mathematics

MDPI

Article

Symmetric Properties of (b,c)-Inverses

Guiqi Shi and Jianlong Chen *

School of Mathematics, Southeast University, Nanjing 210096, China
* Correspondence: jlchen@seu.edu.cn

Abstract: Let b and c be two elements in a semigroup S. The (b,c)-inverse is an important outer inverse because it unifies many common generalized inverses. This paper is devoted to presenting some symmetric properties of (b,c)-inverses and (c,b)-inverses. We first find that S contains a (b,c)-invertible element if and only if it contains a (c,b)-invertible element. Then, for four given elements a,b,c,d in S, we prove that a is (b,c)-invertible and d is (c,b)-invertible if and only if abd is invertible along c and dca is invertible along b. Inspired by this result, the (b,c)-invertibility is characterized by one-sided invertible elements. Furthermore, we show that a is inner (b,c)-invertible and d is inner (c,b)-invertible if and only if c is inner (a,d)-invertible and b is inner (d,a)-invertible.

Keywords: generalized inverse; (b,c)-inverse; inner (b,c)-inverse; outer inverse

MSC: 16U90; 15A09

1. Introduction

An element a in a semigroup S is said to be regular if there exists $x \in S$ such that $axa = a$, in which case x is called an inner inverse (or a $\{1\}$-inverse) of a. Recall that an involution $*$ of S is a self-map such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a,b \in S$. If there exists x satisfying $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$, then it is the unique solution of the previous four equations and is called the Moore–Penrose inverse [1] of a (denoted by $a^\dagger$).

An element a in a semigroup S is Drazin invertible [2] if there exists $x \in S$ such that

$$xa^{m+1} = a^m \text{ for some } m \in \mathbb{N}^+,\ ax^2 = x,\ ax = xa.$$

If such x exists, then it is unique and called the Drazin inverse of a (denoted by a^D). The smallest integer m that makes the above equations hold is called the Drazin index of a and denoted by $\mathrm{ind}(a)$. If $\mathrm{ind}(a) = 1$, x is called the group inverse of a and denoted by $a^\#$.

Let S be any semigroup and $a,b \in S$. Mary [3] defined that the inverse of a along b as the unique element y satisfying the following relations:

$$y \in bS \cap bS,\ yab = b,\ bay = b.$$

In this case, a is said to be invertible along b, and y is denoted by $a^{||b}$. If, moreover, $aa^{||b}a = a$, then $a^{||b}$ is called the inner inverse of a along b. He also proved that the Moore–Penrose inverse of an element a is equal to $a^{||a^*}$, and the group inverse of a is equal to $a^{||a}$. The set of all elements which are invertible along b is denoted by $S^{||b}$.

Let S be any semigroup and $a,b,c \in S$. Drazin [4] defined the (b,c)-inverse of a to be the unique element y satisfying

$$y \in bS \cap Sc,\ yab = b,\ cay = c.$$

In this case, a is said to be (b,c)-invertible, and y is denoted by $a^{||(b,c)}$. When $b = c$, we can see that $a^{||(b,b)} = a^{||b}$. To see the difference between inverses along an element and

Citation: Shi, G.; Chen, J. Symmetric Properties of (b,c)-Inverses. *Mathematics* **2022**, *10*, 2948. https://doi.org/10.3390/math10162948

Academic Editors: Jaan Janno and Michael Voskoglou

Received: 22 May 2022
Accepted: 12 August 2022
Published: 16 August 2022

(b,c)-inverses, we consider the semigroup $\mathbb{C}^{2\times 2}$. Let $a = b = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $c = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Then $a^{||(b,c)} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = a^{||b}$.

Later, Drazin [5] also defined the one-sided version of the (b,c)-inverse in a semigroup S. If $b \in Scab$, or equivalently if there exists y such that $y \in Sc$ and $yab = b$, then a is said to be left (b,c)-invertible. Such y is called a left (b,c)-inverse of a. Dually, a is said to be right (b,c)-invertible if $c \in cabS$, or equivalently if there exists z such that $z \in bS$ and $caz = c$. Such z is called a right (b,c)-inverse of a. Drazin proved that a is (b,c)-invertible if and only if a is left and right (b,c)-invertible. Given any semigroup S and $b, c \in S$, we denote the sets of all left (b,c)-invertible elements, right (b,c)-invertible elements and (b,c)-invertible elements in S by $S_l^{||(b,c)}$, $S_r^{||(b,c)}$ and $S^{||(b,c)}$, respectively.

The motivation of this paper comes from the following facts.

Lemma 1 (Theorem 7 in [3]). *Let S be any semigroup and $a, b \in S$. Then a is invertible along b if and only if ab is group invertible with $b \in Sab$ if and only if ba is group invertible with $b \in baS$, in which case,*

$$a^{||b} = b(ab)^{\#} = (ba)^{\#}b.$$

Lemma 2 (Corollary 2.7 in [6]). *Let S be any semigroup and $a, b \in S$. Then a is inner invertible along b if and only if a is invertible along b and b is invertible along a.*

These are two interesting results with nice symmetry. However, in general cases, the (b,c)-invertibility of a does not imply that ab, ac, ba and ca are group invertible (see Example 2.1 in [7]), and a being (b,c)-invertible with $aa^{||(b,c)}a = a$ does not imply that $b \in S^{||(a,a)}$ and $c \in S^{||(a,a)}$ (see the case of $b = a$ and $c = a^*$).

Wu and Chen [7] had done some interesting work on the case of $a \in S^{||(b,c)} \cap S^{||(c,b)}$. They characterize $a \in S^{||(b,c)} \cap S^{||(c,b)}$ by using group invertible elements and invertible elements, respectively. We find that $S^{||(b,c)} \neq \emptyset$ implies $S^{||(c,b)} \neq \emptyset$, so it seems more natural to consider the situation $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$, which of course includes the case of $a \in S^{||(b,c)} \cap S^{||(c,b)}$. This paper focuses on the case of $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$.

