mathematics

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Stabilization for Stochastic Coupled Kuramoto Oscillators via Nonlinear Distributed Feedback Control

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Abstract: This paper investigates the stabilization for stochastic coupled Kuramoto oscillators (SCKOs) via nonlinear distributed feedback control. An original nonlinear distributed feedback control with the advantages of fast response, no steady-state deviation, and easy implementation is designed to stabilize SCKOs. With the help of the Lyapunov method and stochastic analysis skills, some novel sufficient conditions guaranteeing the stochastic stability for SCKOs are provided by constructing a new and suitable Lyapunov function for SCKOs. Finally, a numerical example is given to illustrate the effectiveness and applicability of the theoretical result.

Keywords: stabilization; stochastic coupled Kuramoto oscillators; nonlinear distributed feedback control; Lyapunov method

MSC: 05C81; 60H10

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Citation: Kang, R.; Gao, S. Stabilization for Stochastic Coupled Kuramoto Oscillators via Nonlinear Distributed Feedback Control. *Mathematics* **2022**, *10*, 3329. https://doi.org/10.3390/math10183329

Academic Editor: Roman Ivanovich Parovik

Received: 30 July 2022
Accepted: 11 September 2022
Published: 14 September 2022

1. Introduction

In recent years, coupled systems have been extensively studied for their applications in physics, ecology, epidemiology, etc. [1–12]. In particular, coupled Kuramoto oscillators (CKOs) have received a lot of attention, and their dynamical properties have been investigated by many scholars [13–16]. In [14], researchers not only studied CKOs' robustness, but also pointed out their research value in the field of deep brain stimulation in neurological diseases. In [15], a novel method of perceptual grouping model is proposed based on CKOs, which can dramatically improve the convergence speed, anti-noise ability and computing performance. So far, CKOs have been applied in various fields broadly, such as engineering, chemistry, biology and mathematics [17–22]. However, CKOs in the real world are always affected by environmental noise inevitably, which may change the dynamical properties of CKOs in general, and stochastic models are more consistent with the actual world relatively [2,23–25]. In some cases, the perturbation intensity to CKOs is linear; please see the literature [23,26]. As is known to all, in many practical situations, such as engineering, physics, mathematics and other fields, CKOs may also be influenced by nonlinear perturbations [27,28]. Consequently, it is necessary to consider stochastic coupled Kuramoto oscillators (SCKOs).

The practical application of SCKOs mainly depends on their dynamical properties, especially stability. Thus, how to analyze and control the stability of SCKOs is an important issue of concern to the scientific community. In this paper, we focus on the stochastic stability of SCKOs model with input control as follows.

$$\mathrm{d}\theta_i(t) = \left[W_i - \frac{k}{n}\sum_{j=1}^{n}\sin\big(\theta_i(t)-\theta_j(t)\big) + u_i\right]\mathrm{d}t + \alpha_i\theta_i^{\beta_i}(t)\mathrm{d}B_i(t), \quad i = 1,2,\ldots,n, \tag{1}$$

where W_i is the vibration frequency of oscillators in the absence of interaction, n is the number of the oscillators, and constant k represents the coupling strength. The phase of oscillator i is denoted as θ_i, which is a variable moving around the limit cycle and increasing with time. $\sin(\theta_i - \theta_j)$ stands for the influence of oscillator j on oscillator i. We use u_i for a control and $B_i(t)$ for a one-dimensional Brownian motion. In addition, the perturbation intensity is represented by $\alpha_i \theta_i^{\beta_i}$, and α_i and β_i are two positive constants.

As we all know, the Lyapunov method is one of the powerful methods to discuss system stability. However, it is very difficult to construct an appropriate Lyapunov function for system (1), because it is nonlinear and takes input control as well as random factors into account. Motivated by the above discussions and some studies in the literature [29,30], we aim to design the controller u_i to ensure that system (1) satisfies the following two points:

1. System (1) has a unique trivial solution $\theta^* = (0, 0, \dots, 0)^{\mathrm{T}}$;
2. The solution θ^* of system (1) is stable.

