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Article

Soft Sets with Atoms

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Abstract: The theory of finitely supported structures is used for dealing with very large sets having a certain degree of symmetry. This framework generalizes the classical set theory of Zermelo-Fraenkel by allowing infinitely many basic elements with no internal structure (atoms) and by equipping classical sets with group actions of the permutation group over these basic elements. On the other hand, soft sets represent a generalization of the fuzzy sets to deal with uncertainty in a parametric manner. In this paper, we study the soft sets in the new framework of finitely supported structures, associating to any crisp set a family of atoms describing it. We prove some finiteness properties for infinite soft sets, some order properties and Tarski-like fixed point results for mappings between soft sets with atoms.

Keywords: finitely supported sets; finitely supported complete lattices; soft sets; un-finite sets; fixed points

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1. Introduction

Finitely supported sets are related to the permutation models of the Zermelo–Fraenkel set theory with atoms (ZFA) which were originally described by Fraenkel and Mostowski in the 1930s to prove the independence of the axiom of choice from the other axioms of the ZFA set theory [1,2]. Since the existence of atoms (that are defined as entities having no internal structure) in ZFA requires the modification of the axiom of extensionality from the Zermelo–Fraenkel set theory (ZF), finitely supported sets were alternatively described and studied in the ZF set theory by equipping ZF sets with actions of the group of finitary permutations (i.e., the group of one-to-one and onto transformations) of some basic elements whose internal structure is ignored. If A is the set of all basic elements (called atoms by analogy with the ZFA approach), a finitely supported element in a ZF set equipped with such a group action is an element having the property that there exists a finite subset of A such that every finitary permutation of A fixing the related subset of A pointwise also leaves it unchanged under the effect of the group action. An invariant set is a ZF set equipped with a group action of the group of all finitary permutations of A having the property that all its elements are finitely supported. A finitely supported set is a subset of an invariant set which is finitely supported as an element in the powerset of the invariant set. A finitely supported structure is a finitely supported set equipped with a finitely supported binary relation (these aspects are detailed in Section 2). A categorical development for finitely supported sets is presented in [3], while a set theoretical approach is presented in [4].

Finitely supported sets were not only used to deal with very large structures [5], but also to model renaming, fresh names and variables binding in the theory of programming [3]. Inductively defined finitely supported sets involving the name-abstraction together with a disjoint union and a Cartesian product are able to encode syntax modulo renaming of bound variables. The concept of structural recursion for defining syntax-manipulating functions can be formalized into this framework, which admits a proving method by structural induction. Some generalizations of finitely supported sets were used in [6] to study automata or Turing machines over infinite alphabets by relaxing the notion of finiteness (instead of assuming finitely many elements, one may assume finitely many orbits/equivalence classes). Finitely supported partially ordered sets were first introduced in [7] to describe a denotational semantics for a functional programming language incorporating facilities for manipulating syntax involving names and binding operations. For this, the author presented the solution of the Scott recursive domain equation $D \cong (D \to D)$ in the framework of finitely supported structures. Later, we used finitely supported partially ordered sets and lattices to describe abstract interpretation, rough sets and fuzzy sets in the new framework and to provide various fixed point and approximation properties for infinite structures [8].

The world of finitely supported structures contains both the family of non-atomic ZF structures which are proved to be trivially invariant (i.e., all their elements are empty supported because, intuitively, they are hierarchically constructed from $\emptyset$) and the family of atomic structures with finite (but possibly non-empty) supports. One of our goals was to check whether the ZF results remain valid when replacing a 'non-atomic ZF structure' with an 'atomic and finitely supported structure'. We emphasized in [4] that results from ZF might lose their validity when transferring them into an atomic framework (such as ZFA). For example, the 'multiple choice principle implies the axiom of choice' is a valid theorem in ZF (see Theorem 5.4 in [9]), but it does not hold in ZFA because the multiple choice principle is valid in the second Fraenkel model of ZFA, namely model N2 from [10] (see Theorem 9.2 of [11]), while the axiom of choice is not valid in the related model.

The meta-theoretical technique for transferring ZF results into the world of finitely supported sets and structures is based on a closure property for finite supports in a (higher-order) hierarchical construction (called 'S-finite support principle') claiming that *"for any finite set S of atoms, anything that can be defined in higher-order logic from structures supported by S, by using each time only constructions supported by S, is itself supported by S"* [4]. The formal involvement of this meta-theoretical principle requires a step-by-step building of the support of a structure by employing, at every step, the previously constructed supports of the substructures of the related structure.

Regarding the motivation of this article, let us recall that the theory of finitely supported structures allows a discrete (finitary) representation of possibly infinite sets containing enough symmetries to be concisely handled. This theory allows us to treat as equivalent the elements in a structure that have a certain degree of similarity and to focus only on those elements that are really different (those forming the support of the structure). The aim of this paper is to define and study the soft sets in the framework of finitely supported structures.

Fuzzy set theory deals with uncertainties [12]. While a crisp set has associated the characteristic function which establishes whether a certain element belongs to the set, a fuzzy set has associated a membership function which models the degree of membership for each element. More exactly, the membership function associated to a fuzzy set X could take any values in the real interval $[0, 1]$, while the classical characteristic function of a set X can only take two values: 0 (for non-membership) and 1 (for membership). More generally, the unit interval $[0, 1]$ could be replaced by an arbitrary complete lattice to define L-fuzzy sets [13]. In [14] and [8], we studied the fuzzy sets and the L-fuzzy sets in the framework of finitely supported structures and provided a discrete presentation of infinite (L-)fuzzy sets.

Soft sets represent a generalization of fuzzy sets [15]. If U is an initial universal set and E is a set of parameters, a soft set is defined as a mapping from a subset of E to the powerset of U. In this paper, we describe the soft sets in the framework of finitely supported sets. After mentioning some preliminary results, we define a soft set with atoms as a finitely supported mapping from a finitely supported set to the finitely supported powerset of the universal set A of all basic elements. In this way, every element in the crisp finitely supported set has a finitely supported subset of atoms associated. We present some Dedekind-finiteness properties of the set of all soft sets with atoms defined on a certain finitely supported set satisfying a Dedekind-finiteness property. Particularly, we are able to prove that whenever X is a finitely supported set having the property that its powerset is Dedekind-finite, the finite powerset of the set of all soft sets with atoms defined on $X \times A^n$ is also Dedekind-finite. Then we organize the set $Soft(X)$ of all soft sets with atoms defined on a finitely supported set X as a finitely supported complete lattice, and by using the S-finite support principle, we prove a Tarski-like fixed point property stating that the set of all fixed points of any finitely supported, order-preserving, self mapping on $Soft(X)$ forms a finitely supported complete lattice. Since the theory of finitely supported sets represents a tool for providing a computational description of very large structures, our approach helps us to provide finiteness properties for infinite fuzzy sets by involving the notion of finite support.

2. Preliminaries

In this section, we present preliminary results regarding finitely supported sets. They were also described by us in other articles and monographs we mention here [4]. They can also be found in a slightly different framework of nominal sets in [3]. The elements of A can be checked only for equality, i.e., their internal structure in not taken into consideration. A *transposition of atoms* is a function that interchanges two atoms, i.e., a function of form $(x\,y) : A \to A$ defined by $(x\,y)(x) = y$, $(x\,y)(y) = x$, and $(x\,y)(z) = z$ for $z \neq x, y$. A *finitary permutation of atoms* is defined as a bijection of A generated by composing finitely many transpositions, i.e., a bijection of A leaving invariant all but finitely many elements of A. The set of all finitary permutations of atoms is denoted by S_A.