In Section 2, we prove that

$$a \in S^{||(b,c)} \text{ and } d \in S^{||(c,b)} \quad \Leftrightarrow \quad abd \in S^{||c} \text{ and } dca \in S^{||b},$$

which allows us to transform many questions on the (b,c)-invertibility and (c,b)-invertibility into those on the invertibility along b and c. As an application of this observation, the (one-sided) (b,c)-invertibility is characterized by one-sided invertible elements.

If $a \in S^{||(b,c)}$ such that $aa^{||(b,c)}a = a$, then $a^{||(b,c)}$ is called the inner (b,c)-inverse of a. In Section 3, we consider some symmetric properties of inner (b,c)-inverses. We prove that a is inner (b,c)-invertible and d is inner (c,b)-invertible if and only if c is inner (a,d)-invertible and b is inner (d,a)-invertible. Especially, a is both inner (b,c)-invertible and inner (c,b)-invertible if and only if both b and c are inner invertible along a if and only if a is inner invertible along b and c, in which case

$$a^{||(b,c)} = a^{||b}aa^{||c} \quad \text{and} \quad a^{||(c,b)} = a^{||c}aa^{||b}.$$

At last, under the assumption that $aa^{||c}a = a$, we characterize the product $a^{||b}aa^{||c}$ by equations and prove that $a^{||b}aa^{||c}$ is equal to the $(b, baa^{||c})$-inverse of a, which generalizes some results on the DMP inverse.

2. Characterizations of $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$

We first recall two basic characterizations of (b,c)-invertibility, which will be frequently used in the sequel discussion.

Lemma 3 (Theorem 2.2 in [4]). *Let S be any semigroup and $a, b, c \in S$. Then a is (b,c)-invertible if and only if $c \in cabS$ and $b \in Scab$. In this case, $a^{||(b,c)} = sc = bt$, where $c = cabt$ and $b = scab$.*

Lemma 4 (Proposition 6.1 in [4]). *Let S be any semigroup and $a, b, c \in S$. Then a is (b, c)-invertible if and only if there exists $y \in S$ satisfying that*

$$yay = y,\ S^1y = S^1c,\ yS^1 = bS^1,$$

where S^1 stands for the monoid generated by S.

From previous two Lemmas, we can immediately obtain a connection between $S^{||(b,c)}$ and $S^{||(c,b)}$.

Proposition 1. *Let S be any semigroup and $b, c \in S$. Then $S^{||(b,c)} \neq \emptyset$ if and only if $S^{||(c,b)} \neq \emptyset$.*

Proof. If $a \in S^{||(b,c)}$, then $cabS^1 = cS^1$ and $S^1cab = S^1b$ by Lemma 3. From Proposition 3.3 in [8], we know that cab is regular. Thus cab is the (c, b)-inverse of $(cab)^-$ by Lemma 4, for any inner inverse $(cab)^-$ of cab.

By symmetry, the converse statement is also true. □

Proposition 2. *Let S be any semigroup and $b, c \in S$. If $S^{||(b,c)} \neq \emptyset$, then the mapping $\phi : x \mapsto cx^-b$ is a bijection from $\{a^{||(b,c)} \mid a \in S^{||(b,c)}\}$ to $\{d^{||(c,b)} \mid d \in S^{||(c,b)}\}$, for any inner inverse x^- of x.*

Proof. At first, we prove that ϕ is well defined. Suppose that $x = a^{||(b,c)}$ for some $a \in S^{||(b,c)}$. For any inner inverse x^- of x, we know that $a^{||(b,c)}$ is the (b, c)-inverse of x^- by Lemma 4. Then

$$cx^-b = (caa^{||(b,c)})x^-b = ca[(x^-)^{||(b,c)}x^-b] = cab.$$

Next we prove that ϕ is a bijection. Define another mapping $\psi : y \mapsto by^-c$ from $\{d^{||(c,b)} \mid d \in S^{||(c,b)}\}$ to $\{a^{||(b,c)} \mid a \in S^{||(b,c)}\}$. Similarly, ψ is well-defined. Then we obtain

$$\psi\phi(x) = b(cab)^-c = a^{||(b,c)} = x,$$

where the last second equality holds because of Theorem 2.7 in [9]. Similarly, $\phi\psi(y) = y$. Thus, ϕ is a bijection. □

Let a, b, c be elements in a semigroup S such that a is (b, c)-invertible. We wonder what conditions are needed to ensure that d is (c, b)-invertible. To handle this question, we consider the following Lemma.

Lemma 5. *Let S be any semigroup and $b, c, d, u, v \in S$. If $uS^1 = bS^1$ and $S^1v = S^1c$, then*

(1) *d is left (b, c)-invertible if and only if d is left (u, v)-invertible;*
(2) *d is right (b, c)-invertible if and only if d is right (u, v)-invertible;*
(3) (Remark 2.2(i) in [10]) *d is (b, c)-invertible if and only if d is (u, v)-invertible, in which case, $a^{||(b,c)} = a^{||(u,v)}$.*

Proof. (1) Suppose that $b = ug$, $c = hv$, $u = bt$ and $v = sc$ for some $g, h, s, t \in S^1$.

