambda_1{}^4 + 240\lambda_1{}^3 l\lambda_2 - 69120 T^2\lambda_1{}^2 + \\ 480\lambda_1{}^3 l + 280\lambda_1{}^2 l\lambda_2 - 69120 T^2\lambda_1 + 380\lambda_1{}^2 l + \\ 120\lambda_1 l\lambda_2 + 140\lambda_1 l + 20 l\lambda_2 + 20l \end{pmatrix}\eta^3 + \\ \begin{pmatrix} -69120 T^2\lambda_1{}^2 - 120\lambda_1{}^3 l - 120\lambda_1{}^2 l\lambda_2 + 17280 T^2\lambda_1 \\ -255\lambda_1{}^2 l - 80\lambda_1 l\lambda_2 + 3456 T^2 - 178\lambda_1 l - 20 l\lambda_2 - 43l \end{pmatrix}\eta^2 + \\ \begin{pmatrix} -138240 T^2\lambda_1{}^3 - 240 l\lambda_1{}^4 - 240\lambda_1{}^3 l\lambda_2 + 34560 T^2\lambda_1{}^2 - 510\lambda_1{}^3 l \\ -160\lambda_1{}^2 l\lambda_2 + 6912 T^2\lambda_1 - 356\lambda_1{}^2 l - 40\lambda_1 l\lambda_2 - 86\lambda_1 l \end{pmatrix}\eta \end{bmatrix} \end{cases} \tag{62b}$$

(V) Fifth-order approximation

The differential equation used for the solution of $S_i(\eta)$ (i = 15, 16, 17, 18, 19, 20) can be obtained from the coefficient of $\alpha_1^5, \alpha_1^4 W_m, \alpha_1^3 W_m^2, \alpha_1^2 W_m^3, \alpha_1 W_m^4, W_m^5$ in Equation (37):

$$\begin{aligned} \frac{d^2}{d\eta^2}[(1-\eta)S_{15}] \quad & + \frac{dW_1}{d\eta}\frac{dW_{10}}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_6}{d\eta} + \frac{1}{2}n_1\left(\frac{dW_3}{d\eta}\right)^2 \\ & + n_1\frac{dW_1}{d\eta}\frac{dW_6}{d\eta} + n_2\frac{dW_1}{d\eta}\frac{dW_3}{d\eta} + \frac{1}{2}n_3\left(\frac{dW_1}{d\eta}\right)^2 = 0 \end{aligned} \tag{63a}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}[(1-\eta)S_{16}] \quad & + \frac{dW_1}{d\eta}\frac{dW_{11}}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_{10}}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_7}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_6}{d\eta} \\ & + n_1\left(\frac{dW_1}{d\eta}\frac{dW_7}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_6}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_4}{d\eta}\right) + n_2\left(\frac{dW_1}{d\eta}\frac{dW_4}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_3}{d\eta}\right) \\ & + n_3\frac{dW_1}{d\eta}\frac{dW_2}{d\eta} = 0 \end{aligned} \tag{63b}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}[(1-\eta)S_{17}] \quad & + \frac{dW_1}{d\eta}\frac{dW_{12}}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_{11}}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_8}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_7}{d\eta} + \frac{dW_5}{d\eta}\frac{dW_6}{d\eta} \\ & + n_1\left(\frac{dW_1}{d\eta}\frac{dW_8}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_7}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_5}{d\eta}\right) + \frac{1}{2}n_1\left(\frac{dW_4}{d\eta}\right)^2 \\ & + n_2\left(\frac{dW_1}{d\eta}\frac{dW_5}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_4}{d\eta}\right) + \frac{1}{2}n_3\left(\frac{dW_2}{d\eta}\right)^2 = 0 \end{aligned} \tag{63c}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}[(1-\eta)S_{18}] \quad & + \frac{dW_1}{d\eta}\frac{dW_{13}}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_{12}}{d\eta} + \frac{dW_3}{d\eta}\frac{dW_9}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_8}{d\eta} + \frac{dW_5}{d\eta}\frac{dW_7}{d\eta} \\ & + n_1\left(\frac{dW_1}{d\eta}\frac{dW_9}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_8}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_5}{d\eta}\right) + n_2\frac{dW_2}{d\eta}\frac{dW_5}{d\eta} = 0 \end{aligned} \tag{63d}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}[(1-\eta)S_{19}] \quad & + \frac{dW_1}{d\eta}\frac{dW_{14}}{d\eta} + \frac{dW_2}{d\eta}\frac{dW_{13}}{d\eta} + \frac{dW_4}{d\eta}\frac{dW_9}{d\eta} + \frac{dW_5}{d\eta}\frac{dW_8}{d\eta} \\ & + n_1\frac{dW_2}{d\eta}\frac{dW_9}{d\eta} + \frac{1}{2}n_1\left(\frac{dW_5}{d\eta}\right)^2 = 0 \end{aligned} \tag{63e}$$

$$\frac{d^2}{d\eta^2}[(1-\eta)S_{20}] + \frac{dW_2}{d\eta}\frac{dW_{14}}{d\eta} + \frac{dW_5}{d\eta}\frac{dW_9}{d\eta} = 0 \tag{63f}$$

which should satisfy the boundary conditions, that is, Equation (45), in which (i = 15, 16, 17, 18, 19, 20); thus, the solution gives

$$\begin{cases} S_{15} = S_{16} = S_{18} = S_{20} = 0 \\ S_{17} = \frac{n_3}{6(2\lambda_1+1)^2}\left[\begin{array}{l} \eta^3 + (4\lambda_1+1)\eta^2 + \\ (6\lambda_1{}^2+4\lambda_1+1)\eta + \lambda_2(6\lambda_1{}^2+4\lambda_1+1) \end{array}\right] \end{cases} \tag{64a}$$