Compared with the previous literature, our main contributions are as follows:

- A novel nonlinear distributed feedback control is designed to stabilize the SCKOs.
- By constructing a new Lyapunov function for SCKOs and applying some stochastic analysis skills, sufficient criteria for the stability of SCKOs which can be easily checked are obtained.
- A numerical simulation is performed to verify the effectiveness and applicability of our theoretical result.

The rest of this paper is organized below. Section 2 introduces the notations and preliminary knowledge. In Section 3, a new nonlinear distributed feedback control is designed, and some novel sufficient conditions are derived to determine the stability of SCKOs by the Lyapunov method and stochastic analysis skills. Section 4 presents an example and numerical simulation to verify the effectiveness and applicability of our theoretical result. Finally, some conclusions are given in Section 5.

2. Notations and Preliminary Knowledge

In this section, some notations which will be used later are introduced in Section 2.1. Some knowledge about stochastic analysis and the definition of stochastic stability are given in Section 2.2.

2.1. Notations

Throughout this paper, we use the following notations unless otherwise specified. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^{*+} = \{1, 2, \dots\}$, and $\mathbb{R}^n$ be the n-dimensional Euclidean space. The $a \vee b$ and $a \wedge b$ represent the maximum and the minimum of a and b, respectively. The indicator function I_D of a set D is defined by

$$I_D(x) = \begin{cases} 1, & for\ x \in D, \\ 0, & for\ x \notin D. \end{cases}$$

Denote the Euclidean norm for matrices by $|\cdot|$ and write the transpose of vectors or matrices as superscript "T" in the full text. Let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$ represent the family of all real-valued functions $V(x, t)$ defined on $\mathbb{R}^d \times \mathbb{R}^+$ which are continuously twice differentiable in x and once differentiable in t. A function α: $\mathbb{R}^+ \to \mathbb{R}^+$ is of class κ, if α is continuous, strictly increasing and $\alpha(0) = 0$. A complete probability space is represented by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, in which Ω is a sample space, $\mathcal{F}$ is a σ-field, $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, and $\mathbb{P}$ is a probability measure. Let $B(t) = (B_1(t), \dots, B_n(t))^{\mathrm{T}}$ be an n-dimensional Brownian motion, which is defined on the probability space. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$.

2.2. Preliminaries

In what follows, some prior knowledge about stochastic differential equations is presented, which can be found in [23]. Consider a general stochastic differential equation, which is described in the following form

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t),$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $B(t)$ is an n-dimensional Brownian motion. In addition, functions $f, g : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ are the drift coefficient as well as the diffusion coefficient, respectively. For any given $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$, we define a differential operator $\mathcal{L}V(x,t)$ as follows.

$$\mathcal{L}V(x,t) = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x,t) + \frac{1}{2}\mathrm{trace}\Big[g^{\mathrm{T}}(x,t)\frac{\partial^2 V(x,t)}{\partial x^2} g(x,t)\Big],$$

where

$$\frac{\partial V(x,t)}{\partial x} = \Big(\frac{\partial V(x,t)}{\partial x_1}, \dots, \frac{\partial V(x,t)}{\partial x_n}\Big), \frac{\partial^2 V(x,t)}{\partial x^2} = \begin{pmatrix} \frac{\partial^2 V(x,t)}{\partial x_1 \partial x_1} & \frac{\partial^2 V(x,t)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V(x,t)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 V(x,t)}{\partial x_2 \partial x_1} & \frac{\partial^2 V(x,t)}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 V(x,t)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 V(x,t)}{\partial x_n \partial x_1} & \frac{\partial^2 V(x,t)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V(x,t)}{\partial x_n \partial x_n} \end{pmatrix}.$$

Definition 1 ([23])**.** *The trivial solution of system (1) is said to be stochastically stable if for any $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$, such that*

$$\mathbb{P}\{|x(t; t_0, x_0)| < r, t \geq t_0\} \geq 1 - \varepsilon,$$

wherever $|x_0| < \delta$.