Definition 1 ([4])**.** *Let X be a ZF set.*

1. *An* S_A-action *on X is a group action of S_A on X, i.e., a function $\cdot : S_A \times X \to X$ having the properties that $Id_A \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$.*
2. *An* S_A-set *is a pair $(X, \cdot)$, where X is a ZF set and $\cdot : S_A \times X \to X$ is an S_A-action on X.*
3. *Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$* supports *x if for each $\pi \in Fix(S)$, we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \in S_A \,|\, \pi(a) = a$ for all $a \in S\}$. An element which is supported by a finite subset of atoms is called* finitely supported*.*
4. *Let $(X, \cdot)$ be an S_A-set. We say that set X is an* invariant set *whenever for each $x \in X$ there is a finite set $S_x \subset A$ supporting x.*
5. *Let X be an S_A-set, and $x \in X$. If there is a finite set supporting x, then there exists a (unique) least finite set $supp(x)$ supporting x [4], defined as the intersection of all sets supporting x, which is called the* support *of x. An empty supported element is* equivariant*; $z \in X$ is equivariant if and only if $\pi \cdot z = z$ for all $\pi \in S_A$.*

Let $(X, \cdot)$ and $(Y, \triangleright)$ be S_A-sets. According to [4], the set A of atoms is an invariant set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. The Cartesian product $X \times Y$ is an S_A-set with the S_A-action $\otimes$ defined by $\pi \otimes (x, y) = (\pi \cdot x, \pi \triangleright y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. For $(X, \cdot)$ and $(Y, \triangleright)$ invariant sets, $(X \times Y, \otimes)$ is also an invariant set. The powerset $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is an S_A-set with the S_A-action $\star$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all $\pi \in S_A$ and $Y \in \wp(X)$. For an invariant set $(X, \cdot)$, $\wp_{fs}(X)$ denotes the set formed from those subsets of X that are finitely supported in the sense of Definition 1(3) as elements of the S_A-set $\wp(X)$ with respect to the S_A-action $\star$ described above; $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$. Non-atomic sets are trivially invariant, i.e., they are equipped with the action $\diamond$ defined by $\pi \diamond x = x$ for all $\pi \in S_A$ and $x \in X$.

Definition 2.

1. *A subset Y of an invariant set $(X, \cdot)$ is called* finitely supported *if and only if $Y \in \wp_{fs}(X)$. In this case, we shortly say that $(Y, \cdot)$ is a* finitely supported set*.*
2. *A subset Y of an invariant set $(X, \cdot)$ is* uniformly supported *if all of its elements are supported by the same finite set of atoms.*

3. *A finitely supported set that does not contain a uniformly supported infinite subset is called* un-finite.

If $\pi \in S_A$, $X \in \wp(Y)$ and Y is an S_A-set, then $\pi \star X = X$ if and only if $\pi \star X \subseteq X$, considering $\star$ defined on $\wp(Y)$. Hence, X is finitely supported by S if and only if $\pi \cdot x \in X$ for all $x \in X$ and $\pi \in Fix(S)$ (this is because finitary permutations of atoms are of finite order). We also note that a finite subset of an invariant set is uniformly supported by the union of the supports of its elements.

It is worth noting that there may exist subsets of an invariant set that are not finitely supported. As presented in [4], a subset of the invariant set A of all atoms is finitely supported if and only if it is finite or it has a finite complement. Furthermore, if $X \subset A$ and X is finite, then $supp(X) = X$. If $Z \subseteq A$ and Z has a finite complement, then $supp(Z) = A \setminus Z$.

Since functions are particular subsets of the Cartesian product of two sets, for two invariant sets $(X, \cdot)$ and $(Y, \triangleright)$, $U \in \wp_{fs}(X)$, $V \in \wp_{fs}(Y)$, we say that a function $f : U \to V$ is *finitely supported* if $f \in \wp_{fs}(X \times Y)$. Note that Y^X is an S_A-set with the S_A-action $\widetilde{\star}$ defined by $(\pi \widetilde{\star} f)(x) = \pi \triangleright (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : U \to V$ is finitely supported (in the sense of the above definition) if and only if it is finitely supported with respect to the permutation action $\widetilde{\star}$. The set of all finitely supported functions from U to V is denoted by V^U_{fs}.

Proposition 1. *Let U be a finitely supported subset of an invariant set $(X, \cdot)$ and V a finitely supported subset of an invariant set $(Y, \triangleright)$. A function $f : U \to V$ is supported by S if and only if for all $u \in U$ and all $\pi \in Fix(S)$ we have $\pi \cdot u \in U$, $\pi \triangleright f(u) \in V$ and $f(\pi \cdot u) = \pi \triangleright f(u)$.*

It is proven in [4] that a finitely supported function $f : A \to A$ is bijective if and only if it is a finitary permutation; thus, finitary permutations are simply called *permutations* in the framework of finitely supported structures.

3. Finitely Supported Soft Sets

In this section, we present the notion of a soft set with atoms by translating the ZF concept of a soft set in the framework of finitely supported structures. A soft set with atoms is a finitely supported function from a finitely supported set to the finitely supported powerset of all atoms. Formally, this is expressed below.

Definition 3. *Let X be a finitely supported set. A* soft set with atoms *on X is a pair (X, f), where $f : X \to \wp_{fs}(A)$ is a finitely supported function, and A is the invariant set of all atoms equipped with the canonical S_A-action $(\pi, a) \mapsto \pi(a)$ for $\pi \in S_A$, $a \in A$.*

Proposition 2. *Let X be a finitely supported set.*

1. *The pair (X, f), where $f : X \to \wp_{fs}(A)$ is defined by $f(x) = supp(x)$ for all $x \in X$, is a soft set with atoms.*
2. *The pair (X, g), where $g : X \to \wp_{fs}(A)$ is defined by $g(x) = A \setminus supp(x)$ for all $x \in X$, is a soft set with atoms.*

Proof. We note that X is a finitely supported set, which means it is a finitely supported subset of an invariant set $(Y, \cdot)$, i.e., X is a finitely supported element of $\wp(Y)$ having a support $supp(X)$ defined as the least finite set of atoms supporting X.

1. Let $\pi \in Fix(supp(X))$ and $x \in X$. Therefore, $\pi \cdot x \in X$. We claim that $S_x := \pi \star supp(x) = \{\pi(a) \mid a \in supp(x)\}$ supports $\pi \cdot x$. Let $\tau \in Fix(S_x)$. This means $\tau(\pi(a)) = \pi(a)$ for all $a \in supp(x)$. Thus, $\pi^{-1}(\tau(\pi(a))) = \pi^{-1}(\pi(a)) = a$ for all $a \in supp(x)$, and so $\pi^{-1} \circ \tau \circ \pi \in Fix(supp(x))$. Since $supp(x)$ supports x, we get that $(\pi^{-1} \circ \tau \circ \pi) \cdot x = x$, and so $\tau \cdot (\pi \cdot x) = (\tau \circ \pi) \cdot x = \pi \cdot x$ from which $supp(\pi \cdot x) \subseteq S_x = \pi \star supp(x)$ for each $x \in X$ and each $\pi \in Fix(supp(X))$ (*). We

can apply the relation (*) for the elements $\pi^{-1} \in Fix(supp(X))$ and $\pi \cdot x \in X$. We have that $supp(x) = supp((\pi^{-1} \circ \pi) \cdot x) = supp(\pi^{-1} \cdot (\pi \cdot x)) \subseteq \pi^{-1} \star supp(\pi \cdot x)$, for which (since the relation $\subseteq$ on $\wp_{fs}(A)$ is equivariant) $\pi \star supp(x) \subseteq \pi \star (\pi^{-1} \star supp(\pi \cdot x)) = (\pi \circ \pi^{-1}) \star supp(\pi \cdot x) = supp(\pi \cdot x)$. Thus, $f(\pi \cdot x) = supp(\pi \cdot x) = \pi \star supp(x) = \pi \star f(x)$ for all $x \in X$ and all $\pi \in Fix(supp(X))$ meaning that f is finitely supported, and so it is a soft set with atoms.

2. Let $\pi \in Fix(supp(X))$ and $x \in X$. Therefore, $\pi \cdot x \in X$. Since A is an invariant set, we get that $\pi \star A = A$. According to item 1, we have $g(\pi \cdot x) = A \setminus supp(\pi \cdot x) = A \setminus (\pi \star supp(x)) = (\pi \star A) \setminus (\pi \star supp(x)) = \pi \star (A \setminus supp(x)) = \pi \star g(x)$ for all $x \in X$, and so g is finitely supported according to Proposition 1.

□

4. Finiteness Properties of Soft Sets with Atoms

The goal of this section is to prove several un-finiteness properties for infinite soft sets with atoms. We start this section with a lemma that can be proved by involving the S-finite support principle.

Lemma 1. *Let $(U, \cdot)$, $(V, \cdot)$ be finitely supported sets.*

1. *There exists a finitely supported bijective function* $\varphi : \{f : U \to \wp_{fs}(V) \mid f \text{ is finitely supported}\} \to \wp_{fs}(U \times V)$.
2. *There exists a finitely supported bijection $f : U \times V \to V \times U$.*
3. *If there is a finitely supported bijection $f : U \to V$, then there exists a finitely supported bijection $g : \wp_{fs}(U) \to \wp_{fs}(V)$.*
4. *If there is a finitely supported bijection $f : U \to V$, then there exists a finitely supported bijection $f^{-1} : V \to U$.*
5. *If there is a finitely supported bijection $f : U \to V$, then U is un-finite if and only if V is un-finite.*

Proof.