$$\begin{aligned} S_{19} &= -\frac{n_1+m_1}{90720(2\lambda_1+1)^5}\times \\ &\left[\begin{array}{l} (18l\lambda_1+9l)\eta^7 + \left(138240T^2\lambda_1 + 204l\lambda_1{}^2 + 69120T^2 + 180l\lambda_1 + 39l\right)\eta^6 + \\ \left(774144T^2\lambda_1{}^2 + 1092l\lambda_1{}^3 + 331776T^2\lambda_1 + 1506l\lambda_1{}^2 - 27648T^2 + 726l\lambda_1 + 123l\right)\eta^5 + \\ \left(\begin{array}{l} 1741824T^2\lambda_1{}^3 + 2772l\lambda_1{}^4 + 1512l\lambda_1{}^3\lambda_2 + 483840T^2\lambda_1{}^2 + 5754l\lambda_1{}^3 + 1764l\lambda_1{}^2\lambda_2 - \\ 248832T^2\lambda_1 + 4656l\lambda_1{}^2 + 756l\lambda_1\lambda_2 - 27648T^2 + 1734l\lambda_1 + 126l\lambda_2 + 249l \end{array}\right)\eta^4 + \\ \left(\begin{array}{l} 1451520T^2\lambda_1{}^4 + 2520l\lambda_1{}^5 + 2520l\lambda_1{}^4\lambda_2 + 1016064T^2\lambda_1{}^3 + \\ 7812l\lambda_1{}^4 + 4452l\lambda_1{}^3\lambda_2 - 725760T^2\lambda_1{}^2 + 8904l\lambda_1{}^3 + 2184l\lambda_1{}^2\lambda_2 - \\ 127872T^2\lambda_1 + 4341l\lambda_1{}^2 + 406l\lambda_1\lambda_2 - 3456T^2 + 698l\lambda_1 - 14l\lambda_2 - 52l \end{array}\right)\eta^3 + \\ \left(\begin{array}{l} 1451520T^2\lambda_1{}^4 + 2520l\lambda_1{}^5 + 2520l\lambda_1{}^4\lambda_2 - 919296T^2\lambda_1{}^3 + 4452l\lambda_1{}^4 + \\ 1092l\lambda_1{}^3\lambda_2 - 241920T^2\lambda_1{}^2 + 1764l\lambda_1{}^3 - 56l\lambda_1{}^2\lambda_2 - 31104T^2\lambda_1 - \\ 643l\lambda_1{}^2 - 154l\lambda_1\lambda_2 - 3456T^2 - 506l\lambda_1 - 14l\lambda_2 - 52l \end{array}\right)\eta^2 + \\ \left(\begin{array}{l} -1451520T^2\lambda_1{}^4 - 2520l\lambda_1{}^5 - 2520l\lambda_1{}^4\lambda_2 - 193536T^2\lambda_1{}^3 - 6258l\lambda_1{}^4 - \\ 2268l\lambda_1{}^3\lambda_2 - 96768T^2\lambda_1{}^2 - 5712l\lambda_1{}^3 - 896l\lambda_1{}^2\lambda_2 - 31104T^2\lambda_1 - \\ 2449l\lambda_1{}^2 - 154l\lambda_1\lambda_2 - 3456T^2 - 506l\lambda_1 - 14l\lambda_2 - 52l \end{array}\right)(\eta+\lambda_2) \end{array}\right] \end{aligned} \tag{64b}$$

Given that the next solving process becomes more complex and the calculation accuracy has been satisfied, we end the computation here. After summarizing the results, we have

$$\begin{aligned} \frac{P}{16} &= \frac{4[1 - m_1\alpha_1 - (m_2 - m_1{}^2)\alpha_1{}^2 - (m_3 - 2m_1m_2 + m_1{}^3)\alpha_1{}^3]}{l(2\lambda_1+1)}W_m \\ &+[1+(n_1+m_1)\alpha_1]P_9W_m^3 \end{aligned} \tag{65}$$

$$W = \frac{\eta^2 + 2\lambda_1\eta}{2\lambda_1+1}W_m + [1+(n_1+m_1)\alpha_1]W_9W_m^3 \tag{66}$$

$$S = (1 + n_1\alpha_1 + n_2\alpha_1{}^2 + n_3\alpha_1{}^3)S_5W_m^2 + [1+(2n_1+m_1)\alpha_1]S_{14}W_m^4 \tag{67}$$

in which S_5, P_9, and W_9, as well as S_{14}, are shown in Equations (51), (58a), (58b), and (60b), respectively.

3.3. Biparametric Perturbation on α_1 with P_m

Another group of parameter combination, α_1 and P_m, is selected as the perturbation parameters, in which α_1 is still the graded index in the tensile zone and P_m is the dimensionless load, that is,

$$P_m = \frac{qa^4}{16E_0t^4} = \frac{P}{16}. \tag{68}$$

W and S in Equations (30) and (31) are expressed in the perturbation parameters,

$$W = W(\alpha_1, P_m, \eta),\ S = S(\alpha_1, P_m, \eta). \tag{69}$$

Thus, W and S are expanded in the form of the power series of α_1 and P_m,

$$\begin{aligned} W &= \overline{W_1}\alpha_1 + \overline{W_2}P_m + \overline{W_3}\alpha_1{}^2 + \overline{W_4}\alpha_1P_m + \overline{W_5}P_m{}^2 + \overline{W_6}\alpha_1{}^3 + \overline{W_7}\alpha_1{}^2P_m + \overline{W_8}\alpha_1P_m{}^2 + \overline{W_9}P_m{}^3 \\ &+\overline{W_{10}}\alpha_1{}^4 + \overline{W_{11}}\alpha_1{}^3P_m + \overline{W_{12}}\alpha_1{}^2P_m{}^2 + \overline{W_{13}}\alpha_1P_m{}^3 + \overline{W_{14}}P_m{}^4 + \overline{W_{15}}\alpha_1{}^5 + \overline{W_{16}}\alpha_1{}^4P_m \\ &+\overline{W_{17}}\alpha_1{}^3P_m{}^2 + \overline{W_{18}}\alpha_1{}^2P_m{}^3 + \overline{W_{19}}\alpha_1P_m{}^4 + \overline{W_{20}}P_m{}^5 + \cdots \end{aligned} \tag{70}$$

and

$$\begin{aligned} S &= \overline{S_1}\alpha_1 + \overline{S_2}P_m + \overline{S_3}{\alpha_1}^2 + \overline{S_4}\alpha_1 P_m + \overline{S_5}{P_m}^2 + \overline{S_6}{\alpha_1}^3 + \overline{S_7}{\alpha_1}^2 P_m + \overline{S_8}\alpha_1 {P_m}^2 + \overline{S_9}{P_m}^3 \\ &+ \overline{S_{10}}{\alpha_1}^4 + \overline{S_{11}}{\alpha_1}^3 P_m + \overline{S_{12}}{\alpha_1}^2 {P_m}^2 + \overline{S_{13}}\alpha_1 {P_m}^3 + \overline{S_{14}}{P_m}^4 + \overline{S_{15}}{\alpha_1}^5 + \overline{S_{16}}{\alpha_1}^4 P_m \\ &+ \overline{S_{17}}{\alpha_1}^3 {P_m}^2 + \overline{S_{18}}{\alpha_1}^2 {P_m}^3 + \overline{S_{19}}\alpha_1 {P_m}^4 + \overline{S_{20}}{P_m}^5 + \cdots \end{aligned} \tag{71}$$

In addition, in the governing equations, Equation (37) remains unchanged, while Equation (38) needs to be slightly adjusted due to diffident perturbation parameters we select here. For this purpose, substituting Equation (68) into Equation (38) yields

$$\begin{aligned} &\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{dW}{d\eta} - 2T^2(1-\eta)^2\left(\frac{dW}{d\eta}\right)^3\right] \\ &= l\left(1 + m_1\alpha_1 + m_2{\alpha_1}^2 + m_3{\alpha_1}^3 + \cdots\right)\left[\frac{S}{4}\frac{dW}{d\eta} - P_m - \frac{T^2 S}{2}\left(\frac{dW}{d\eta}\right)^3\right] \end{aligned} \tag{72}$$

The next perturbation steps may follow the process of Section 3.2.