If d is left (b, c)-invertible, then there exists $x \in S$ such that $b = xcdb$. It follows that

$$u = bt = xcdbt = xcdu = xhvdu \in Svdu.$$

Conversely, suppose that $u = yvdu$ for some $y \in S$. We have that

$$b = ug = yvdug = yvdb = yscdb \in Scdb.$$

(2) It can be proved similarly.
(3) It can be proved by combining (1) and (2). □

Proposition 3. *Let S be any semigroup and $a, b, c, d \in S$. If a is (b, c)-invertible, then we have the following:*

(1) *d is left (c, b)-invertible if and only if d is left invertible along cab;*
(2) *d is right (c, b)-invertible if and only if d is right invertible along cab;*
(3) *d is (c, b)-invertible if and only if d is invertible along cab, in which case, $d^{||(c,b)} = d^{||cab}$.*

Proof. If $a \in S^{||(b,c)}$, then $cabS^1 = cS^1$ and $S^1cab = S^1b$ by Lemma 3. Taking $u = v = cab$ and exchanging the position of b and c in Lemma 5, then the proposition follows. □

Now we give the main result of this section, which presents a necessary and sufficient condition for any semigroup S and $a, b, c, d \in S$ such that $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$.

Theorem 1. *Let S be any semigroup and $a, b, c, d \in S$. Then $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$ if and only if $abd \in S^{||c}$ and $dca \in S^{||b}$. In this case,*

$$a^{||(b,c)} = bd(abd)^{||c} = (dca)^{||b}dc,$$

$$d^{||(c,b)} = (abd)^{||c}ab = ca(dca)^{||b}.$$

Proof. If $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$, then we know that

$$c \in cabS, \quad b \in Scab, \quad b \in bdcS, \quad \text{and} \quad c \in Sbdc$$

by Lemma 3. It follows that

$$c \in cabS \subseteq cabdcS \quad \text{and} \quad c \in Sbdc \subseteq Scabdc,$$

which means that $abd \in S^{||c}$. Similarly, $dca \in S^{||b}$.

Conversely, if $abd \in S^{||c}$ and $dca \in S^{||b}$, then we have

$$c = cabd(abd)^{||c} \in cabS \quad \text{and} \quad b = (dca)^{||b}dcab \in Scab.$$

So $a \in S^{||(b,c)}$ by Lemma 3. Similarly, $d \in S^{||(c,b)}$. The formulae of $a^{||(b,c)}$ and $d^{||(c,d)}$ follow from Lemma 3. □

From above proof, we can see that the one-sided version of Theorem 1 is also true. We list it below and omit its proof.

Proposition 4. *Let S be any semigroup and $a, b, c, d \in S$. Then*

(1) *$a \in S_l^{||(b,c)}$ and $d \in S_l^{||(c,b)}$ if and only if $abd \in S_l^{||c}$ and $dca \in S_l^{||b}$;*
(2) *$a \in S_r^{||(b,c)}$ and $d \in S_r^{||(c,b)}$ if and only if $abd \in S_r^{||c}$ and $dca \in S_r^{||b}$.*

Let S be any semigroup and $a, b \in S$. Lemma 1 shows that $a \in S^{||b}$ if and only if $b \in Sab$ and $ab \in S^{\#}$ if and only if $b \in baS$ and $ba \in S^{\#}$, in which case $a^{||(b,b)} = b(ab)^{\#} = (ba)^{\#}b$. By Theorem 1, we can also characterize the (b, c)-inverse and (c, b)-inverse by the group inverses.

Proposition 5. *Let S be any semigroup and $a, b, c, d \in S$. If $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$, then $abdc, bdca, dcab$ and $cabd$ are group invertible. In this case,*

$$a^{||(b,c)} = bdc(abdc)^{\#} = bd(cabd)^{\#}c = b(dcab)^{\#}dc = (bdca)^{\#}bdc,$$

$$d^{||(c,b)} = cab(dcab)^{\#} = ca(bdca)^{\#}b = c(abdc)^{\#}ab = (cabd)^{\#}cab.$$

Proof. If $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$, then $abd \in S^{||c}$ and $dca \in S^{||b}$ by Theorem 1. According to Lemma 1, $abdc, bdca, dcab$ and $cabd$ are group invertible with

$$(abd)^{||c} = c(abdc)^{\#} = (cabd)^{\#}c \text{ and } (dca)^{||b} = b(dcab)^{\#} = (bdca)^{\#}b.$$

Substituting them into the formulae for $a^{||(b,c)}$ and $d^{||(c,b)}$ in Theorem 1, the formulae in terms of the group inverses follow. □

Proposition 6. *Let S be any semigroup and $a, b, c, d \in S$. If u is any one of $abdc$, $bdca$, $dcab$, $cabd$, then the following conditions are equivalent:*
(1) $a \in S^{||(b,c)}$ *and* $d \in S^{||(c,b)}$;
(2) *u is group invertible,* $a \in S_l^{||(b,c)}$ *and* $d \in S_l^{||(c,b)}$;
(3) *u is group invertible,* $a \in S_r^{||(b,c)}$ *and* $d \in S_r^{||(c,b)}$;
(4) *u is Drazin invertible,* $a \in S_l^{||(b,c)}$ *and* $d \in S_l^{||(c,b)}$;
(5) *u is Drazin invertible,* $a \in S_r^{||(b,c)}$ *and* $d \in S_r^{||(c,b)}$.

Proof. (1) ⇒ (2). By Proposition 5.
(2) ⇒ (4). It is obvious.
(4) ⇒ (1). If u is Drazin invertible, then $abdc$ and $dcab$ are Drazin invertible by Cline's formula [11]. Meanwhile, from $b \in Scab$ and $c \in Sbdc$, we know that

$$Sdcab \subseteq Scab \subseteq Sbdcab \subseteq Scabdcab \subseteq Sbdcabdcab \subseteq S(dcab)^2 \subseteq Sdcab.$$

It follows that $\mathrm{ind}(dcab) = 1$, which means that $dcab$ is group invertible. Similarly, $abdc$ is group invertible.
Noting that $b \in Scab \subseteq Sbdcab \subseteq Sdcab$ and $c \in Sbdc \subseteq Scabdc \subseteq Sabdc$, we have

$$b = bdcab(dcab)^{\#} \in bdcS \text{ and } c = cabdc(abdc)^{\#} \in cabS.$$

(1) ⇒ (3) ⇒ (5) ⇒ (1) can be proved dually. □

Let R be any associative ring with 1 and $a, b \in R$ such that b is regular with an inner inverse b^-. Theorem 3.2 in [12] proved that a is invertible along b if and only if $ab + 1 - b^-b$ is invertible if and only if $ba + 1 - bb^-$ is invertible. Denoting the set of all invertible (resp., left and right invertible) elements in R by R^{-1} (resp., R_l^{-1} and R_r^{-1}), we characterize the (one-sided) (b,c)-inverse and (one-sided) (c,b)-inverse by using (one-sided) invertible elements as follows.