1. Let us define φ as follows:

$$\varphi(f) = \begin{cases} \{(u,v) \mid u \in U, v \in f(u)\}, & \text{if } \exists u \in U.\ f(u) \neq \emptyset; \\ \emptyset, & \text{if } \forall u \in U.\ f(u) = \emptyset. \end{cases}$$

- We remark that $\emptyset$ is equivariant under the S_A-action on the powerset of an invariant set. Indeed, since $\subseteq$ is equivariant, we have $\emptyset \subseteq \pi \star \emptyset \subseteq \pi^2 \star \emptyset \subseteq \ldots \subseteq \pi^n \star \emptyset \subseteq \ldots$ for all $\pi \in S_A$. Since there is $m \in \mathbb{N}$ such that $\pi^m = Id_A$, we get that $\emptyset = \pi \star \emptyset$ for all $\pi \in S_A$.
- φ is well-defined. Let $f : U \to \wp_{fs}(V)$ be a finitely supported function. We claim that $\varphi(f)$ is supported by $supp(U) \cup supp(V) \cup supp(f)$. Let $\pi \in Fix(supp(U) \cup supp(V) \cup supp(f))$. Then $\pi \star U = U$, $\pi \star V = V$, $f(\pi \cdot z) = \pi \star f(z)$ for all $z \in U$. We have $\pi \star \varphi(f) = \pi \star \emptyset$ (if $\forall u \in U.f(u) = \emptyset$) $= \emptyset$ (if $\forall u \in U.f(u) = \emptyset$) $\overset{\pi \cdot u := u'}{=} \emptyset$ (if $\forall u' \in U.f(\pi^{-1} \cdot u') = \emptyset$) $= \emptyset$ (if $\forall u' \in U.\pi^{-1} \star f(u') = \emptyset$) $= \emptyset$ (if $\forall u' \in U.f(u') = \emptyset$). Moreover, $\pi \star \varphi(f) = \pi \star \{(u,v) \mid u \in U, f(u) \neq \emptyset, v \in f(u)\}$ (if $\exists u \in U.f(u) \neq \emptyset$) $= \{\pi \otimes (u,v) \mid u \in U, f(u) \neq \emptyset, v \in f(u)\}$ (if $\exists u \in U.f(u) \neq \emptyset$) $= \{(\pi \cdot u, \pi \cdot v) \mid u \in U, f(u) \neq \emptyset, v \in f(u)\}$ (if $\exists u \in U.f(u) \neq \emptyset$) $\overset{\pi \cdot u := u', \pi \cdot v := v'}{=} \{(u', v') \mid \pi^{-1} \cdot u' \in U, f(\pi^{-1} \cdot u') \neq \emptyset, \pi^{-1} \cdot v' \in f(\pi^{-1} \cdot u')\}$ (if $\exists u' \in U.f(\pi^{-1} \cdot u') \neq \emptyset$) $= \{(u', v') \mid u' \in \pi \star U, \pi^{-1} \star f(u') \neq \emptyset, v' \in \pi \star f(\pi^{-1} \cdot u')\}$ (if $\exists u' \in U.\pi^{-1} \star f(u') \neq \emptyset$) $= \{(u', v') \mid u' \in U, f(u') \neq \pi \star \emptyset, v' \in f(\pi \cdot (\pi^{-1} \cdot u'))\}$ (if $\exists u' \in U.f(u') \neq \pi \star \emptyset$). Hence, $\pi \star \varphi(f) = \begin{cases} \{(u', v') \mid u' \in U, f(u') \neq \emptyset, v' \in f(u'))\}, & \text{if } \exists u' \in U.\ f(u') \neq \emptyset; \\ \emptyset, & \text{if } \forall u' \in U.\ f(u') = \emptyset \end{cases}$ $= \varphi(f)$.

- φ is finitely supported. Let $\pi \in Fix(supp(U) \cup supp(V))$. We have $\varphi(\pi\widetilde{\star}f) = \emptyset$ (if $\forall u \in U.(\pi\widetilde{\star}f)(u) = \emptyset$) $= \emptyset$ (if $\forall u \in U.\pi \star f(\pi^{-1} \cdot u) = \emptyset$) $= \emptyset$ (if $\forall u \in U.f(\pi^{-1} \cdot u) = \pi^{-1} \star \emptyset$) $= \emptyset$ (if $\forall u \in U.f(\pi^{-1} \cdot u) = \emptyset$) $\overset{\pi^{-1}\cdot u := u''}{=} \emptyset$ (if $\forall u'' \in U.f(u'') = \emptyset$) $= \pi \star \emptyset$ (if $\forall u'' \in U.f(u'') = \emptyset$). Moreover, in the other case, $\varphi(\pi\widetilde{\star}f) = \{(u,v) \mid u \in U, (\pi\widetilde{\star}f)(u) \neq \emptyset, v \in (\pi\widetilde{\star}f)(u)\}$ (if $\exists u \in U.(\pi\widetilde{\star}f)(u) \neq \emptyset$) $= \{(u,v) | u \in U,\ \pi \star f(\pi^{-1} \cdot u) \neq \emptyset, v \in \pi \star f(\pi^{-1} \cdot u)\}$(if $\exists u \in U.\pi \star f(\pi^{-1} \cdot u) \neq \emptyset$) $= \{(u,v) \mid u \in U, f(\pi^{-1} \cdot u) \neq \pi^{-1} \star \emptyset, \pi^{-1} \cdot v \in f(\pi^{-1} \cdot u)\}$ (if $\exists u \in U.f(\pi^{-1} \cdot u) \neq \pi^{-1} \star \emptyset$) $= \{(u,v) \mid u \in U, f(\pi^{-1} \cdot u) \neq \emptyset, \pi^{-1} \cdot v \in f(\pi^{-1} \cdot u)\}$ (if $\exists u \in U.f(\pi^{-1} \cdot u) \neq \emptyset$) $\overset{\pi^{-1}\cdot u := u'', \pi^{-1}\cdot v := v''}{=} \{(\pi \cdot u'', \pi \cdot v'') \mid u'' \in \pi^{-1} \star U, f(u'') \neq \emptyset, v'' \in f(u'')\}$ (if $\exists u'' \in U.f(u'') \neq \emptyset$). Thus, we get that $\varphi(\pi\widetilde{\star}f) = \begin{cases} \{\pi \otimes (u'',v'') \mid u'' \in U, f(u'') \neq \emptyset, v'' \in f(u'')\}, & \text{if } \exists u'' \in U.f(u'') \neq \emptyset; \\ \pi \star \emptyset, & \text{if } \forall u'' \in U.f(u'') = \emptyset \end{cases} = \pi \star \varphi(f)$.
- φ is injective. Assume $\varphi(f) = \varphi(f')$. If $\varphi(f) = \varphi(f') = \emptyset$, then $f(u) = f'(u) = \emptyset$ for all $u \in U$. Otherwise, $(u,v) \in \varphi(f) \Leftrightarrow (u,v) \in \varphi(f')$. Therefore, $v \in f(u) \Leftrightarrow v \in f'(u)$, $f(u) = \emptyset \Leftrightarrow f'(u) = \emptyset$, and so $f(u) = f'(u)$ for all $u \in U$.
- φ is surjective. Let $T \in \wp_{fs}(U \times V)$. If $T = \emptyset$, we consider $g : U \to \wp_{fs}(V)$, $g(u) = \emptyset$ for all $u \in U$. Clearly for $\pi \in Fix(supp(U) \cup supp(V))$ we have $\pi \cdot u \in U$ and $g(\pi \cdot u) = \emptyset = \pi \star \emptyset = \pi \star g(u)$, meaning that g is finitely supported. Clearly, $\varphi(g) = T$. Now, assume $T \neq \emptyset$ and let $U_1 = \{u \in U \mid \exists v \in V$ such that $(u,v) \in T\}$. We claim that U_1 is supported by $supp(U) \cup supp(V) \cup supp(T)$. Let $\pi \in Fix(supp(U) \cup supp(V) \cup supp(T))$. Since $supp(T)$ supports T, we get that $(\pi \cdot u, \pi \cdot v) = \pi \otimes (u,v) \in T$ for all $(u,v) \in T$. Let $u_1 \in U_1$. Then there is $v_1 \in V$ such that $(u_1,v_1) \in T$. Hence, there exists $\pi \cdot v_1 \in V$ such that $(\pi \cdot u_1, \pi \cdot v_1) \in T$, and so $\pi \cdot u_1 \in U_1$, which means U_1 is supported by $supp(U) \cup supp(V) \cup supp(T)$, i.e., $supp(U_1) \subseteq supp(U) \cup supp(V) \cup supp(T)$. We also remark that $supp(U \setminus U_1) \subseteq supp(U) \cup supp(V) \cup supp(T)$. We define $h : U \to \wp_{fs}(V)$ as follows: $h(u) = \begin{cases} \{v \mid (u,v) \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases}$.