(I) First-order approximation

(i) The differential equation used for the solution of $\overline{S_i}(\eta)(i = 1, 2)$ can be obtained from the coefficient of α_1 and P_m in Equation (37):

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_i}\right] = 0, \tag{73}$$

which should satisfy the boundary conditions ($i = 1, 2$)

$$\begin{cases} \lambda_2\frac{d\overline{S_i}}{d\eta} - \overline{S_i} = 0 \ \ at\ \eta = 0 \\ \overline{S_i} \neq \infty\ at\ \eta = 1 \end{cases}; \tag{74}$$

thus, the solution gives

$$\overline{S_1} = \overline{S_2} = 0. \tag{75}$$

(ii) The differential equation used for the solution of $\overline{W_i}(\eta)$ ($i = 1, 2$) can be obtained from the coefficient of α_1 and P_m in Equation (72):

$$\begin{cases} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_1}}{d\eta}\right] = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_2}}{d\eta}\right] = -l \end{cases} \tag{76}$$

which should be solved under ($i = 1, 2$)

$$\begin{cases} \overline{W_i} = 0,\ \lambda_1\frac{d^2\overline{W_i}}{d\eta^2} - \frac{d\overline{W_i}}{d\eta} = 0\ at\ \eta = 0 \\ \frac{d\overline{W_i}}{d\eta} \neq \infty\ at\ \eta = 1 \end{cases}; \tag{77}$$

Thus, we obtain

$$\overline{W_1} = 0, \overline{W_2} = \frac{l(\eta^2 + 2\lambda_1\eta)}{4}. \tag{78}$$

(II) Second-order approximation

(i) The differential equation used for the solution of $\overline{S_i}(\eta)(i = 3, 4, 5)$ can be obtained from the coefficient of $\alpha_1^2, \alpha_1 P_m, P_m^2$ in Equation (37):

$$\begin{cases} \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_3}\right] + \frac{1}{2}\left(\frac{d\overline{W_1}}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_4}\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_2}}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_5}\right] + \frac{1}{2}\left(\frac{d\overline{W_2}}{d\eta}\right)^2 = 0 \end{cases}, \tag{79}$$

which should satisfy the boundary conditions, that is, Equation (74), in which (i = 3, 4, 5); thus, the solution gives

$$\begin{cases} \overline{S_3} = \overline{S_4} = 0 \\ \overline{S_5} = \frac{l^2}{96}[\eta^3 + (4\lambda_1 + 1)\eta^2 + (6\lambda_1^2 + 4\lambda_1 + 1)\eta + \lambda_2(6\lambda_1^2 + 4\lambda_1 + 1)] \end{cases}. \tag{80}$$

(ii) The differential equation used for the solution of $\overline{W_\mathrm{i}}(\eta)$ (i = 3, 4, 5) can be obtained from the coefficient of $\alpha_1^2, \alpha_1 P_m, P_m^2$ in Equation (72):

$$\begin{cases} \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_3}}{d\eta}\right] = l\frac{\overline{S_1}}{4}\frac{d\overline{W_1}}{d\eta} \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_4}}{d\eta}\right] = l\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_2}}{d\eta} + \frac{\overline{S_2}}{4}\frac{d\overline{W_1}}{d\eta}\right) - lm_1 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_5}}{d\eta}\right] = l\frac{\overline{S_2}}{4}\frac{d\overline{W_2}}{d\eta} \end{cases} \tag{81}$$

which should be solved under Equation (77), in which (i = 3, 4, 5); thus, we obtain

$$\overline{W_3} = \overline{W_5} = 0, \overline{W_4} = \frac{lm_1(\eta^2 + 2\lambda_1\eta)}{4}. \tag{82}$$

(III) Third-order approximation

(i) The differential equation used for the solution of $\overline{S_\mathrm{i}}(\eta)$($i$ = 6, 7, 8, 9) can be obtained from the coefficient of $\alpha_1^3, \alpha_1^2 P_m, \alpha_1 P_m^2, P_m^3$ in Equation (37):

$$\begin{cases} \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_6}\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_3}}{d\eta} + \frac{1}{2}n_1\left(\frac{d\overline{W_1}}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_7}]\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_4}}{d\eta} + \frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_2}}{d\eta} + n_1\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_2}}{d\eta} = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_8}\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_5}}{d\eta} + \frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_2}}{d\eta} + \frac{1}{2}n_1\left(\frac{d\overline{W_2}}{d\eta}\right)^2 = 0 \\ \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_9}\right] + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_5}}{d\eta} = 0 \end{cases} \tag{83}$$

which should satisfy the boundary conditions, that is, Equation (74), in which (i = 6, 7, 8, 9); thus, the solution gives

$$\begin{cases} \overline{S_6} = \overline{S_7} = \overline{S_9} = 0 \\ \overline{S_8} = n_1\overline{S_5} = \frac{n_1 l^2}{96}[\eta^3 + (4\lambda_1 + 1)\eta^2 + (6\lambda_1^2 + 4\lambda_1 + 1)\eta + \lambda_2(6\lambda_1^2 + 4\lambda_1 + 1)] \end{cases}. \tag{84}$$

(ii) The differential equation used for the solution of $\overline{W_\mathrm{i}}(\eta)$ (i = 6, 7, 8, 9) can be obtained from the coefficient of $\alpha_1^3, \alpha_1^2 P_m, \alpha_1 P_m^2, P_m^3$ in Equation (72):

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_6}}{d\eta} - 2T^2(1-\eta)^2\left(\frac{d\overline{W_1}}{d\eta}\right)^3\right] = l\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_3}}{d\eta} + \frac{\overline{S_3}}{4}\frac{d\overline{W_1}}{d\eta} + m_1\frac{\overline{S_1}}{4}\frac{d\overline{W_1}}{d\eta}\right) \tag{85a}$$

$$\begin{aligned} &\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_7}}{d\eta} - 6T^2(1-\eta)^2\frac{d\overline{W_2}}{d\eta}\left(\frac{d\overline{W_1}}{d\eta}\right)^2\right] \\ &= l\left[\frac{\overline{S_1}}{4}\frac{d\overline{W_4}}{d\eta} + \frac{\overline{S_2}}{4}\frac{d\overline{W_3}}{d\eta} + \frac{\overline{S_3}}{4}\frac{d\overline{W_2}}{d\eta} + \frac{\overline{S_4}}{4}\frac{d\overline{W_1}}{d\eta} + m_1\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_2}}{d\eta} + \frac{\overline{S_2}}{4}\frac{d\overline{W_1}}{d\eta}\right)\right] - lm_2 \end{aligned} \tag{85b}$$

$$\begin{aligned} &\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_8}}{d\eta} - 6T^2(1-\eta)^2\frac{d\overline{W_1}}{d\eta}\left(\frac{d\overline{W_2}}{d\eta}\right)^2\right] \\ &= l\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_5}}{d\eta} + \frac{\overline{S_2}}{4}\frac{d\overline{W_4}}{d\eta} + \frac{\overline{S_4}}{4}\frac{d\overline{W_2}}{d\eta} + \frac{\overline{S_5}}{4}\frac{d\overline{W_1}}{d\eta} + m_1\frac{\overline{S_2}}{4}\frac{d\overline{W_2}}{d\eta}\right) \end{aligned} \tag{85c}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_9}}{d\eta} - 2T^2(1-\eta)^2\left(\frac{d\overline{W_2}}{d\eta}\right)^3\right] = l\left(\frac{\overline{S_2}}{4}\frac{d\overline{W_5}}{d\eta} + \frac{\overline{S_5}}{4}\frac{d\overline{W_2}}{d\eta}\right) \tag{85d}$$