Proposition 7. *Let R be any associative ring with 1 and $a, b, c, d \in R$ such that b and c are regular. If b^- is an inner inverse of b and c^- is an inner inverse of c, denote*

$$u = cabd + 1 - cc^-, \quad v = bdca + 1 - bb^-,$$

$$s = abdc + 1 - c^-c, \quad t = dcab + 1 - b^-b.$$

Then
(1) $a \in R_l^{||(b,c)}$ *and* $d \in R_l^{||(c,b)}$ *if and only if* $u \in R_l^{-1}$ *and* $v \in R_l^{-1}$ *if and only if* $s \in R_l^{-1}$ *and* $t \in R_l^{-1}$*, in which case* $u_l^{-1}cab$ *is a left* (c,b)*-inverse of d and* $v_l^{-1}bdc$ *is a left* (b,c)*-inverse of a, where* u_l^{-1} *and* v_l^{-1} *are left inverses of u and v, respectively;*
(2) $a \in R_r^{||(b,c)}$ *and* $d \in R_r^{||(c,b)}$ *if and only if* $u \in R_r^{-1}$ *and* $v \in R_r^{-1}$ *if and only if* $s \in R_r^{-1}$ *and* $t \in R_r^{-1}$*, in which case* $bdcs_r^{-1}$ *is a right* (b,c)*-inverse of a and* $cabt_r^{-1}$ *is a right* (c,b)*-inverse of d, where* s_r^{-1} *and* t_r^{-1} *are right inverses of s and t, respectively;*

(3) $a \in R^{||(b,c)}$ *and* $d \in R^{||(c,b)}$ *if and only if* $u \in R^{-1}$ *and* $v \in R^{-1}$ *if and only if* $s \in R^{-1}$ *and* $t \in R^{-1}$*, in which case,*

$$a^{||(b,c)} = v^{-1}bdc = bdcs^{-1} \text{ and } d^{||(c,b)} = u^{-1}cab = cabt^{-1}.$$

Proof. (1) By Proposition 4, $a \in R_l^{||(b,c)}$ and $d \in R_l^{||(c,b)}$ if and only if $abd \in R_l^{||c}$ and $dca \in R_l^{||b}$. Additionally, $abd \in R_l^{||c}$ and $dca \in R_l^{||b}$ if and only if $u \in R_l^{-1}$ and $v \in R_l^{-1}$ by Theorem 3.2 in [13], which is equivalent to $s \in R_l^{-1}$ and $t \in R_l^{-1}$ by Jacobson's lemma.

If $u \in R_l^{-1}$, multiplying by c on the right of $u = cabd + 1 - cc^-$ yields that $uc = cabdc$. It follows that $c = u_l^{-1}uc = u_l^{-1}cabdc \in Rbdc$, which means that $u_l^{-1}cab$ is a left (c,b)-inverse of d. Similarly, one can prove that $v_l^{-1}bdc$ is a left (b,c)-inverse of a.

(2) Similarly by using Theorem 3.4 in [13].

(3) Combining (1) and (2), it follows. □

If $a \in S^{||(b,c)}$, we showed in the proof of Proposition 1 that $(cab)^- \in S^{||(c,b)}$ for any inner inverse $(cab)^-$ of cab. Suppose that b, c and cab are regular. then $a \in R^{||(b,c)}$ if and only if $u = cab(cab)^- + 1 - cc^- \in R^{-1}$ and $v = (cab)^-cab + 1 - b^-b \in R^{-1}$ by replacing d by $(cab)^-$ in Proposition 7. However, characterizing the left (b,c)-invertibility of a only requires that b, cab are regular and v is left invertible.

Proposition 8. *Let R be any associative ring with* 1 *and* $a, b, c \in R$ *such that b and cab are regular. If* b^- *is an inner inverse of b and* $(cab)^-$ *is an inner inverse of cab, then the following conditions are equivalent:*

(1) *a is left* (b,c)*-invertible;*
(2) $v = (cab)^-cab + 1 - b^-b \in R_l^{-1}$;
(3) $t = b(cab)^-ca + 1 - bb^- \in R_l^{-1}$.

In this case, $t_l^{-1}b(cab)^-c$ *is a left* (b,c)*-inverse of a, where* t_l^{-1} *is a left inverse of t.*

Proof. (1) ⇒ (2). If a is left (b,c)-invertible, then $Rcab = Rb$. It follows that $b(cab)^-cab = b$. Then we have

$$\begin{aligned} & [b^-b + 1 - (cab)^-cab][(cab)^-cab + 1 - b^-b] \\ = \; & b^-b(cab)^-cab + b^-b(1 - b^-b) + (1 - (cab)^-cab)(cab)^-cab \\ & + (1 - (cab)^-cab)(1 - b^-b) \\ = \; & b^-b + 0 + 0 + 1 - b^-b - (cab)^-cab + (cab)^-cabb^-b \\ = \; & 1. \end{aligned}$$

So $v = (cab)^-cab + 1 - b^-b$ is left invertible.