3. Main Result

In this section, a nonlinear distributed feedback control is designed, and sufficient criteria for the stability of system (1) are obtained by constructing a suitable Lyapunov function and applying some stochastic analysis skills.

Theorem 1. *A nonlinear distributed feedback control*

$$u_i = -\Big(3k + \alpha_i^2\Big)\mathrm{sgn}\theta_i - W_i \tag{2}$$

is designed, where W_i is the vibration frequency of oscillators without interaction, and k denotes the coupling strength. The phase of oscillator i is represented by θ_i, which is a variable that moves around the limit cycle and increases over time. In addition, α_i is the positive constant in the perturbation intensity $\alpha_i \theta_i^{\beta_i}$. If $\beta_i \in (0, \frac{1}{2}]$, then system (1) has a unique global asymptotic stochastic stable trivial solution $\theta^* = (0, 0, \dots, 0)^{\mathrm{T}}$.

Proof. For the aim of this paper, let

$$V_i(\theta_i, t) = \frac{1}{2}\theta_i^2 + \frac{k}{n}\sum_{j=1}^{n}\Big[1 - \cos\Big(\int_0^t \theta_i(s)\mathrm{d}s - \int_0^t \theta_j(s)\mathrm{d}s\Big)\Big]. \tag{3}$$

By the definition of operator $\mathcal{L}V(x,t)$, it follows that

$$\begin{aligned}
\mathcal{L}V_i(\theta_i,t) \quad &= \quad \theta_i\left[W_i - \frac{k}{n}\sum_{j=1}^{n}\sin(\theta_i-\theta_j) - \left(3k+\alpha_i^2\right)\mathrm{sgn}\theta_i - W_i\right] \\
&\quad + \frac{k}{n}\sum_{j=1}^{n}\sin\left(\int_0^t\theta_i(s)\mathrm{d}s - \int_0^t\theta_j(s)\mathrm{d}s\right)(\theta_i-\theta_j) + \alpha_i^2\left(\theta_i^{\beta_i}\right)^2 \\
&\leq \quad -\frac{k}{n}\sum_{j=1}^{n}\sin(\theta_i-\theta_j)\theta_i - \left(3k+\alpha_i^2\right)|\theta_i| + \alpha_i^2|\theta_i| + \frac{k}{n}\sum_{j=1}^{n}(|\theta_i|-|\theta_j|) \\
&\leq \quad \left(3k+\alpha_i^2\right)|\theta_i| - \left(3k+\alpha_i^2\right)|\theta_i| + \frac{k}{n}\sum_{j=1}^{n}(|\theta_j|-|\theta_i|) \\
&\leq \quad \frac{k}{n}\sum_{j=1}^{n}(|\theta_j|-|\theta_i|). \qquad (4)
\end{aligned}$$

We describe system (1) by a weighted directed digraph $(\mathcal{G},A)$ with $n(n\geq 2)$ vertices, where $A=(a_{ij})_{n\times n}=(k/n)_{n\times n}$ is a weight matrix which stands for the coupled strength of oscillators of system (1). Each vertex of the directed digraph $\mathcal{G}$ denotes an oscillator of system (1), and the directed arcs represent the inter-connections and interactions among vertex systems.