  We prove that h is well-defined. Let us consider $u \in U_1$, and $\pi \in Fix(supp(U) \cup supp(V) \cup supp(T) \cup supp(u))$. We have $\pi \cdot u = u$, and $\pi \star h(u) =$
  $\begin{cases} \pi \star \{v \mid (u,v) \in T\}, & \text{if } u \in U_1; \\ \pi \star \emptyset, & \text{if } u \notin U_1 \end{cases} =$
  $\begin{cases} \{\pi \cdot v \mid (u,v) \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases} \overset{\pi\cdot v := v'}{=}$
  $\begin{cases} \{v' \mid (\pi^{-1} \cdot u, \pi^{-1} \cdot v') \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases} =$
  $\begin{cases} \{v' \mid \pi^{-1} \otimes (u,v') \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases} =$
  $\begin{cases} \{v' \mid (u,v') \in \pi \star T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases} =$
  $\begin{cases} \{v' \mid (u,v') \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \notin U_1 \end{cases} = h(u)$, which means that $h(u) \in \wp_{fs}(V)$ for all $u \in U_1$. We used $\pi \star T = T$ because π fixes $supp(T)$ pointwise. For $u \in U \setminus U_1$, we have $h(u) = \emptyset \in \wp_{fs}(V)$. Let $\pi \in Fix(supp(U) \cup supp(V) \cup supp(T))$, and so $\pi \star T = T$, $\pi \star U_1 = U_1$. Then, $h(\pi \cdot u) =$
  $\begin{cases} \{v \mid (\pi \cdot u, v) \in T\}, & \text{if } \pi \cdot u \in U_1; \\ \emptyset, & \text{if } \pi \cdot u \notin U_1 \end{cases} =$
  $\begin{cases} \{\pi \cdot (\pi^{-1} \cdot v) \mid \pi \otimes (u, \pi^{-1} \cdot v) \in T\}, & \text{if } u \in \pi^{-1} \star U_1; \\ \emptyset, & \text{if } u \in \pi^{-1} \star (U \setminus U_1) \end{cases} \overset{\pi^{-1}\cdot v := w}{=}$
  $\begin{cases} \{\pi \cdot w \mid \pi \otimes (u,w) \in T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \in U \setminus U_1 \end{cases} =$
  $\begin{cases} \{\pi \cdot w \mid (u,w) \in \pi^{-1} \star T\}, & \text{if } u \in U_1; \\ \emptyset, & \text{if } u \in U \setminus U_1 \end{cases} =$
  $\begin{cases} \{\pi \cdot w \mid (u,w) \in T\}, & \text{if } u \in U_1; \\ \pi \star \emptyset, & \text{if } u \in U \setminus U_1 \end{cases} = \pi \star h(u)$, which means that h is finitely supported (according to Proposition 1). Moreover, $\varphi(h) = T$, and so φ is surjective.

2. Let us define $f : U \times V \to V \times U$ by $f(u,v) = (v,u)$ for all $u \in U$ and $v \in V$. Let $\pi \in Fix(supp(U) \cup supp(V))$. Then $\pi \cdot u \in U$ for all $u \in U$, and $\pi \cdot v \in V$ for all $v \in V$. Hence $\pi \otimes (u,v) \in U \times V$ and $\pi \otimes (v,u) \in V \times U$ for all $u \in U$ and $v \in V$. Furthermore, $f(\pi \otimes (u,v)) = f(\pi \cdot u, \pi \cdot v) = (\pi \cdot v, \pi \cdot u) = \pi \otimes (v,u) = \pi \otimes f(u,v)$ for all $u \in U$ and $v \in V$. According to Proposition 1, f is finitely supported.
3. We define $g(X) = \{f(x) \,|\, x \in X\} = f(X)$ for $X \in \wp_{fs}(U)$, $X \neq \emptyset$ and $g(\emptyset) = \emptyset$. First, we prove that for $X \in \wp_{fs}(U)$ and $Y \in \wp_{fs}(V)$, $f(X)$ is supported by $supp(U) \cup supp(V) \cup supp(X) \cup supp(f)$ and $f^{-1}(Y)$ is supported by $supp(U) \cup supp(V) \cup supp(Y) \cup supp(f)$, that is $f(X) \in \wp_{fs}(V)$ and $f^{-1}(Y) \in \wp_{fs}(U)$. Let $\sigma \in Fix(supp(U) \cup supp(V) \cup supp(X) \cup supp(f))$. Let $v \in f(X)$, meaning that $v = f(x)$ for some $x \in X$. We have $\sigma \cdot x \in X$ and $\sigma \cdot v = \sigma \cdot f(x) = f(\sigma \cdot x) \in f(X)$, meaning that $\sigma \star f(X) = f(X)$. Now, let $\pi \in Fix(supp(U) \cup supp(V) \cup supp(Y) \cup supp(f))$. We have $\pi \star Y = Y$ and $\pi \star f^{-1}(Y) = \{\pi \cdot u \,|\, u \in f^{-1}(Y)\} = \{\pi \cdot u \,|\, f(u) \in Y\} \overset{\pi \cdot u := z}{=} \{z \,|\, f(\pi^{-1} \cdot z) \in Y\} = \{z \,|\, \pi^{-1} \cdot f(z) \in Y\} = \{z \,|\, f(z) \in \pi \star Y\} = \{z \,|\, f(z) \in Y\} = f^{-1}(Y)$. Clearly, g is injective because $g(X) = g(Z)$ implies $f(X) = f(Z)$, and so $f^{-1}(f(X)) = f^{-1}(f(Z))$, from which $X = Z$. We also have that g is surjective because $Y = g(f^{-1}(Y))$ for all $Y \in \wp_{fs}(V)$.
   It remains to prove that g is finitely supported. Let $\pi \in Fix(supp(f) \cup supp(U) \cup supp(V))$. We have $g(\pi \star X) = \{f(x) \,|\, x \in \pi \star X\} = \{f(x) \,|\, \pi^{-1} \cdot x \in X\} \overset{\pi^{-1} \cdot x = z}{=} \{f(\pi \cdot z) \,|\, z \in X\} = \{\pi \cdot f(z) \,|\, z \in X\} = \pi \star \{f(z) \,|\, z \in X\} = \pi \star g(X)$, and so g is finitely supported according to Proposition 1.
4. We prove that $supp(f) \cup supp(U) \cup supp(V)$ supports f^{-1}. Let π be a permutation of atoms fixing this set pointwise. Let fix an arbitrary $v \in V$. Then $f(\pi^{-1} \cdot u) = \pi^{-1} \cdot f(u)$ for all $u \in U$ (according to Proposition 1). We have $f^{-1}(\pi \cdot v) = u' \Leftrightarrow f(u') = \pi \cdot v \Leftrightarrow \pi^{-1} \cdot f(u') = v \Leftrightarrow f(\pi^{-1} \cdot u') = v \Leftrightarrow f^{-1}(v) = \pi^{-1} \cdot u' \Leftrightarrow \pi \cdot f^{-1}(v) = u'$. Thus, $f^{-1}(\pi \cdot v) = \pi \cdot f^{-1}(v)$ for all $v \in V$, meaning that f^{-1} is finitely supported (according to Proposition 1).
5. According to items 3 and 4, there exists a finitely supported bijection $g : \wp_{fs}(U) \to \wp_{fs}(V)$ and a finitely supported bijection $g^{-1} : \wp_{fs}(V) \to \wp_{fs}(U)$. Assume that V is un-finite. Let $\mathcal{F}$ be a uniformly supported (by S) family of elements in $\wp_{fs}(U)$. Let $\pi \in Fix(S \cup supp(g))$. Then π fixes pointwise the support of each member of $\mathcal{F}$, and so $\pi \star X = X$ for all $X \in \mathcal{F}$. For $X \in \mathcal{F}$ we have $\pi \star g(X) = g(\pi \star X) = g(X)$; thus, $\{g(X) \,|\, X \in \mathcal{F}\}$ is a uniformly supported subset of V which should be finite. Since g is injective, we have that $\mathcal{F}$ is finite. Conversely, assume that U is un-finite. Let $\mathcal{G}$ be a uniformly supported (by S') family of elements in $\wp_{fs}(V)$. Let $\sigma \in Fix(S' \cup supp(g^{-1}))$. Then σ fixes pointwise the support of each member of $\mathcal{G}$. For $Y \in \mathcal{G}$, we have $\sigma \star g^{-1}(Y) = g^{-1}(\sigma \star Y) = g^{-1}(Y)$, and so $\{g^{-1}(Y) \,|\, Y \in \mathcal{G}\}$ is a uniformly supported subset of U which should be finite. Since g^{-1} is injective, we have that $\mathcal{G}$ is finite.