(2) ⇒ (3). By Jacobson's lemma.

(3) ⇒ (1). Multiplying by b on the right of $t = b(cab)^-ca + 1 - bb^-$ yields that $tb = b(cab)^-cab$. It follows that

$$b = t_l^{-1}tb = t_l^{-1}b(cab)^-cab \in Rcab.$$

Then $t_l^{-1}b(cab)^-c$ is a left (b,c)-inverse of a. □

Dually, we have a characterization for right (b,c)-invertibility as follows.

Proposition 9. *Let R be any associative ring with* 1 *and* $a, b, c \in R$ *such that c and cab are regular. If* c^- *is an inner inverse of c and* $(cab)^-$ *is an inner inverse of cab, then the following conditions are equivalent:*

(1) *a is right* (b,c)*-invertible;*
(2) $u = cab(cab)^- + 1 - cc^- \in R_r^{-1}$;

(3) $s = ab(cab)^{-}c + 1 - c^{-}c \in R_r^{-1}$.

In this case, $b(cab)^{-}cs_r^{-1}$ *is a right* (b,c)*-inverse of* a*, where* s_r^{-1} *is a right inverse of* s.

Combining Propositions 8 and 9, we have the following characterization for (b,c)-invertibility.

Theorem 2. *Let* R *be any associative ring with* 1 *and* $a,b,c \in R$ *such that* b*,* c *and* cab *are regular. If* $b^{-}, c^{-}, (cab)^{-}$ *are inner inverses of* b, c, cab*, respectively, then the following conditions are equivalent:*

(1) *a is (b,c)-invertible;*

(2) $u = cab(cab)^{-} + 1 - cc^{-} \in R_r^{-1}$ *and* $v = (cab)^{-}cab + 1 - b^{-}b \in R_l^{-1}$;

(3) $s = ab(cab)^{-}c + 1 - c^{-}c \in R_r^{-1}$ *and* $t = b(cab)^{-}ca + 1 - bb^{-} \in R_l^{-1}$.

In this case,

$$a^{||(b,c)} = t_l^{-1}b(cab)^{-}c = b(cab)^{-}cs_r^{-1},$$

where t_l^{-1} *is a left inverse of* t *and* s_r^{-1} *is a right inverse of* s.

3. Symmetric Properties of Inner (b,c)-Invertible Elements

Let S be any semigroup and $a,b,c \in S$. If $a \in S^{||(b,c)}$ such that $aa^{||(b,c)}a = a$, then $a^{||(b,c)}$ is called the inner (b,c)-inverse of a. For arbitrary $a \in S^{||(b,c)}$, it is easy to verify that $a^{||(b,c)}$ is the inner (b,c)-inverse of $aa^{||(b,c)}a$. Theorem 2.13 in [14] proved that a is inner (b,c)-invertible if and only if $b \in Sab$, $c \in caS$ and $a \in abS \cap Sca$.

Let R be any associative ring with 1 and $a,b,c \in R$. Theorem 3.16 in [15] proved that a is inner (b,c)-invertible if and only if a is regular, $R = a^{\circ} \oplus bR$ and $R = {}^{\circ}a \oplus Rc$. We give a characterization for inner (b,c)-invertible elements as follows.

Proposition 10. *Let* S *be any semigroup and* $a,b,c \in S$*. Then the following conditions are equivalent:*

(1) *a is inner (b,c)-invertible;*

(2) *a is (b,c)-invertible and $a \in abS$;*

(3) *a is (b,c)-invertible and $S \in Sca$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $a^{||(b,c)} = bt$ for some $t \in S$. Then

$$a = aa^{||(b,c)}a = abta \in abS.$$

(2) $\Rightarrow$ (1). Assume that $a = aby$ for some $y \in S$. Then

$$aa^{||(b,c)}a = aa^{||(b,c)}aby = aby = a.$$

(1) $\Leftrightarrow$ (3) can be proved similarly. □

Let S be any semigroup and $a,d \in S$. Lemma 2 shows that $a \in S^{||b}$ and $b \in S^{||a}$ if and only if a is inner invertible along b. It follows immediately that a is inner invertible along b if and only if b is inner invertible along a. We consider to generalize this fact to the case of $a \in S^{||(b,c)}$ and $d \in S^{||(c,b)}$.

Proposition 11. *Let* S *be any semigroup and* $a,b,c \in S$*. If* $a \in S^{||(b,c)}$ *and* $d \in S^{||(c,b)}$*, then* c *is inner* (a',d')*-invertible and* b *is inner* (d',a')*-invertible, where* $a' = aa^{||(b,c)}a$ *and* $d' = dd^{||(c,b)}d$.

Proof. Let $a' = aa^{||(b,c)}a$ and $d' = dd^{||(c,b)}d$. We first prove that c is (a', d')-invertible. In fact, supposing that $a^{||(b,c)} = sc$ for some $s \in S$,

$$\begin{aligned} aa^{||(b,c)}a &= aa^{||(b,c)}aa^{||(b,c)}a \\ &= ascaa^{||(b,c)}a \\ &= asd^{||(c,b)}dcaa^{||(b,c)}a \\ &= asd^{||(c,b)}dd^{||(c,b)}dcaa^{||(b,c)}a \in Sdd^{||(c,b)}dcaa^{||(b,c)}a. \end{aligned}$$

Similarly, $dd^{||(c,b)}d \in dd^{||(c,b)}dcaa^{||(b,c)}aS$.