which should be solved under Equation (77), in which (i = 6, 7, 8, 9); thus, we obtain

$$\left\{\begin{array}{l} \overline{W_6} = \overline{W_8} = 0, \overline{W_7} = \frac{lm_2\left(\eta^2+2\lambda_1\eta\right)}{4} \\ \overline{W_9} = \frac{-l^4}{276840}\left\{\begin{array}{l} 2\eta^6 + \left[13824T^2/l + 6(3\lambda_1+1)\right]\eta^5 + \\ \left[51840T^2\lambda_1/l - 17280T^2/l + 15\left(5{\lambda_1}^2 + 4\lambda_1 + 1\right)\right]\eta^4 + \\ \left[69120(\lambda_1-1)T^2\lambda_1/l + 20\left(6{\lambda_1}^3 + 9{\lambda_1}^2 + 5\lambda_1 + 1 + \lambda_2\left(6{\lambda_1}^2 + 4\lambda_1 + 1\right)\right)\right]\eta^3 + \\ \left[\begin{array}{l} 34560(3\lambda_1-1)T^2{\lambda_1}^3/l + \\ 30\left(6{\lambda_1}^3 + 9{\lambda_1}^2 + 5\lambda_1 + 1 + \lambda_2\left(18{\lambda_1}^3 + 18{\lambda_1}^2 + 7\lambda_1 + 1\right)\right) \end{array}\right]\left(\eta^2 + 2\lambda_1\eta\right) \end{array}\right\} \end{array}\right. \tag{86}$$

(IV) Fourth-order approximation

(i) The differential equation used for the solution of $\overline{S_{\mathrm{i}}}(\eta)(i$ = 10, 11, 12, 13, 14) can be obtained from the coefficient of $\alpha_1^4, \alpha_1^3 P_m, \alpha_2^2 P_m^2, \alpha_1 P_m^3, P_m^4$ in Equation (37):

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{10}}\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_6}}{d\eta} + \frac{1}{2}\left(\frac{d\overline{W_3}}{d\eta}\right)^2 + n_1\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_3}}{d\eta} + \frac{1}{2}n_2\left(\frac{d\overline{W_1}}{d\eta}\right)^2 = 0, \tag{87a}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{11}}\right] \quad & + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_7}}{d\eta} + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_6}}{d\eta} + \frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_4}}{d\eta} \\ & + n_1\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_4}}{d\eta} + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_3}}{d\eta}\right) + n_2\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_2}}{d\eta} = 0 \end{aligned} \tag{87b}$$

$$\begin{aligned} \frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{12}}\right] \quad & + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_8}}{d\eta} + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_7}}{d\eta} + \frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_5}}{d\eta} + \frac{1}{2}\left(\frac{d\overline{W_4}}{d\eta}\right)^2 \\ & + n_1\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_5}}{d\eta} + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_4}}{d\eta}\right) + \frac{1}{2}n_2\left(\frac{d\overline{W_2}}{d\eta}\right)^2 = 0 \end{aligned} \tag{87c}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{13}}\right] + \frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_9}}{d\eta} + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_8}}{d\eta} + \frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_5}}{d\eta} + n_1\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_5}}{d\eta} = 0, \tag{87d}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{14}}\right] + \frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_9}}{d\eta} + \frac{1}{2}\left(\frac{d\overline{W_5}}{d\eta}\right)^2 = 0, \tag{87e}$$

which should satisfy the boundary conditions, that is, Equation (74), in which (i = 10, 11, 12, 13, 14); thus, the solution gives

$$\left\{\begin{array}{l} \overline{S_{10}} = \overline{S_{11}} = \overline{S_{13}} = 0 \\ \overline{S_{12}} = n_2\overline{S_5} = \frac{n_2 l^2}{96}\left[\eta^3 + (4\lambda_1+1)\eta^2 + (6\lambda_1^2 + 4\lambda_1 + 1)\eta + \lambda_2(6\lambda_1^2 + 4\lambda_1 + 1)\right] \end{array}\right., \tag{88a}$$

$$\overline{S_{14}} = \frac{-l^5}{7741440} \times$$
$$\left\{\begin{array}{l} 3\eta^7 + \\ \left(23040T^2/l + 34\lambda_1 + 13\right)\eta^6 + \\ \left[9216(14\lambda_1-1)T^2/l + 182{\lambda_1}^2 + 160\lambda_1 + 41\right]\eta^5 + \\ \left[\begin{array}{l} \left(290304{\lambda_1}^2 - 64512\lambda_1 - 9216\right)T^2/l + 462{\lambda_1}^3 \\ +728{\lambda_1}^2 + 412\lambda_1 + 83 + 42\lambda_2\left(6{\lambda_1}^2 + 4\lambda_1 + 1\right) \end{array}\right]\eta^4 + \\ \left[\begin{array}{l} 9216\left(35{\lambda_1}^3 - 21{\lambda_1}^2 - 7\lambda_1 - 1\right)T^2/l + 420{\lambda_1}^4 + 1512{\lambda_1}^3 + \\ 1708{\lambda_1}^2 + 832\lambda_1 + 153 + 56\lambda_2\left(30{\lambda_1}^3 + 32{\lambda_1}^2 + 13\lambda_1 + 2\right) \end{array}\right]\eta^3 + \\ \left[\begin{array}{l} 9216\left(35{\lambda_1}^4 - 7{\lambda_1}^3 - 21{\lambda_1}^2 - 7\lambda_1 - 1\right)T^2/l + 2100{\lambda_1}^4 + 4032{\lambda_1}^3 + 3108{\lambda_1}^2 + \\ 1112\lambda_1 + 153 + 56\lambda_2\left(90{\lambda_1}^4 + 120{\lambda_1}^3 + 67{\lambda_1}^2 + 18\lambda_1 + 2\right) \end{array}\right]\eta^2 + \\ \left[\begin{array}{l} \left(483840{\lambda_1}^5 - 1128960{\lambda_1}^4 - 645120{\lambda_1}^3 - 193536{\lambda_1}^2 - 64512\lambda_1 - 9216\right)T^2/l \\ +2520{\lambda_1}^5 + 5880{\lambda_1}^4 + 6132{\lambda_1}^3 + 3528{\lambda_1}^2 + 1112\lambda_1 + 153 \\ +28\lambda_2\left(270{\lambda_1}^5 + 450{\lambda_1}^4 + 345{\lambda_1}^3 + 149{\lambda_1}^2 + 35\lambda_1 + 4\right) \end{array}\right]\eta + \\ \left[\begin{array}{l} \left(483840{\lambda_1}^5 - 1128960{\lambda_1}^4 - 645120{\lambda_1}^3 - 193536{\lambda_1}^2 - 64512\lambda_1 - 9216\right)T^2/l \\ +3528{\lambda_1}^2 + 7560{\lambda_1}^5\lambda_2 + 2520{\lambda_1}^5 + 1112\lambda_1 + 12600{\lambda_1}^4\lambda_2 + 1008\lambda_1\lambda_2 \\ +4172{\lambda_1}^2\lambda_2 + 5880{\lambda_1}^4 + 6132{\lambda_1}^3 + 9660{\lambda_1}^3\lambda_2 + 112\lambda_2 + 153\lambda_2 \end{array}\right]\lambda_2 \end{array}\right\} \tag{88b}$$