□

Theorem 1. *Let X be a finitely supported set such that $\wp_{fs}(X)$ is un-finite. Then the set $Soft(X) = \{f : X \to \wp_{fs}(A) \,|\, f \text{ is finitely supported}\}$ is also un-finite.*

Proof. According to Lemma 1(1), there exists a finitely supported bijection $\varphi : Soft(X) \to \wp_{fs}(X \times A)$. According to Lemma 1(2) and (3), there exists a finitely supported bijection $\psi : \wp_{fs}(X \times A) \to \wp_{fs}(A \times X)$. According to Lemma 1(4), there exists a finitely supported bijection $\rho : \wp_{fs}(A \times X) \to \{f : A \to \wp_{fs}(X) \,|\, f \text{ is finitely supported}\}$. Thus, there is a bijection $\rho \circ \psi \circ \varphi : Soft(X) \to \{f : A \to \wp_{fs}(X) \,|\, f \text{ is finitely supported}\}$ supported by $supp(\rho) \cup supp(\psi) \cup supp(\varphi) \cup supp(X)$. We prove that $\{f : A \to \wp_{fs}(X) \,|\, f$ is finitely supported $\}$ is un-finite, which will complete our proof.

Assume by contradiction that, for a certain $S \in \wp_{fin}(A)$, there exist infinitely many functions $f : A \to \wp_{fs}(X)$ that are supported by S. For each function f, there exist and are unique two functions $f|_S$ and $f|_{A\setminus S}$ (i.e., the restrictions of f to S and $A \setminus S$, respectively) that are supported by S. This follows from Proposition 1 since $supp(S) = supp(A \setminus S) = S$, and so for $\pi \in Fix(S)$ we have $\pi(x) = x \in S$, $f|_S(\pi(x)) = f(\pi(x)) = \pi \star f(x) = \pi \star f|_S(x)$, $\forall x \in S$, and $\pi(y) \in A \setminus S$, $f|_{A\setminus S}(\pi(y)) = f(\pi(y)) = \pi \star f(y) = \pi \star f|_{A\setminus S}(y)$ for all $y \in A \setminus S$.

We prove that there is a finitely supported injection φ from $\{h : S \to \wp_{fs}(X) \mid h$ finitely supported $\}$ into $\wp_{fs}(X)^{|S|}$, where $|S|$ represents the cardinality of S. If $S = \{x_1, \ldots, x_n\}$, we may define $\varphi(h) = (h(x_1), \ldots, h(x_n))$. Let π be a permutation fixing $\{x_1, \ldots, x_n\} \cup supp(X)$ pointwise. We have $\varphi(\pi\widetilde{\star}h) = ((\pi\widetilde{\star}h)(x_1), \ldots, (\pi\widetilde{\star}h)(x_n)) = (\pi \star h(\pi^{-1}(x_1)), \ldots, \pi \star h(\pi^{-1}(x_n))) = (\pi \star h(x_1), \ldots, \pi \star h(x_n)) = \pi \otimes \varphi(h)$ for all $h \in \wp_{fs}(X)^S_{fs}$, and so φ is finitely supported. Using the relation $supp(U_1) \cup \ldots \cup supp(U_n) = supp((U_1, \ldots, U_n))$ for all $U_1, \ldots, U_n \in \wp_{fs}(X)$, we have that the $|S|$-times Cartesian product of $\wp_{fs}(X)$ does not contain a uniformly supported infinite subset; otherwise, $\wp_{fs}(X)$ should itself contain a uniformly supported infinite subset, contradicting our hypothesis.

Since we proved that $f|_S$ can be defined in at most finitely many ways, there must exist an infinite family $\mathcal{U}$ of functions $g : (A \setminus S) \to \wp_{fs}(X)$ which are supported by the same S. Let us fix an element $z \in A \setminus S$, and let $g \in \mathcal{U}$. For $\pi \in Fix(S \cup \{z\})$, according to Proposition 1, we have $\pi \star g(z) = g(\pi(z)) = g(z)$ meaning that $supp(g(z)) \subseteq S \cup \{z\}$. However, in $\wp_{fs}(X)$ there exist only finitely many elements supported by $S \cup \{z\}$. Thus, there is $k \in \mathbb{N}$ such that $g_1(z), \ldots, g_k(z)$ are distinct elements in $\wp_{fs}(X)$ with $g_1, \ldots, g_k \in \mathcal{U}$, and $g(z) \in \{g_1(z), \ldots, g_k(z)\}$ for all $g \in \mathcal{U}$. Let us take $g \in \mathcal{U}$ and an arbitrary $w \in A \setminus S$, meaning that the transposition $(z\,w)$ fixes S pointwise. We have that there exists $i \in \{1, \ldots, k\}$ such that $g(z) = g_i(z)$. Since g, g_i are both supported by S and $(z\,w) \in Fix(S)$, according to Proposition 1 we get that $g(w) = g((z\,w)(z)) = (z\,w) \star g(z) = (z\,w) \star g_i(z) = g_i((z\,w)(z)) = g_i(w)$, which finally leads to $g = g_i$ on their entire domain of definition $A \setminus S$. Therefore, $\mathcal{U} \subseteq \{g_1, \ldots, g_m\}$ meaning that $\mathcal{U}$ is finite, a contradiction with our assumption that $\mathcal{U}$ is infinite. □

Corollary 1. *The set $Soft(A^m)$ is un-finite for every $m \in \mathbb{N}$.*

Proof. We have to prove that for any $m \in \mathbb{N}$, $\wp_{fs}(A^m)$ is un-finite (and the result follows from Theorem 1). For $m = 1$, this is obvious because the subsets of A (i.e., the elements of $\wp_{fs}(A)$) supported by a certain $S \in \wp_{fin}(A)$ are precisely the supersets of $A \setminus S$ and the subsets of S, being at most $2^{(|S|+1)}$ such subsets. We proceed by induction on m. Assume that $\wp_{fs}(A^m)$ is un-finite. According to Theorem 1, $Soft(A^m)$ is un-finite. According to Lemma 1(1), there exists a finitely supported bijection $\varphi : Soft(A^m) \to \wp_{fs}(A^m \times A) = \wp_{fs}(A^{m+1})$, and so $\wp_{fs}(A^{m+1})$ is also un-finite. □

Corollary 2. *Let X be a finitely supported set such that $\wp_{fs}(X)$ is un-finite. Then the set $Soft(X \times A^n)$ is also un-finite for all $n \in \mathbb{N}^*$.*

Proof. From the proof of Theorem 1, we have that the set $\{f : A \to \wp_{fs}(Y) \mid f$ is finitely supported $\}$ is un-finite whenever Y is a finitely supported set such that $\wp_{fs}(Y)$ is un-finite (1). We prove that $\wp_{fs}(X \times A^n)$ is un-finite by induction on n. For $n = 1$ we have to prove that $\wp_{fs}(X \times A)$ is un-finite. According to Lemma 1, we have that there is a finitely supported bijection between $\wp_{fs}(A \times X)$ and the set $\{f : A \to \wp_{fs}(X) \mid f$ is finitely supported $\}$. Hence, by (1), $\wp_{fs}(A \times X)$ is un-finite, and so $\wp_{fs}(X \times A)$ is also un-finite (according to Lemma 1). Now, assume that $\wp_{fs}(X \times A^k)$ is un-finite for some $k \in \mathbb{N}^*$. We have that there is a finitely supported bijection (by $supp(X)$) between $X \times A^{k+1}$ and $(X \times A^k) \times A$; according to Lemma 1(2), there is a finitely supported bijection (also by $supp(X)$) between $(X \times A^k) \times A$ and $A \times (X \times A^k)$. From Lemma 1(3), there is a finitely supported bijection between $\wp_{fs}((X \times A^k) \times A)$ and $\wp_{fs}(A \times (X \times A^k))$, and a finitely supported bijection between $\wp_{fs}(X \times A^{k+1})$ and $\wp_{fs}(A \times (X \times A^k))$. However, from Lemma 1(1), there is a finitely supported bijection between $\wp_{fs}(A \times (X \times A^k))$ and the set $Z = \{f : A \to \wp_{fs}(X \times A^k) \mid f \text{ is finitely supported}\}$. Since $\wp_{fs}(X \times A^k)$ is un-finite according to the inductive hypothesis, from (1) we have that Z is un-finite, and so $\wp_{fs}(X \times A^{k+1})$ is un-finite, from which $\wp_{fs}(X \times A^n)$ is un-finite for all $n \in \mathbb{N}^*$. From Theorem 1, we get that $Soft(X \times A^n)$ is un-finite for all $n \in \mathbb{N}^*$. $\square$