Meanwhile, we have

$$\begin{aligned} cc^{||(a',d')}c &= d^{||(c,b)}dcc^{||(a',d')}c \\ &= d^{||(c,b)}dd^{||(c,b)}dcc^{||(a',d')}c \\ &= d^{||(c,b)}dc \\ &= c. \end{aligned}$$

By symmetry, we have that b is inner (d', a')-invertible. □

Lemma 6. *Let S be any semigroup and $a, b, c, d \in S$. If $a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$, $c \in S^{||(a,d)}$ and $b \in S^{||(d,a)}$, then*

$$a^{||(b,c)}a = bb^{||(d,a)}, \quad aa^{||(b,c)} = c^{||(a,d)}c,$$
$$dd^{||(c,b)} = b^{||(d,a)}b, \quad d^{||(c,b)}d = cc^{||(a,d)}.$$

Proof. If $a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$, $c \in S^{||(a,d)}$ and $b \in S^{||(d,a)}$, then we have

$$a^{||(b,c)}a = a^{||(b,c)}abb^{||(d,a)} = bb^{||(d,a)}.$$

Similarly, $aa^{||(b,c)} = c^{||(a,d)}c$, $dd^{||(c,b)} = b^{||(d,a)}b$ and $d^{||(c,b)}d = cc^{||(a,d)}$. □

Now we have the main result of this section.

Theorem 3. *Let S be any semigroup and $a, b, c, d \in S$. Then the following conditions are equivalent:*

(1) *a is inner (b, c)-invertible and d is inner (c, b)-invertible;*
(2) *c is inner (a, d)-invertible and b is inner (d, a)-invertible;*
(3) *$a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$ and $b \in S^{||(d,a)}$;*
(4) *$a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$ and $c \in S^{||(a,d)}$.*

Proof. (1) ⇒ (2). If a is inner (b, c)-invertible and d is inner (c, b)-invertible, then c is inner (a, d)-invertible and b is inner (d, a)-invertible by Proposition 11.

(2) ⇒ (1). It is similar to the proof of (1) ⇒ (2).

(1) ⇒ (3). If a is inner (b, c)-invertible and d is inner (c, b)-invertible, then $aa^{||(b,c)}a = a$ and $dd^{||(c,b)}d = d$. It follows that b is (d, a)-invertible by Proposition 11.

(3) ⇒ (1). If $a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$ and $b \in S^{||(d,a)}$, then $aa^{||(b,c)}a = abb^{||(d,a)} = a$ and $dd^{||(c,b)}d = b^{||(d,a)}bd = d$ by Lemma 6.

The equivalence of (1) and (4) can be proved similarly. □

Corollary 1. *Let S be any semigroup and $a, b \in S$. Then a is inner invertible along b if and only if b is inner invertible along a.*

If $a \in S^{||b}$ and $b \in S^{||a}$, then $a^{||b} = (ba)^{\#}b$ and $b^{||a} = a(ba)^{\#}$ by Lemma 1. It follows that

$$a^{||b}b^{||a} = (ba)^{\#}ba(ba)^{\#} = (ba)^{\#}.$$

By symmetry, $b^{||a}a^{||b} = (ab)^{\#}$. We generalized this result to the case of (b,c)-inverses.

Proposition 12. *Let S be any semigroup and $a,b,c,d \in S$. If $a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$, $b \in S^{||(d,a)}$ and $c \in S^{||(a,d)}$, then $abdc, bdca, dcab, cabd \in S^{\#}$ with*

$$(dcab)^{\#} = b^{||(d,a)}a^{||(b,c)}c^{||(a,d)}d^{||(c,b)},$$

$$(cabd)^{\#} = d^{||(c,b)}b^{||(d,a)}a^{||(b,c)}c^{||(a,d)},$$

$$(abdc)^{\#} = c^{||(a,d)}d^{||(c,b)}b^{||(d,a)}a^{||(b,c)},$$

$$(bdca)^{\#} = a^{||(b,c)}c^{||(a,d)}d^{||(c,b)}b^{||(d,a)}.$$

Proof. If $a \in S^{||(b,c)}$, $d \in S^{||(c,b)}$, $b \in S^{||(d,a)}$ and $c \in S^{||(a,d)}$, then $abdc, dcab \in S^{\#}$ with $a^{||(b,c)} = bdc(abdc)^{\#}$ and $c^{||(a,d)} = (abdc)^{\#}abd$ by Proposition 5, then we have

$$\begin{aligned} & b^{||(d,a)}a^{||(b,c)}c^{||(a,d)}d^{||(c,b)} \\ = \; & b^{||(d,a)}bdc(abdc)^{\#}(abdc)^{\#}abdd^{||(c,b)} \\ = \; & dc(abdc)^{\#}(abdc)^{\#}ab \\ = \; & (dcab)^{\#}, \end{aligned}$$

where the last equality follows by Cline's formula [11]. The remaining three equalities can be verified similarly. □

Proposition 13. *Let S be any semigroup and $a,b,c \in S$. Then the following conditions are equivalent:*

(1) *a is both inner (b,c)-invertible and inner (c,b)-invertible;*
(2) *both b and c are inner invertible along a;*
(3) *a is inner invertible along b and c.*

In this case,

$$a^{||(b,c)} = bb^{||a}a^{||c} = a^{||b}c^{||a}c = a^{||b}aa^{||c}$$

and

$$a^{||(c,b)} = cc^{||a}a^{||b} = a^{||c}b^{||a}b = a^{||c}aa^{||b}.$$

Proof. (1) ⇔ (2). Taking $a = d$ in Theorem 3, then the equivalence between (1) and (2) follows.

(2) ⇔ (3). By Corollary 13.

In this case, noting that $a^{||(b,c)}a = bb^{||a} = a^{||b}a$ and $aa^{||(b,c)} = c^{||a}c = aa^{||c}$ by Lemma 6, we have

$$a^{||(b,c)} = a^{||(b,c)}aa^{||(b,c)} = a^{||(b,c)}c^{||a}c = a^{||(b,c)}aa^{||c} = bb^{||a}a^{||c} = a^{||b}aa^{||c} = a^{||b}c^{||a}c.$$

Similarly, we can obtain the formula of $a^{||(c,b)}$. □

Let S be any semigroup and $a \in S$. Theorem 4.4 in [16] proved that a is core invertible if and only if a is (a,a^*)-invertible, and a is dual core invertible if and only if a is (a^*,a)-invertible. Taking $b = a$ and $c = a^*$ in Proposition 13, we have the following result.