(ii) The differential equation used for the solution of $\overline{W_i}(\eta)$ (i = 10, 11, 12, 13, 14) can be obtained from the coefficient of $\alpha_1^4, \alpha_1^3 P_m, \alpha_2^2 P_m^2, \alpha_1 P_m^3, P_m^4$ in Equation (72):

$$\begin{aligned}&\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_{10}}}{d\eta}-6T^2(1-\eta)^2\frac{d\overline{W_3}}{d\eta}\left(\frac{d\overline{W_1}}{d\eta}\right)^2\right]\\&=l\left[\frac{\overline{S_1}}{4}\frac{d\overline{W_6}}{d\eta}+\frac{\overline{S_3}}{4}\frac{d\overline{W_3}}{d\eta}+\frac{\overline{S_6}}{4}\frac{d\overline{W_1}}{d\eta}+m_1\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_3}}{d\eta}+\frac{\overline{S_3}}{4}\frac{d\overline{W_1}}{d\eta}\right)+m_2\frac{\overline{S_1}}{4}\frac{d\overline{W_1}}{d\eta}-\frac{T^2\overline{S_1}}{2}\left(\frac{d\overline{W_1}}{d\eta}\right)^3\right]\end{aligned} \tag{89a}$$

$$\begin{aligned}&\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_{11}}}{d\eta}-6T^2(1-\eta)^2\frac{d\overline{W_4}}{d\eta}\left(\frac{d\overline{W_1}}{d\eta}\right)^2\right]\\&=l\left\{\begin{array}{l}\frac{\overline{S_1}}{4}\frac{d\overline{W_7}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_6}}{d\eta}+\frac{\overline{S_3}}{4}\frac{d\overline{W_4}}{d\eta}+\frac{\overline{S_4}}{4}\frac{d\overline{W_3}}{d\eta}+\frac{\overline{S_6}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_7}}{4}\frac{d\overline{W_1}}{d\eta}+\\ m_1\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_4}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_3}}{d\eta}+\frac{\overline{S_3}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_4}}{4}\frac{d\overline{W_1}}{d\eta}\right)+m_2\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_1}}{d\eta}\right)-\\ \frac{T^2}{2}\left[\overline{S_2}\left(\frac{d\overline{W_1}}{d\eta}\right)^3+3\overline{S_1}\frac{d\overline{W_2}}{d\eta}\left(\frac{d\overline{W_1}}{d\eta}\right)^2\right]\end{array}\right\}-lm_3\end{aligned} \tag{89b}$$

$$\begin{aligned}&\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_{12}}}{d\eta}-6T^2(1-\eta)^2\frac{d\overline{W_1}}{d\eta}\left(\frac{d\overline{W_2}}{d\eta}\right)^2\right]\\&=l\left\{\begin{array}{l}\frac{\overline{S_1}}{4}\frac{d\overline{W_8}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_7}}{d\eta}+\frac{\overline{S_3}}{4}\frac{d\overline{W_5}}{d\eta}+\frac{\overline{S_4}}{4}\frac{d\overline{W_4}}{d\eta}+\frac{\overline{S_5}}{4}\frac{d\overline{W_3}}{d\eta}+\frac{\overline{S_7}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_8}}{4}\frac{d\overline{W_1}}{d\eta}+\\ m_1\left(\frac{\overline{S_1}}{4}\frac{d\overline{W_5}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_4}}{d\eta}+\frac{\overline{S_4}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_5}}{4}\frac{d\overline{W_1}}{d\eta}\right)+m_2\frac{\overline{S_2}}{4}\frac{d\overline{W_2}}{d\eta}-\\ \frac{T^2}{2}\left[3\overline{S_1}\frac{d\overline{W_1}}{d\eta}\left(\frac{d\overline{W_2}}{d\eta}\right)^2+3\overline{S_2}\frac{d\overline{W_2}}{d\eta}\left(\frac{d\overline{W_1}}{d\eta}\right)^2\right]\end{array}\right\}\end{aligned} \tag{89c}$$

$$\begin{aligned}&\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_{13}}}{d\eta}-6T^2(1-\eta)^2\frac{d\overline{W_4}}{d\eta}\left(\frac{d\overline{W_2}}{d\eta}\right)^2\right]\\&=l\left\{\begin{array}{l}\frac{\overline{S_1}}{4}\frac{d\overline{W_9}}{d\eta}+\frac{\overline{S_2}}{4}\frac{d\overline{W_8}}{d\eta}+\frac{\overline{S_4}}{4}\frac{d\overline{W_5}}{d\eta}+\frac{\overline{S_5}}{4}\frac{d\overline{W_4}}{d\eta}+\frac{\overline{S_8}}{4}\frac{d\overline{W_2}}{d\eta}+\frac{\overline{S_9}}{4}\frac{d\overline{W_1}}{d\eta}+\\ m_1\left(\frac{\overline{S_2}}{4}\frac{d\overline{W_5}}{d\eta}+\frac{\overline{S_5}}{4}\frac{d\overline{W_2}}{d\eta}\right)-\frac{T^2}{2}\left[\overline{S_1}\left(\frac{d\overline{W_2}}{d\eta}\right)^3+3\overline{S_2}\frac{d\overline{W_1}}{d\eta}\left(\frac{d\overline{W_2}}{d\eta}\right)^2\right]\end{array}\right\}\end{aligned} \tag{89d}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\frac{d\overline{W_{14}}}{d\eta}\right]=l\left[\frac{\overline{S_2}}{4}\frac{d\overline{W_9}}{d\eta}+\frac{\overline{S_5}}{4}\frac{d\overline{W_5}}{d\eta}+\frac{\overline{S_9}}{4}\frac{d\overline{W_2}}{d\eta}-\frac{T^2}{2}\overline{S_2}\left(\frac{d\overline{W_2}}{d\eta}\right)^3\right] \tag{89e}$$

which should be solved under Equation (77), in which (i = 10, 11, 12, 13, 14); thus, we obtain