Corollary 3. *Let X be a finitely supported set such that $\wp_{fs}(X)$ is un-finite. Then, for any $n \in \mathbb{N}$, every injective function $\varphi : Soft(X \times A^n) \to Soft(X \times A^n)$ is also surjective.*

Proof. Fix some $n \in \mathbb{N}$. According to Theorem 1 and Corollary 2, $Soft(X \times A^n)$ is un-finite. Assume by contradiction that there is an injective function $\varphi : Soft(X \times A^n) \to Soft(X \times A^n)$ having the property that $Im(\varphi) \subsetneq Soft(X \times A^n)$. Therefore, there is $f_0 \in Soft(X \times A^n)$ such that $f_0 \notin Im(\varphi)$, and hence, $f_0 \neq \varphi^m(f_0)$ for all $m \in \mathbb{N}^*$. Using an induction on k, we show that $\varphi^k(f_0)$ is supported by $supp(\varphi) \cup supp(f_0) \cup supp(X)$ for all $k \in \mathbb{N}$. For $k = 0$, we have that $\varphi^0(f_0) = f_0$ is supported by $supp(f_0) \cup supp(X)$ according to the definition of the support. Let $\pi \in Fix(supp(\varphi) \cup supp(f_0) \cup supp(X))$. For $k = 1$ we have $\pi \widetilde{\star} \varphi(f_0) = \varphi(\pi \widetilde{\star} f_0) = \varphi(f_0)$, and so $\varphi^1(f_0)$ is supported by $supp(\varphi) \cup supp(f_0) \cup supp(X)$. From the inductive hypothesis and since $supp(\varphi^k(f_0))$ is the intersection of all finite sets supporting $\varphi^k(f_0)$, we have that $supp(\varphi^k(f_0)) \subseteq supp(\varphi) \cup supp(f_0) \cup supp(X)$. Therefore, we also have $\pi \in Fix(supp(\varphi^k(f_0))$, that is $\pi \widetilde{\star} \varphi^k(f_0) = \varphi^k(f_0)$. Since $Soft(X \times A^n)$ is supported by $supp(X)$ and π fixes $supp(X)$ pointwise, we have $\pi \widetilde{\star} f \in Soft(X \times A^n)$ for all $f \in Soft(X \times A^n)$. According to Proposition 1, $\pi \widetilde{\star} \varphi^{k+1}(f_0) = \pi \widetilde{\star} \varphi(\varphi^k(f_0)) = \varphi(\pi \widetilde{\star} \varphi^k(f_0)) = \varphi(\varphi^k(f_0)) = \varphi^{k+1}(f_0)$, and so $\varphi^{k+1}(f_0)$ is supported by $supp(\varphi) \cup supp(f_0) \cup supp(X)$, from which our claim follows. Since φ is injective and $f_0 \neq \varphi^m(f_0)$ for all $m \in \mathbb{N}^*$, we get that $\varphi^p(f_0) \neq \varphi^q(f_0)$ for all $p, q \in \mathbb{N}, p \neq q$. Therefore, the sequence $(\varphi^k(f_0))_{k \in \mathbb{N}} \subseteq Soft(X \times A^n)$ is infinite and uniformly supported by $supp(\varphi) \cup supp(f_0) \cup supp(X)$, which represents a contradiction. $\square$

Corollary 4. *Let X be a finitely supported set such that $\wp_{fs}(X)$ is un-finite. Then $\wp_{fin}(Soft(X \times A^n))$ is also un-finite for all $n \in \mathbb{N}$.*

Proof. Let $\{f_1, \ldots, f_k\} \in \wp_{fin}(Soft(X \times A^n))$. First, we prove that $supp(\{f_1, \ldots, f_k\}) \cup supp(X) = supp(f_1) \cup \ldots \cup supp(f_k)$. Let us take $\pi \in Fix(supp(f_1) \cup \ldots \cup supp(f_k))$. Then $\pi \in Fix(supp(f_i))$, and so $\pi \widetilde{\star} f_i = f_i$ for all $i \in \{1, \ldots, k\}$. Hence, $\pi \star \{f_1, \ldots, f_k\} = \{\pi \widetilde{\star} f_1, \ldots, \pi \widetilde{\star} f_k\} = \{f_1, \ldots, f_k\}$ meaning that $supp(f_1) \cup \ldots \cup supp(f_k)$ supports $\{f_1, \ldots, f_k\}$, and so we have $supp(\{f_1, \ldots, f_k\}) \subseteq supp(f_1) \cup \ldots \cup supp(f_k)$. According to Proposition 1, we have that $\pi \otimes (x, (a_1, \ldots a_n)) \in X \times A^n$ for all $x \in X$ and $a_1, \ldots, a_n \in A$, and so $\pi \star X = X$ meaning that $supp(X) \subseteq supp(f_1) \cup \ldots \cup supp(f_k)$. Hence, $supp(\{f_1, \ldots, f_k\}) \cup supp(X) \subseteq supp(f_1) \cup \ldots \cup supp(f_k)$.

Conversely, let $a \in supp(f_1) \cup \ldots \cup supp(f_k)$. If $a \in supp(X)$, the problem is solved; thus, assume $a \notin supp(X)$. There exists $j \in \{1, \ldots, k\}$ such that $a \in supp(f_j)$. Let $b \in A$ such that $b \notin supp(X)$, $b \notin supp(\{f_1, \ldots, f_k\})$ and $b \notin supp(f_1) \cup \ldots \cup supp(f_k)$. We prove that $(b\ a) \widetilde{\star} f_j \neq f_1, \ldots, f_k$. Indeed, assume by contradiction that $(b\ a) \widetilde{\star} f_j = f_i$ for some $i \in \{1, \ldots, k\}$. Since $b, a \notin supp(X)$, we have $(b\, a) \in Fix(supp(X))$, and so $(b, a) \in Fix(supp(X \times A^n))$. According to Proposition 2, since $a \in supp(f_j)$ we have $b = (b\ a)(a) \in (b\ a)(supp(f_j)) = supp((b\ a) \widetilde{\star} f_j) = supp(f_i)$ contradicting the choice of b. Therefore, $(b\, a) \star \{f_1, \ldots, f_k\} \neq \{f_1, \ldots, f_k\}$. Since $b \notin supp(\{f_1, \ldots, f_k\})$, we should have $a \in supp(\{f_1, \ldots, f_k\})$; otherwise, we would have $(b\, a) \in Fix(supp(\{f_1, \ldots, f_k\}))$ and $(b\, a) \star \{f_1, \ldots, f_k\} = \{f_1, \ldots, f_k\}$ - a contradiction.

It is easy to note that we have $supp(X) \subseteq supp(\{f_1, \ldots, f_k\})$. Let $\sigma \in Fix(supp(\{f_1, \ldots, f_k\}))$. Then, for each $l \in \{1, \ldots, k\}$, there is a unique $m \in \{1, \ldots, k\}$ such that $\sigma \widetilde{\star} f_l = f_m$. Thus, whenever $(x, (a_1, \ldots, a_n)) \in X \times A^n$, we have that it belongs to the domain of definition of some f_l, and so $(\sigma \cdot x, (\sigma(a_1), \ldots, \sigma(a_n)))$ belongs to the domain of definition of f_m, meaning that $\sigma \cdot x \in X$ for all $x \in X$. We get that

$supp(\{f_1,\dots,f_k\}) = supp(f_1)\cup\ldots\cup supp(f_k)$. Thus, whenever $supp(\{f_1,\dots,f_k\})\subseteq S$, we have $supp(f_i)\subseteq S$ for all $i\in\{1,\dots,k\}$. Therefore, if $\wp_{fin}(Soft(X\times A^n))$ had an infinite uniformly supported subset, then $Soft(X\times A^n)$ would have an infinite uniformly supported subset. However, from Corollary 2, we know that $Soft(X\times A^n)$ is un-finite. This ends the proof. □

5. Order Properties of Soft Sets with Atoms

In this section, we define a finitely supported order relation in the family of all soft sets with atoms (on a certain finitely supported set), and prove that this family can be organized as a finitely supported complete lattice. Furthermore, we prove a Tarski-like fixed point theorem.