Corollary 2 (Theorem 5.6 in [17])**.** *Let S be any semigroup and $a \in S$. Then a is both core invertible and dual core invertible if and only if a is both groups are invertible and Moore–Penrose invertible. In this case, $a^{\#}aa^{\dagger}$ is the core inverse of a and $a^{\dagger}aa^{\#}$ is the dual core inverse of a.*

The reason why the (b,c)-inverse of a is equal to $a^{||(b,b)}aa^{||(c,c)}$ in Theorem 13 is based on the following fact.

Proposition 14. *Let S be any semigroup and $a, b, c \in S$. If $a \in S^{||c} \cap S^{||(c,b)} \cap S^{||b}$, then $aa^{||(c,b)}a \in S^{||(b,c)}$ with*
$$(aa^{||(c,b)}a)^{||(b,c)} = a^{||b}aa^{||c}.$$

Proof. It is clear that $a^{||b}aa^{||c} \in bS \cap Sc$. We have
$$a^{||b}aa^{||c}aa^{||(c,b)}ab = a^{||b}aa^{||(c,b)}ab = a^{||b}ab = b$$
and
$$caa^{||(c,b)}aa^{||b}aa^{||c} = caa^{||(c,b)}aa^{||c} = caa^{||c} = c.$$
So $aa^{||(c,b)}a \in S^{||(b,c)}$ with $(aa^{||(c,b)}a)^{||(b,c)} = a^{||b}aa^{||c}$. □

If a is invertible along b and c, then the (c, b)-invertibility can be characterized by $a^{||b}aa^{||c}$.

Proposition 15. *Let S be any semigroup and $a, b, c \in S$. If a is invertible along b and c, then*

(1) *$a \in S_l^{||(c,b)}$ if and only if $S^1a^{||b}aa^{||c} = S^1c$;*
(2) *$a \in S_r^{||(c,b)}$ if and only if $a^{||b}aa^{||c}S^1 = bS^1$;*
(3) *$a \in S^{||(c,b)}$ if and only if $a^{||b}aa^{||c}S^1 = bS^1$ and $S^1a^{||b}aa^{||c} = S^1c$.*

Proof. (1) Noting that $S^1a^{||b} = S^1b$ and $a^{||c}S^1 = cS^1$, we have $a \in S_l^{||(c,b)}$ if and only if $a \in S_l^{||(a^{||c},a^{||b})}$ by Lemma 5. Additionally, $a \in S_l^{||(a^{||c},a^{||b})}$ if and only if $S^1a^{||b}aa^{||c} = S^1a^{||c} = S^1c$ by definition.

(2) Can be proved similarly.

(3) Combining (1) and (2). □

Let $A \in \mathbb{C}^{n\times n}$. Malik and Thome [18] defined the matrix $A^{D,\dagger} = A^DAA^\dagger$ to be the DMP inverse of A and $A^{\dagger,D} = A^\dagger AA^D$ to be the dual DMP inverse of A. Later, Mehdipour and Salemi [19] defined the matrix $A^{c\dagger} = A^\dagger AA^DAA^\dagger$ to be the CMP inverse of A. We know that $A^\dagger = A^{||A^*}$ and $A^D = a^{||A^m}$, where $m = \mathrm{ind}(A)$, it is natural to consider the properties of $a^{||b}aa^{||c}$, $a^{||c}aa^{||b}$ and $a^{||c}aa^{||b}aa^{||c}$, under the assumption that $aa^{||c}a = a$.

Proposition 16. *Let S be any semigroup and $a, b, c \in S$. If a is invertible along b and c such that $aa^{||c}a = a$, then*

(1) *$a^{||b}aa^{||c}$ is the unique solution of the following equations*
$$xax = x, \;\; bax = baa^{||c}, \;\; xa = a^{||b}a;$$
(2) *$a^{||c}aa^{||b}$ is the unique solution of the following equations*
$$xax = x, \;\; ax = aa^{||b}, \;\; xab = a^{||c}ab;$$
(3) *$a^{||c}aa^{||b}aa^{||c}$ is the unique solution of the following equations*
$$xax = x, \;\; axa = aa^{||b}a \;\; bax = baa^{||c}, \;\; xab = a^{||c}ab.$$

Proof. (1) We first check that $a^{||b}aa^{||c}$ satisfies these three equations. Actually, we have
$$a^{||b}aa^{||c}aa^{||b}aa^{||c} = a^{||b}aa^{||b}aa^{||c} = a^{||b}aa^{||c},$$
$$baa^{||b}aa^{||c} = baa^{||c} \;\; \text{and} \;\; a^{||b}aa^{||c}a = a^{||b}a.$$

If y also satisfies these equations, supposing that $a^{||b} = sb$ for some $s \in S$, then
$$y = yay = a^{||b}ay = sbay = sbaa^{||c} = a^{||b}aa^{||c}.$$

(2) and (3) can be proved similarly. □

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = m$. Taking $b = A^m$ and $c = A^*$ in Proposition 16, we recover the characterizations of the DMP inverse ([18], Theorem 2.2), dual DMP inverse and CMP inverse ([19], Theorem 2.1).

Particularly, $a^{||b}aa^{||c}$, $a^{||c}aa^{||b}$ and $a^{||c}aa^{||b}aa^{||c}$ can be expressed as the $(\_,\_)$-inverses of a.