Let $V(\theta,t)=\sum_{i=1}^{n}C_iV_i(\theta_i,t)$, where $C_i=\frac{k^n(n-1)}{n^2}$ $(i=1,2,\ldots,n)$ is the cofactor of the i-th principal diagonal element of the Laplace matrix

$$\begin{pmatrix}
\frac{k(n-1)}{n} & -\frac{k}{n} & \cdots & -\frac{k}{n} \\
-\frac{k}{n} & \frac{k(n-1)}{n} & \cdots & -\frac{k}{n} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{k}{n} & -\frac{k}{n} & \cdots & \frac{k(n-1)}{n}
\end{pmatrix}$$

of weight matrix A. In addition, using inequality (4) and combinatorial identity (see Theorem 2.2 in [31]), we can derive that

$$\begin{aligned}
\mathcal{L}V(\theta,t) &= \sum_{i=1}^{n}C_i\mathcal{L}V_i(\theta_i,t) \leq \sum_{i,j=1}^{n}\frac{k}{n}C_i(|\theta_j|-|\theta_i|) \\
&= \sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})}(|\theta_s|-|\theta_r|) \\
&\leq 0,
\end{aligned}$$

where $\mathbb{Q}$ is the set of all spanning unicyclic graphs, $W(\mathcal{Q})$ denotes the weight of $\mathcal{Q}$, $\mathcal{C}_\mathcal{Q}$ is the directed cycle of $\mathcal{Q}$, and $(s,r)\in E(\mathcal{C}_\mathcal{Q})$ is a set of arcs (s,r) leading from initial vertex s to terminal vertex r. For more detailed knowledge of graph theory, readers can refer to the following studies [32,33]. Furthermore, by applying the definition of $V(\theta,t)$ and Equation (3), we can immediately obtain that

$$V(\theta,t)\geq\frac{k^n(n-1)}{2n^2}\sum_{i=1}^{n}\theta_i^2\geq\frac{k^n(n-1)}{2n^2}|\theta|^2\triangleq\mu(|\theta|),$$

in which $\mu(\cdot)=\frac{k^n(n-1)}{2n^2}(\cdot)^2\in\kappa$. Moreover, let $\varepsilon\in(0,1)$ and $0<r<n$. From the continuity of $V(\theta,t)$ and $V(0,0)=0$, we learn that there exists a $\delta\geq 0$, such that

$$\frac{1}{\varepsilon}\sup_{\theta\in S_\delta}V(\theta,0)\leq\mu(r),$$

in which $S_\delta = \{\theta \| \theta| < \delta\}$. Then, we can easily see that $\delta < r$. Note $\theta(t;0,0) = \theta(t)$, and assume that $\tau = \inf\{t \geq 0; \theta(t) \notin S_r\}$. According to the Itô's formula, it is not hard to deduce that when $t \geq 0$,

$$V(\theta(\tau \wedge t), \tau \wedge t) = V(0,0) + \int_0^{\tau \wedge t} \mathcal{L}V(\theta(s), s)\mathrm{d}s + \int_0^{\tau \wedge t} \frac{\partial V(\theta(s), s)}{\partial \theta} \alpha_i \theta_i^{\beta_i}(s)\mathrm{d}s.$$

Then, it derives that

$$\mathbb{E}V(\theta(\tau \wedge t), \tau \wedge t) \leq V(0,0).$$

For simplicity, we write that when $\tau \leq t$,

$$|\theta(\tau \wedge t)| = |\theta(\tau)| = r.$$

Therefore,

$$\mathbb{E}V(\theta(\tau \wedge t), \tau \wedge t) \geq \mathbb{E}\Big[I_{\{\tau \leq t\}} V(\theta(\tau), \tau)\Big] \geq \mu(V)\mathbb{P}\{\tau \leq t\}.$$

To here, we have obtained that

$$\mathbb{P}\{\tau \leq t\} \leq \varepsilon.$$

Let $t \to \infty$. Thus, it obviously follows that

$$\mathbb{P}\{\tau < \infty\} \leq \varepsilon.$$

From what has been discussed above, it is straightforward to show that

$$\mathbb{P}\{|\theta(t)| < r, t \geq 0\} \geq 1 - \varepsilon.$$

In other words, system (1) is stochastically stable, which means that the proof is complete. □