$$\left\{\begin{array}{l}\overline{W_{10}}=\overline{W_{12}}=\overline{W_{14}}=0\\ \overline{W_{11}}=\frac{lm_3\left(\eta^2+2\lambda_1\eta\right)}{4}\\ \overline{W_{13}}=(n_1+m_1)\overline{W_9}\\ \quad=-\frac{l^4(n_1+m_1)}{276840}\left\{\begin{array}{l}2\eta^6+\left[13824T^2/l+6(3\lambda_1+1)\right]\eta^5+\\ \left[51840\lambda_1T^2/l-17280T^2/l+15\left(5{\lambda_1}^2+4\lambda_1+1\right)\right]\eta^4+\\ \left[\begin{array}{l}69120\lambda_1(\lambda_1-1)T^2/l+\\ 20\left(6{\lambda_1}^3+9{\lambda_1}^2+5\lambda_1+1+\lambda_2\left(6{\lambda_1}^2+4\lambda_1+1\right)\right)\end{array}\right]\eta^3+\\ \left[\begin{array}{l}34560{\lambda_1}^3(3\lambda_1-1)T^2/l+\\ 30\left(6{\lambda_1}^3+9{\lambda_1}^2+5\lambda_1+1+\lambda_2\left(18{\lambda_1}^3+18{\lambda_1}^2+7\lambda_1+1\right)\right)\end{array}\right]\left(\eta^2+2\lambda_1\eta\right)\end{array}\right\}\end{array}\right. \tag{90}$$

(V) Fifth-order approximation

The differential equation used for the solution of $\overline{S_i}(\eta)(i = 15, 16, 17, 18, 19, 20)$ can be obtained from the coefficient of $\alpha_1^5, \alpha_1^4 P_m, \alpha_2^3 P_m^2, \alpha_2^2 P_m^3, \alpha_1 P_m^4, P_m^5$ in Equation (37):

$$\begin{aligned}\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{15}}\right] \quad & +\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_{10}}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_6}}{d\eta}+n_1\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_6}}{d\eta} \\ & +n_2\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_3}}{d\eta}+\frac{1}{2}n_1\left(\frac{d\overline{W_3}}{d\eta}\right)^2+\frac{1}{2}n_3\left(\frac{d\overline{W_1}}{d\eta}\right)^2=0\end{aligned}, \tag{91a}$$

$$\begin{aligned}\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{16}}\right] \quad & +\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_{11}}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_{10}}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_7}}{d\eta}+\frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_6}}{d\eta} \\ & +n_1\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_7}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_6}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_4}}{d\eta}\right) \\ & +n_2\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_4}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_3}}{d\eta}\right)+n_3\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_2}}{d\eta}=0\end{aligned} \tag{91b}$$

$$\begin{aligned}\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{17}}\right] \quad & +\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_{12}}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_{11}}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_8}}{d\eta}+\frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_7}}{d\eta}+\frac{d\overline{W_5}}{d\eta}\frac{d\overline{W_6}}{d\eta} \\ & +n_1\left[\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_8}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_7}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_5}}{d\eta}+\frac{1}{2}\left(\frac{d\overline{W_4}}{d\eta}\right)^2\right] \\ & +n_2\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_5}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_4}}{d\eta}\right)+\frac{1}{2}n_3\left(\frac{d\overline{W_2}}{d\eta}\right)^2=0\end{aligned} \tag{91c}$$

$$\begin{aligned}\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{18}}\right] \quad & +\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_{13}}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_{12}}}{d\eta}+\frac{d\overline{W_3}}{d\eta}\frac{d\overline{W_9}}{d\eta}+\frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_8}}{d\eta}+\frac{d\overline{W_5}}{d\eta}\frac{d\overline{W_7}}{d\eta} \\ & +n_1\left(\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_9}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_8}}{d\eta}+\frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_5}}{d\eta}\right)+n_2\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_5}}{d\eta}=0\end{aligned} \tag{91d}$$

$$\begin{aligned}\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{19}}\right] \quad & +\frac{d\overline{W_1}}{d\eta}\frac{d\overline{W_{14}}}{d\eta}+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_{13}}}{d\eta}+\frac{d\overline{W_4}}{d\eta}\frac{d\overline{W_9}}{d\eta}+\frac{d\overline{W_5}}{d\eta}\frac{d\overline{W_8}}{d\eta} \\ & +n_1\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_9}}{d\eta}+\frac{1}{2}n_1\left(\frac{d\overline{W_5}}{d\eta}\right)^2=0\end{aligned} \tag{91e}$$

$$\frac{d^2}{d\eta^2}\left[(1-\eta)\overline{S_{20}}\right]+\frac{d\overline{W_2}}{d\eta}\frac{d\overline{W_{14}}}{d\eta}+\frac{d\overline{W_5}}{d\eta}\frac{d\overline{W_9}}{d\eta}=0, \tag{91f}$$

which should satisfy the boundary conditions, that is, Equation (74), in which (i = 15, 16, 17, 18, 19, 20); thus, the solution gives

$$\begin{cases}\overline{S_{15}}=\overline{S_{16}}=\overline{S_{18}}=\overline{S_{20}}=0 \\ \overline{S_{17}}=n_3\overline{S_5}=\frac{n_3 l^2}{96}\left[\eta^3+(4\lambda_1+1)\eta^2+(6\lambda_1^2+4\lambda_1+1)\eta+\lambda_2(6\lambda_1^2+4\lambda_1+1)\right]\end{cases} \tag{92a}$$

$$\begin{aligned}&\overline{S_{19}}=-\frac{l^5(n_1+m_1)}{7741440}\times \\ &\left\{\begin{array}{l} 3\eta^7+\left(23040T^2/l+34\lambda_1+13\right)\eta^6+\left[9216(14\lambda_1-1)T^2/l+182{\lambda_1}^2+160\lambda_1+41\right]\eta^5+ \\ \left[\begin{array}{l}\left(290304{\lambda_1}^2-64512\lambda_1-9216\right)T^2/l+462{\lambda_1}^3 \\ +728{\lambda_1}^2+412\lambda_1+83+42\lambda_2\left(6{\lambda_1}^2+4\lambda_1+1\right)\end{array}\right]\eta^4+ \\ \left[\begin{array}{l}9216\left(35{\lambda_1}^3-21{\lambda_1}^2-7\lambda_1-1\right)T^2/l+420{\lambda_1}^4+1512{\lambda_1}^3+ \\ 1708{\lambda_1}^2+832\lambda_1+153+56\lambda_2\left(30{\lambda_1}^3+32{\lambda_1}^2+13\lambda_1+2\right)\end{array}\right]\eta^3+ \\ \left[\begin{array}{l}9216\left(35{\lambda_1}^4-7{\lambda_1}^3-21{\lambda_1}^2-7\lambda_1-1\right)T^2/l+2100{\lambda_1}^4+4032{\lambda_1}^3+3108{\lambda_1}^2+ \\ 1112\lambda_1+153+56\lambda_2\left(90{\lambda_1}^4+120{\lambda_1}^3+67{\lambda_1}^2+18\lambda_1+2\right)\end{array}\right]\eta^2+ \\ \left[\begin{array}{l}\left(483840{\lambda_1}^5-1128960{\lambda_1}^4-645120{\lambda_1}^3-193536{\lambda_1}^2-64512\lambda_1-9216\right)T^2/l \\ +2520{\lambda_1}^5+5880{\lambda_1}^4+6132{\lambda_1}^3+3528{\lambda_1}^2+1112\lambda_1+153 \\ +28\lambda_2\left(270{\lambda_1}^5+450{\lambda_1}^4+345{\lambda_1}^3+149{\lambda_1}^2+35\lambda_1+4\right)\end{array}\right]\eta+ \\ \left[\begin{array}{l}\left(483840{\lambda_1}^5-1128960{\lambda_1}^4-645120{\lambda_1}^3-193536{\lambda_1}^2-64512\lambda_1-9216\right)T^2/l \\ +3528{\lambda_1}^2+7560{\lambda_1}^5\lambda_2+2520{\lambda_1}^5+1112\lambda_1+12600{\lambda_1}^4\lambda_2+1008\lambda_1\lambda_2 \\ +4172{\lambda_1}^2\lambda_2+5880{\lambda_1}^4+6132{\lambda_1}^3+9660{\lambda_1}^3\lambda_2+112\lambda_2+153\lambda_2\end{array}\right]\lambda_2 \end{array}\right\}\end{aligned} \tag{92b}$$