Proposition 17. *Let S be any semigroup and $a, b, c \in S$. If a is invertible along b and c such that $aa^{||c}a = a$, then*

(1) *$a^{||b}aa^{||c}$ is the $(b, baa^{||c})$-inverse of a;*
(2) *$a^{||(c,c)}aa^{||(b,b)}$ is the $(a^{||c}ab, b)$-inverse of a;*
(3) *$a^{||(c,c)}aa^{||(b,b)}aa^{||c}$ is the $(a^{||c}ab, baa^{||c})$-inverse of a.*

Proof. (1) It is obvious that $a^{||b}aa^{||c} \in bS \cap baa^{||c}S$. Meanwhile, we have

$$a^{||b}aa^{||c}ab = a^{||b}ab = b,$$

$$baa^{||c}aa^{||b}aa^{||c} = baa^{||b}aa^{||c} = baa^{||c}.$$

So $a^{||b}aa^{||c}$ is the $(b, baa^{||c})$-inverse of a.

(2) and (3) can be proved in a similar way. □

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = m$. Taking $b = A^m$ and $c = A^*$ in Proposition 17, we have $A^{D,\dagger} = A^{||(A^m, A^mA^\dagger)} = A^{||(A^D, A^mA^\dagger)}$, which are Theorem 3.2 in [20] and Theorem 3.6 in [21].

Corollary 3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = m$. Then $A^{c\dagger}$ is the $(A^\dagger A^m, A^mA^\dagger)$-inverse of A.*

Author Contributions: Conceptualization, G.S. and J.C.; methodology, G.S. and J.C.; validation G.S. and J.C.; writing—original draft preparation, G.S.; writing—review and editing, G.S. and J.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (No. 12171083, 11871145, 12071070), the Qing Lan Project of Jiangsu Province, the Fundamental Research Funds for the Central Universities, the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX20_0074).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the editor and reviewers sincerely for their constructive comments and suggestions that have improved the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Penrose, R. A generalized inverse for matrices. *Proc. Camb. Philos. Soc.* **1955**, *51*, 406–413. [CrossRef]
2. Drazin, M.P. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Mon.* **1958**, *65*, 506–514. [CrossRef]
3. Mary, X. On generalized inverses and Green's relations. *Linear Algebra Appl.* **2011**, *434*, 1836–1844. [CrossRef]
4. Drazin, M.P. A class of outer generalized inverses. *Linear Algebra Appl.* **2012**, *436*, 1909–1923. [CrossRef]
5. Drazin, M.P. Left and right generalized inverses. *Linear Algebra Appl.* **2016**, *510*, 64–78. [CrossRef]
6. Mary, X. Classes of semigroups modulo Green's relation $\mathcal{H}$. *Semigroup Forum* **2014**, *88*, 647–669. [CrossRef]
7. Wu, C.; Chen, J.L. On (b, c)-inverses and (c, b)-inverses. *Comm. Algebra* **2021**, *49*, 4313–4323. [CrossRef]
8. Wang, L.; Castro-González, N.; Chen, J.L. Characterizations of outer generalized inverses. *Canad. Math. Bull.* **2017**, *60*, 861–871. [CrossRef]
9. Wang, L.; Mosíc, D. The one-sided inverse along two elements in rings. *Linear Multilinear Algebra* **2021**, *69*, 2410–2422. [CrossRef]
10. Boasso, E.; Kantún-Montiel, G. The (b, c)-inverses in rings and in the Banach context. *Mediterr. J. Math.* **2017**, *14*, 112. [CrossRef]

11. Cline, R.E. An application of representation for the generalized inverse of a matrix. In *Proceedings of the Technology Summary Report;* Mathematics Research Center, U.S. Army, University of Wisconsin: Madison, WI, USA, 1965; p. 592.
12. Mary, X.; Patrício, P. Generalized inverses modulo $\mathcal{H}$ in semigroups and rings. *Linear Multilinear Algebra* **2012**, *61*, 1130–1135. [CrossRef]
13. Zhu, H.H.; Chen, J.L.; Patrício, P. Further results on the inverse along an element in semigroups and rings. *Linear Multilinear Algebra* **2016**, *64*, 393–403. [CrossRef]
14. Mosić, D.; Zou, H.L.; Chen, J.L. On the (b,c)-inverse in rings. *Filomat* **2018**, *32*, 1221–1231. [CrossRef]
15. Xu, S.Z.; Benítez, J. Existence criteria and expressions of the (b,c)-inverse in rings and its applications. *Mediterr. J. Math.* **2018**, *15*, 14. [CrossRef]
16. Rakić, D.S.; Diničić, N.Č.; Djordjević, D.S. Group, Moore-penrose, core and dual core inverse in rings with involution. *Linear Algebra Appl.* **2014**, *463*, 115–133. [CrossRef]
17. Chen, J.L.; Zhu, H.H.; Patrício, P.; Zhang, Y.L. Characterizations and representation of core and dual core inverses. *Canad. Math. Bull.* **2017**, *60*, 269–282. [CrossRef]
18. Malik, S.B.; Thome, N. On a new generalized inverse for matrices of an arbitrary index. *Appl. Math. Comput.* **2014**, *226*, 575–580. [CrossRef]
19. Mehdipour, M.; Salemi, A. On a new generalized inverse of matrices. *Linear Multilinear Algebra* **2018**, *66*, 1046–1053. [CrossRef]
20. Ferreyra, D.E.; Levis, E.E.; Thome, N. Chrarcterizations of k-commutative equations for some outer generalized inverses. *Linear Multilinear Algebra* **2020**, *68*, 177–192. [CrossRef]
21. Zuo, K.Z.; Cvetković-Ilić, D.S.; Cheng, Y.J. Different chareaterizations of DMP inverses of matrices. *Linear Multilinear Algebra* **2022**, *70*, 411–418. [CrossRef]