Definition 4.

1. *A* finitely supported partially ordered set $(P,\leq,\cdot)$ *is a finitely supported set* $(P,\cdot)$ *equipped with a partial order relation* $\leq$ *that is finitely supported as a subset of* $P\times P$.
2. *A* finitely supported lattice $(L,\leq,\cdot)$ *is a finitely supported set* $(L,\cdot)$ *equipped with a lattice order relation* $\leq$ *that is finitely supported as a subset of* $L\times L$.
3. *A* finitely supported complete lattice *is a finitely supported lattice* $(L,\leq,\cdot)$ *with the property that every finitely supported subset of* L *has a greatest lower bound and a least upper bound in* L.

Lemma 2. *Let* $\mathcal{F}$ *be a finitely supported family of elements from* $\wp_{fs}(A)$. *Then,*

1. $\underset{X\in\mathcal{F}}{\cup} X\in\wp_{fs}(A)$;
2. $\pi\star\underset{X\in\mathcal{F}}{\cup} X = \underset{X\in\mathcal{F}}{\cup}(\pi\star X)$ *for all* $\pi\in S_A$.

Proof.

1. Let $\pi\in Fix(supp(\mathcal{F}))$. Then $\pi\star X\in\mathcal{F}$ for all $X\in\mathcal{F}$. Let $x\in\underset{X\in\mathcal{F}}{\cup} X$, that is $x\in Y$ for some $Y\in\mathcal{F}$. We have $\pi\star Y\in\mathcal{F}$, and so $\pi(x)\in\pi\star Y$ with $\pi\star Y\in\mathcal{F}$. This means $\pi(x)\in\underset{X\in\mathcal{F}}{\cup} X$, and hence, $\underset{X\in\mathcal{F}}{\cup} X$ is supported by $supp(\mathcal{F})$.
2. Since $\mathcal{F}$ is finitely supported, we have that $\pi\star\mathcal{F} = \{\pi\star X\,|\,X\in\mathcal{F}\}$ is supported by $\pi\star supp(\mathcal{F})$ (from the proof of Proposition 2), and its union exists (by item 1). Let $z\in\pi\star\underset{X\in\mathcal{F}}{\cup} X$. Then $z=\pi(x)$ with $x\in\underset{X\in\mathcal{F}}{\cup} X$. Then there is $Y\in\mathcal{F}$ such that $x\in Y$, and so $z\in\pi\star Y\subseteq\underset{X\in\mathcal{F}}{\cup}(\pi\star X)$. Conversely, if $z\in\pi\star Y$ for some $Y\in\mathcal{F}$, we have that $z=\pi(y)$ with $y\in Y\subseteq\underset{X\in\mathcal{F}}{\cup} X$, i.e., $z\in\pi\star\underset{X\in\mathcal{F}}{\cup} X$.

□

Theorem 2. *Let* $(X,\cdot)$ *be a finitely supported set. On* $Soft(X)$, *we define the relation* $f\leq g$ *if and only if* $f(x)\subseteq g(x)$ *for all* $x\in X$. *Then* $(Soft(X),\leq,\widetilde{\star})$ *is a finitely supported complete lattice.*

Proof. Obviously, $Soft(X)$ is supported by $supp(X)$. This is because for $\sigma\in Fix(supp(X))$ and $f\in Soft(X)$ we have that $\sigma\widetilde{\star}f:\sigma\star X = X\to\wp_{fs}(A)$ is finitely supported (this follows from the proof of Proposition 2), and so $\sigma\widetilde{\star}f\in Soft(X)$. Let $\pi\in Fix(supp(X))$, meaning that $\pi\cdot x\in X$ for all $x\in X$. It is easy to note that the order relation $\subseteq$ on $\wp_{fs}(A)$ is equivariant. Let $f,g\in Soft(X)$ such that $f\leq g$. Thus, $f(x)\subseteq g(x)$ for all $x\in X$, and so, since $\pi^{-1}\cdot x\in X$, we get that $f(\pi^{-1}\cdot x)\subseteq g(\pi^{-1}\cdot x)$ for all $x\in X$. Finally, since $\subseteq$ is equivariant, we get that $\pi\star f(\pi^{-1}\cdot x)\subseteq\pi\star g(\pi^{-1}\cdot x)$ for all $x\in X$, which means $(\pi\widetilde{\star}f)(x)\subseteq(\pi\widetilde{\star}g)(x)$ for all $x\in X$, and so $\pi\widetilde{\star}f\leq\pi\widetilde{\star}g$. Therefore, $\leq$ is supported by $supp(X)$.

Let $\mathcal{F}$ be a finitely supported family of elements in $Soft(X)$. Let us fix $x\in X$. We prove that the family $\mathcal{F}_x = \{f(x)\,|\,f\in\mathcal{F}\}$ is finitely supported. Let $\pi\in Fix(supp(X)\cup supp(\mathcal{F})\cup supp(x))$. Then $\pi\widetilde{\star}f\in\mathcal{F}$ for all $f\in\mathcal{F}$, and $\pi\cdot x = x$. Let $f(x)\in\mathcal{F}_x$. Then

$f \in \mathcal{F}$, and so there is a unique $g \in \mathcal{F}$ such that $\pi \widetilde{\star} f = g$, i.e., $f = \pi^{-1} \widetilde{\star} g$. Thus, $f(x) = (\pi^{-1} \widetilde{\star} g)(x) = \pi^{-1} \star g((\pi^{-1})^{-1} \cdot x) = \pi^{-1} \star g(\pi \cdot x) = \pi^{-1} \star g(x)$, and so $\pi \star f(x) = g(x) \in \mathcal{F}_x$.

According to Lemma 2, we can define $\underset{f \in \mathcal{F}}{\vee} f : X \to \wp_{fs}(A)$ by $(\underset{f \in \mathcal{F}}{\vee} f)(x) = \underset{f \in \mathcal{F}}{\cup} f(x)$ for all $x \in X$. According to Lemma 2(1), since $\mathcal{F}_x$ is finitely supported, we get that $\underset{f \in \mathcal{F}}{\vee} f$ is well-defined (its image is contained in $\wp_{fs}(A)$). A simple calculation shows that $\underset{f \in \mathcal{F}}{\vee} f$ is the least upper bound of $\mathcal{F}$ in $(Soft(X), \leq)$. Let $\pi \in Fix(supp(X) \cup supp(\mathcal{F}))$. According to Proposition 1, in order to prove that $\underset{f \in \mathcal{F}}{\vee} f$ is finitely supported, we should prove the relation $(\underset{f \in \mathcal{F}}{\vee} f)(\pi \cdot x) = \pi \star (\underset{f \in \mathcal{F}}{\vee} f)(x)$ for all $x \in X$, or equivalently, $\underset{f \in \mathcal{F}}{\cup} f(\pi \cdot x) = \pi \star \underset{f \in \mathcal{F}}{\cup} f(x)$. For every $f \in \mathcal{F}$ there is a unique $g \in \mathcal{F}$ such that $\pi \widetilde{\star} f = g$, i.e., $f = \pi^{-1} \widetilde{\star} g$, and $f(x) = (\pi^{-1} \widetilde{\star} g)(x) = \pi^{-1} \star g((\pi^{-1})^{-1} \cdot x) = \pi^{-1} \star g(\pi \cdot x)$ for all $x \in X$. Thus, for all $x \in X$ we have the following equality of finitely supported sets: $\{f(\pi \cdot x) \mid f \in \mathcal{F}\} = \{g(\pi \cdot x) \mid g \in \mathcal{F}\} = \{\pi \star f(x) \mid f \in \mathcal{F}\}$. By using Lemma 2(2), we get that $\underset{f \in \mathcal{F}}{\cup} f(\pi \cdot x) = \underset{f \in \mathcal{F}}{\cup} (\pi \star f(x)) = \pi \star \underset{f \in \mathcal{F}}{\cup} f(x)$ for all $x \in X$, and so $\underset{f \in \mathcal{F}}{\vee} f$ is finitely supported.