Similarly, we end the computation here. After summarizing the results, we have

$$W = \left(1 + m_1\alpha_1 + m_2{\alpha_1}^2 + m_3{\alpha_1}^3\right)\left(\eta^2 + 2\lambda_1\eta\right)\frac{l}{4}P_m + [1 + (n_1 + m_1)\alpha_1]\overline{W_9}P_m^3 \tag{93}$$

$$S = \left(1 + n_1\alpha_1 + n_2{\alpha_1}^2 + n_3{\alpha_1}^3\right)\overline{S_5}P_m^2 + [1 + (2n_1 + m_1)\alpha_1]\overline{S_{14}}P_m^4 \tag{94}$$

in which $\overline{W_9}$, $\overline{S_5}$, $\overline{S_{14}}$ are shown in Equations (86), (80), and (88b), respectively.

4. Comparisons and Discussions

4.1. Comparison of Two Biparametric Perturbation Solutions

The biparametric perturbation solution based on α_1 and W_m is shown in Equations (65)–(67), while another biparametric perturbation solution based on α_1 and P_m is shown in Equations (93) and (94). It is interesting to compare the two solutions and try to find the difference due to the different parameter combinations.

Let us focus our attention on the relation of load vs. central deflection, that is, P~W_m, which is of importance for the analysis and design of thin plates. We first notice that, for the solution based on α_1 and W_m, the relation of P~W_m has been obtained since this relation is explicitly given in Equation (65), but for the solution based on α_1 and P_m, this relation in Equation (93) is implicit. Thus, we may obtain this relation by further mathematical treatment. To this end, we let η = 1 in Equation (93), also substituting $\overline{W_9}$ at η = 1 into it, thus yielding

$$\begin{aligned} W_m &= (1 + m_1\alpha_1 + m_2{\alpha_1}^2 + m_3{\alpha_1}^3)\tfrac{l(1+2\lambda_1)}{4}P_m \\ &\quad -\tfrac{1+(n_1+m_1)\alpha_1}{276480}\left[\begin{array}{l} 3456(20\lambda_1^4 - 50\lambda_1^3 - 10\lambda_1^2 - 5\lambda_1 - 1)T^2/l \\ +360\lambda_1^4 + 840\lambda_1^3 + 825\lambda_1^2 + 388\lambda_1 + 73 \\ +10\lambda_2(108\lambda_1^4 + 162\lambda_1^3 + 108\lambda_1^2 + 35\lambda_1 + 5) \end{array}\right]P_m^3 \end{aligned} \tag{95}$$

The inversion transform of Equation (95) will give, after ignoring higher-order terms and also considering Equation (68),

$$\begin{aligned} P_m = \tfrac{P}{16} &= \tfrac{4}{(1+m_1\alpha_1+m_2{\alpha_1}^2+m_3{\alpha_1}^3)l(1+2\lambda_1)}W_m \\ &\quad +\tfrac{1+(n_1+m_1)\alpha_1}{1080(1+2\lambda_1)^4(1+m_1\alpha_1+m_2{\alpha_1}^2+m_3{\alpha_1}^3)^4}\times \\ &\quad \left[\begin{array}{l} 3456(20\lambda_1^4 - 50\lambda_1^3 - 10\lambda_1^2 - 5\lambda_1 - 1)T^2/l \\ +360\lambda_1^4 + 840\lambda_1^3 + 825\lambda_1^2 + 388\lambda_1 + 73 \\ +10\lambda_2(108\lambda_1^4 + 162\lambda_1^3 + 108\lambda_1^2 + 35\lambda_1 + 5) \end{array}\right]W_m^3 \end{aligned} \tag{96}$$

while, at the same time, Equation (65) gives, after substituting P_9 into it,

$$\begin{aligned} \tfrac{P}{16} &= \tfrac{4[1-m_1\alpha_1-(m_2-{m_1}^2){\alpha_1}^2-(m_3-2m_1m_2+{m_1}^3){\alpha_1}^3]}{l(2\lambda_1+1)}W_m \\ &\quad +\tfrac{1+(n_1+m_1)\alpha_1}{1080l(2\lambda_1+1)^4}\left[\begin{array}{l} (69120T^2 + 1080l\lambda_2 + 360l){\lambda_1}^4 + \\ (-172800T^2 + 1620l\lambda_2 + 840l){\lambda_1}^3 + \\ (-34560T^2 + 1080l\lambda_2 + 825l){\lambda_1}^2 + \\ (-17280T^2 + 350l\lambda_2 + 388l)\lambda_1 - 3456T^2 + 50l\lambda_2 + 73l \end{array}\right]W_m^3 \end{aligned} \tag{97}$$

It is easy to see that the structural forms of the two solutions are the same, both giving the similar relation; that is, the load is the sum of the terms of W_m and ${W_m}^3$, only with the coefficient differences of W_m and ${W_m}^3$. For the coefficient of W_m, if we spread, in the form of power series, $(1 + m_1\alpha_1 + m_2\alpha_1^2 + m_3\alpha_1^3)^{-1}$ in Equation (96), then the expression in the numerator of the coefficient of W_m in Equation (97) may be obtained, thus verifying the consistency of the term W_m. For the coefficient of ${W_m}^3$, due to the fact that m_i (i = 1, 2, 3) and α_1 all are small quantities, $(1 + m_1\alpha_1 + m_2\alpha_1^2 + m_3\alpha_1^3)^4$ of ${W_m}^3$ in Equation (96) may be approximated as 1; thus, the only difference is embodied in expressions in square brackets, which may be caused by different perturbation parameter combinations. We have to accept the existence of this difference; at least the difference is not very big.