Remark 1. *In recent years, many researchers have used feedback control to study the stability of a system; see for example [34–37]. A novel nonlinear distributed feedback control is proposed to stabilize system (1). Compared with general feedback control, the new nonlinear feedback control has a faster response because it is a distributed control. Furthermore, the proposed controller causes no deviation of oscillators' vibration frequency in steady state, so it is unnecessary to introduce an integral term to eliminate the deviation. Hence, the controller for system (1) is easy to design. To summarize, the controller that we design has the advantage of quick response, no steady-state error and being easy to implement.*

Remark 2. *In fact, by a sign function, the designed controller can perform power instruction via making a comparison with instantaneous frequency θ_i and the given equilibrium frequency. The designed nonlinear distributed feedback control can be regarded as a sliding mode control. Here, u_i has a variable structure, which always changes purposefully based on the current state of the system. This forces the system to slide into a predetermined state trajectory such that the state of Kuramoto oscillators converges to an equilibrium point.*

Remark 3. *It is very difficult to find a suitable Lyapunov function for system (1) due to its complex structure. From the proof of Theorem 1, the Lyapunov function $V(\theta, t)$ for system (1) is constructed as $V(\theta, t) = \sum_{i=1}^n C_i V_i(\theta_i, t)$. It is easy to see that C_i depends on not only the coupling strength between the individual oscillators but also on the topological structure of system (1). Therefore, finding $V_i(\theta_i, t)$ is the key in the study of stabilization for system (1). The problem is solved successfully by constructing*

$$V_i(\theta_i, t) = \frac{1}{2}\theta_i^2 + \frac{k}{n}\sum_{j=1}^{n}\left[1 - \cos\left(\int_0^t \theta_i(s)\mathrm{d}s - \int_0^t \theta_j(s)\mathrm{d}s\right)\right].$$

4. Numerical Simulation

In order to illustrate the effectiveness of the theoretical result, a numerical simulation is given in this section. For system (1), we suppose $n = 7$. Other corresponding parameters are listed in Table 1.

Table 1. Corresponding parameters of system (1).

No.	W_i	α_i	β_i
1	2.2343	3	0.1667
2	−4.8676	1	0.3333
3	5.5776	2	0.2500
4	−1.7688	4	0.2000
5	5.1792	5	0.1667
6	2.4505	3	0.2000
7	−3.9546	1	0.3333

Choose $k = 50$. Let the values of α_i and W_i be the same as shown in Table 1. Obviously, we have $0 < \beta_i \leq \frac{1}{2}$ for $i = 1, 2, \ldots, 7$. Thus, the values of all the parameters we set satisfy the condition of Theorem 1. According to Theorem 1, system (1) has a unique global stochastic stable trivial solution $\theta^* = (0, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}$.

In addition, we choose the initial phase position as $\theta_0 = (25, -27, 25, -23, 28, -29, 28)^{\mathrm{T}}$. The sample path of system (1) is described in Figure 1.

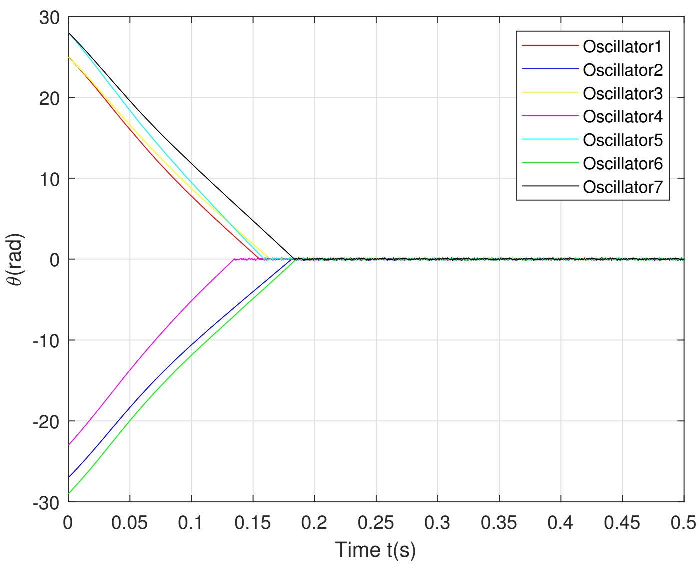

Figure 1. The sample path of the solution to system (1).

From Figure 1, it is easy to find that the solution of system (1) is stochastically stable. Hence, the simulation result confirms the effectiveness of our theoretical result.

Remark 4. *Stability is a quite important characteristic of system (1). Only under the premise of maintaining stability can system (1) work normally and meet the requirements of other performance indicators. From Figure 1, we can easily find that system (1) is stable, which is largely due to the controller we have designed. For readers to understand and convince, as a comparison, we remove the controller and the stochastic perturbation to confirm that the robustness of our designed controller is excellent. According to the idea of control variables, the values of all relevant parameters are still the same as those in Table 1 above, $n = 7$, $k = 50$, and the selection of the initial phase* *position is also the same. Then, we give the numerical simulation result, which is shown in Figure 2. From Figure 2, we can clearly see that the system is not stable, and it may even generate chaos in some cases.*

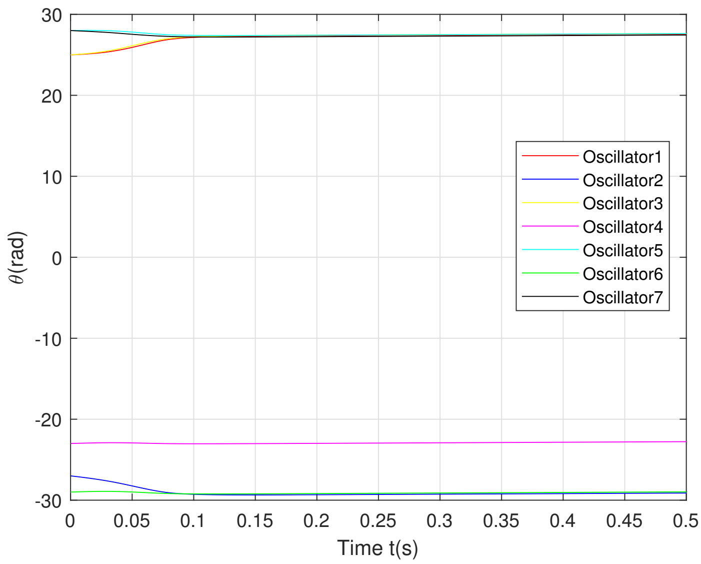

Figure 2. The trajectory of the solution to system (1) without the controller and the stochastic perturbation.

5. Conclusions

In this paper, we have discussed the stabilization of SCKOs by designing a novel nonlinear distributed feedback control successfully, which can guarantee that system (1) has a stable equilibrium solution $\theta^* = (0, 0, \ldots, 0)^{\mathrm{T}}$. In view of the Lyapunov method, some original sufficient criteria have been obtained. A numerical simulation has also been given to verify the applicability and effectiveness of our theoretical result. In fact, our approach can also be applied to explore the stabilization for SCKOs perturbed by some other types of noises, such as SCKOs with Markovian switching and SCKOs with Lévy noise. We will be committed to solving these problems by using our approach in the future.

Author Contributions: Conceptualization, S.G. and R.K.; Methodology, S.G. and R.K.; Software, S.G.; Validation, S.G. and R.K.; Formal analysis, R.K.; Writing—original draft preparation, S.G. and R.K.; Writing—review and editing, R.K.; Visualization, R.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Fundamental Research Funds for the Central Universities (No. 2572022BC04).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by the Fundamental Research Funds for the Central Universities (No. 2572022BC04). The authors sincerely thank anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CKOs	Coupled Kuramoto oscillators
SCKOs	Stochastic coupled Kuramoto oscillators

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