4.2. Comparison with Single-Parameter Perturbation Solution

In our previous study [40], only one parameter W_m is selected to carry out the perturbation solving; the result of P~W_m is as follows:

$$\begin{aligned}\frac{P}{16} &= \frac{4K}{2\lambda_1+1}W_m \\ &+ \frac{W_m^3}{1080(2\lambda_1+1)^4}\left[\begin{array}{l} V\left(\begin{array}{l} 1080\,\lambda_1{}^4\lambda_2+360\,\lambda_1{}^4\;+1620\,\lambda_1{}^3\lambda_2+840\,\lambda_1{}^3+1080\,\lambda_1{}^2\lambda_2 \\ +825\,\lambda_1{}^2+350\,\lambda_1\lambda_2+388\,\lambda_1+50\lambda_2+73 \end{array}\right) \\ +KT^2\left(69120\,\lambda_1{}^4-172800\,\lambda_1{}^3-34560\,\lambda_1{}^2-17280\,\lambda_1-3456\right) \end{array}\right]\end{aligned} \tag{98}$$

in which K and V have the same meaning as this study, but without the expansions with respect to α_1. If we substitute the expansions of K and V, that is, Equation (35), into Equation (98), we will have

$$\begin{aligned}\frac{P}{16} &= \frac{4}{2\lambda_1+1}\frac{1}{l\left(1+m_1\alpha_1+m_2\alpha_1^2+\ldots\right)}W_m \\ &+ \frac{W_m^3}{1080(2\lambda_1+1)^4}\left[\begin{array}{l} \left(1+n_1\alpha_1+n_2\alpha_1^2+\ldots\right)\left(\begin{array}{l} 1080\lambda_1{}^4\lambda_2+360\lambda_1{}^4+1620\lambda_1{}^3\lambda_2+ \\ 840\lambda_1{}^3+1080\lambda_1{}^2\lambda_2+825\lambda_1{}^2+ \\ 350\lambda_1\lambda_2+388\lambda_1+50\lambda_2+73 \end{array}\right) \\ +\frac{T^2}{l\left(1+m_1\alpha_1+m_2\alpha_1^2+\ldots\right)}\left(\begin{array}{l} 69120\lambda_1{}^4-172800\lambda_1{}^3- \\ 34560\lambda_1{}^2-17280\lambda_1-3456 \end{array}\right) \end{array}\right]\end{aligned} \tag{99}$$

If we compare Equation (99) with Equation (96) or (97), once again, it is easy to see that the term of W_m is the same, and the term of $W_m{}^3$ has slight differences, which is mainly reflected in the introduction of K and V expansions and may be caused by the biparametric perturbation.

4.3. Regression Verification

In the improved Föppl–von Kármán equations, the parameter T plays an important role, which serves as a bridge to connect the Föppl–von Kármán equations considering precise curvature with its counterpart considering approximate curvature. Simply speaking, letting T = 0 in Equations (96) and (97) will yield, respectively,

$$\begin{aligned}P_m = \frac{P}{16} &= \frac{4}{(1+m_1\alpha_1+m_2\alpha_1{}^2+m_3\alpha_1{}^3)l(1+2\lambda_1)}W_m \\ &+\frac{1+(n_1+m_1)\alpha_1}{1080(1+2\lambda_1)^4(1+m_1\alpha_1+m_2\alpha_1{}^2+m_3\alpha_1{}^3)^4}\times \\ &\left[\begin{array}{l} +360\lambda_1^4+840\lambda_1^3+825\lambda_1^2+388\lambda_1+73 \\ +10\lambda_2(108\lambda_1^4+162\lambda_1^3+108\lambda_1^2+35\lambda_1+5) \end{array}\right]W_m^3\end{aligned} \tag{100}$$

and

$$\begin{aligned}\frac{P}{16} &= \frac{4[1-m_1\alpha_1-(m_2-m_1{}^2)\alpha_1{}^2-(m_3-2m_1m_2+m_1{}^3)\alpha_1{}^3]}{l(2\lambda_1+1)}W_m \\ &+\frac{1+(n_1+m_1)\alpha_1}{1080l(2\lambda_1+1)^4}\left[\begin{array}{l} (1080l\lambda_2+360l)\lambda_1{}^4+(1620l\lambda_2+840l)\lambda_1{}^3+ \\ (1080l\lambda_2+825l)\lambda_1{}^2+(350l\lambda_2+388l)\lambda_1+50l\lambda_2+73l \end{array}\right]W_m^3\end{aligned} \tag{101}$$

which agrees with the solutions of the Föppl–von Kármán equations considering approximate curvature, that is, Equations (73) and (103) from [18]. This agreement indicates that the biparametric perturbation solutions derived in this study are basically correct, from the viewpoint of regression of solution.

5. Concluding Remarks

In this study, we applied the biparametric perturbation method to solve the improved Föppl–von Kármán equations considering the precise curvature formulas from deformation and the bimodular functionally graded properties from materials. To clearly demonstrate the application of the biparametric perturbation method, two groups of parameter combinations are adopted to obtain two different forms of biparametric perturbation solutions. The following three conclusions can be drawn:

(i) The biparametric perturbation solution based on α_1 and W_m is consistent with another solution based on α_1 and P_m, but two groups of parameter combinations have their own advantages. From the solution on α_1 and W_m, the important relation of P~W_m is easily obtained since it has been explicit in the solution, while for the solution on α_1 and P_m, the relation of P~W_m is existed implicitly in the solution.
(ii) The biparametric perturbation solution is consistent with the single-parameter perturbation solution. Although the selection for multiple parameters makes the perturbation process more complex, the participation of multiple parameters enables us to capture the influence of each factor. This undoubtedly brings convenience to the subsequent parameter analysis.
(iii) Parameters should be chosen with caution, thus combining to achieve an effective perturbation. However, this effectiveness is limited by the respective properties of each parameter and their interrelationships. For example, α_1 and W_m are a set of successful combination in this study, achieving a better convergence, in which α_1 stands for material property and W_m stands for geometrical characteristic, and they are independent of each other. Another set of α_1 and P_m has the same asymptotic effect, and they are not dependent of each other. However, if we try to combine W_m and P_m as a set of parameter combination, we will inevitably encounter undesired results. The reason is the fact that the two parameters are not essentially independent of each other, so the perturbation is doomed to fail.

There is no denying that the calculation process of the parametric perturbation method is somewhat lengthy and annoying. At the same time, this disadvantage is exacerbated by the introduction of multiple perturbation parameters. These two facts may be regarded as the limitations of the multiparameter perturbation proposed. However, the solution is analytical and asymptotic in nature. With the approximation of each order, the influence of a parameter on the solution is shown step by step, and the property of the solution is gradually revealed, which is the intrinsic charm of the perturbation method.

The multiparameter perturbation method proposed in this work attributes to a regular perturbation method, which is one of two important divisions of perturbation. Another is the singular perturbation (including the matching asymptotic expansion method and the deformation coordinate method), which is widely used in the field of fluid mechanics [45–48], for example, the boundary layer problem. When the singular perturbation method is applied to these problems, there exists also a parameter selection issue. From this perspective, the multiparameter perturbation can also be attempted to generalize to the singular perturbation method.

In many industrial applications, the large deformation problem of flexible elements can be found everywhere, and the materials that compose them are also diverse. This work is helpful in analyzing the large deformation problem of flexible platelike elements with bimodular functionally graded properties, and the multiparameter perturbation formulas may be served as a theoretical reference for designers and engineers from applied fields.

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