Similarly, $\underset{f \in \mathcal{F}}{\wedge} f : X \to \wp_{fs}(A)$ defined by $(\underset{f \in \mathcal{F}}{\wedge} f)(x) = \underset{f \in \mathcal{F}}{\cap} f(x)$ for all $x \in X$ is finitely supported, and it is the greatest lower bound of $\mathcal{F}$. $\square$

Theorem 3. *Let $(X, \cdot)$ be a non-empty, finitely supported set. Let $\varphi : Soft(X) \to Soft(X)$ be a finitely supported, order-preserving function. The set $Fix_\varphi = \{f \in Soft(X) \mid \varphi(f) = f\}$ is itself a non-empty finitely supported complete lattice supported by $supp(X) \cup supp(\varphi)$.*

Proof. We prove that Fix_φ is a non-empty, finitely supported set. Let us consider $\pi \in Fix(supp(X) \cup supp(\varphi))$. Since $supp(Soft(X)) \subseteq supp(X)$, we have that $\pi \widetilde{\star} f \in Soft(X)$ for all $f \in Soft(X)$. Now, let us consider $f \in Fix_\varphi$, namely $\varphi(f) = f$. According to Proposition 1, since π fixes $supp(\varphi)$ pointwise, we have $\varphi(\pi \widetilde{\star} f) = \pi \widetilde{\star} \varphi(f) = \pi \widetilde{\star} f$ meaning $\pi \widetilde{\star} f \in Fix_\varphi$, and so Fix_φ is supported by $supp(X) \cup supp(\varphi)$.

Let $Pre_\varphi = \{f \in Soft(X) \mid f \leq \varphi(f)\}$. Pre_φ is non-empty because it contains the least element of $Soft(X)$. Let $\pi \in Fix(supp(X) \cup supp(\varphi))$, and $f \in Pre_\varphi$. Then $f \leq \varphi(f)$. Since $supp(\leq) \subseteq supp(X)$, we have that π fixes $supp(\leq)$ pointwise, and so $\pi \widetilde{\star} f \leq \pi \widetilde{\star} \varphi(f) = \varphi(\pi \widetilde{\star} f)$, which means $\pi \widetilde{\star} f \in Pre_\varphi$. Thus, Pre_φ is supported by $supp(X) \cup supp(\varphi)$, and since $Soft(X)$ is a complete lattice, there must exist $\underset{g \in Pre_\varphi}{\vee} g$. For each $f \in Pre_\varphi$, we have $f \leq \underset{g \in Pre_\varphi}{\vee} g$. Since φ is order preserving, we get that $\varphi(f) \leq \varphi(\underset{g \in Pre_\varphi}{\vee} g)$, and so $f \leq \varphi(f) \leq \varphi(\underset{g \in Pre_\varphi}{\vee} g)$, from which $\underset{g \in Pre_\varphi}{\vee} g \leq \varphi(\underset{g \in Pre_\varphi}{\vee} g)$, i.e $\underset{g \in Pre_\varphi}{\vee} g \in Pre_\varphi$. Since φ is order preserving, we should have $\varphi(h) \in Pre_\varphi$ for all $h \in Pre_\varphi$. Since $\underset{g \in Pre_\varphi}{\vee} g \in Pre_\varphi$, we have $\varphi(\underset{g \in Pre_\varphi}{\vee} g) \in Pre_\varphi$, and so (according to the definition of the least upper bound of Pre_φ) we get that $\varphi(\underset{g \in Pre_\varphi}{\vee} g) \leq \underset{g \in Pre_\varphi}{\vee} g$, which finally leads to $\underset{g \in Pre_\varphi}{\vee} g = \varphi(\underset{g \in Pre_\varphi}{\vee} g)$, and so Fix_φ is non-empty.

Let U be a finitely supported subset of Fix_φ. Since $Soft(X)$ is a finitely supported complete lattice, we have that $\underset{g \in U}{\vee} g$ exists in $Soft(X)$. We prove that $\underset{g \in U}{\vee} g \in Fix_\varphi$. Considering $f \in U$, we have $f \leq \underset{g \in U}{\vee} g$, and so $\varphi(f) \leq \varphi(\underset{g \in U}{\vee} g)$. Since $U \subseteq Fix_\varphi$, we have $\varphi(f) = f$, and so $f \leq \varphi(\underset{g \in U}{\vee} g)$, from which it follows that $\underset{g \in U}{\vee} g \leq \varphi(\underset{g \in U}{\vee} g)$. Now, let $h \in Soft(X)$ such that $\underset{g \in U}{\vee} g \leq h$, and so $\varphi(\underset{g \in U}{\vee} g) \leq \varphi(h)$, from which $\underset{g \in U}{\vee} g \leq \varphi(h)$. Thus, $\underset{g \in U}{\vee} g \leq h$ implies $\underset{g \in U}{\vee} g \leq \varphi(h)$ (†).

Let $Post_\varphi^U = \{f \in Soft(X) \,|\, \varphi(f) \leq f$ and $\underset{g\in U}{\vee} g \leq f\}$. Let $\pi \in Fix(supp(X) \cup supp(\varphi) \cup supp(\underset{g\in U}{\vee} g))$, and let $f \in Post_\varphi^U$. Then $\varphi(f) \leq f$. Since $\leq$ is supported by $supp(X)$, we have $\pi \widetilde{\star} \varphi(f) \leq \pi \widetilde{\star} f$. However, according to Proposition 1, since π fixes $supp(\varphi)$ pointwise, we get that $\pi \widetilde{\star} \varphi(f) = \varphi(\pi \widetilde{\star} f)$, and so $\varphi(\pi \widetilde{\star} f) \leq \pi \widetilde{\star} f$. Since $f \in Post_\varphi^U$, we have $\underset{g\in U}{\vee} g \leq f$. Thus, since π fixes $supp(\underset{g\in U}{\vee} g)$ pointwise and $\leq$ is supported by $supp(X)$, we get that $\underset{g\in U}{\vee} g = \pi \widetilde{\star} \underset{g\in U}{\vee} g \leq \pi \widetilde{\star} f$. Thus, $\pi \widetilde{\star} f \in Post_\varphi^U$, meaning that $Post_\varphi^U$ is supported by $supp(X) \cup supp(\varphi) \cup supp(\underset{g\in U}{\vee} g)$. Therefore, there exists a greatest lower bound of $Post_\varphi^U$, namely $\underset{g\in Post_\varphi^U}{\wedge} g$. For each $f \in Post_\varphi^U$, we have $\underset{g\in Post_\varphi^U}{\wedge} g \leq f$, and so $\varphi(\underset{g\in Post_\varphi^U}{\wedge} g) \leq \varphi(f) \leq f$. Hence, $\varphi(\underset{g\in Post_\varphi^U}{\wedge} g) \leq f$ for all $f \in Post_\varphi^U$, and so $\varphi(\underset{g\in Post_\varphi^U}{\wedge} g) \leq \underset{g\in Post_\varphi^U}{\wedge} g$. Moreover, $\underset{g\in U}{\vee} g \leq f$ for each $f \in Post_\varphi^U$, and so $\underset{g\in U}{\vee} g \leq \underset{g\in Post_\varphi^U}{\wedge} g$ meaning that $\underset{g\in Post_\varphi^U}{\wedge} g \in Post_\varphi^U$. Since φ is order preserving, by using (†), we get that $\varphi(f) \in Post_\varphi^U$ whenever $f \in Post_\varphi^U$. Since $\underset{g\in Post_\varphi^U}{\wedge} g \in Post_\varphi^U$, we get that $\varphi(\underset{g\in Post_\varphi^U}{\wedge} g) \in Post_\varphi^U$, and so $\underset{g\in Post_\varphi^U}{\wedge} g \leq \varphi(\underset{g\in Post_\varphi^U}{\wedge} g)$ from the definition of a greatest lower bound.

We established that $\underset{g\in Post_\varphi^U}{\wedge} g \in Fix_\varphi$ with $\underset{g\in U}{\vee} g \leq \underset{g\in Post_\varphi^U}{\wedge} g$. Hence, $\underset{g\in Post_\varphi^U}{\wedge} g$ is an upper bound for U; now we prove that it is the least one. Let $h \in Fix_\varphi$ be another upper bound of U. Then $\underset{g\in U}{\vee} g \leq h$ (because $\underset{g\in U}{\vee} g$ is the least upper bound of U in $Soft(X)$, and h is an upper bound of U in $Soft(X)$). Thus, $h \in Post_\varphi^U$, from which we get that $\underset{g\in Post_\varphi^U}{\wedge} g \leq h$. We have that $\underset{g\in Post_\varphi^U}{\wedge} g$ is the least upper bound of U in Fix_φ.

We proved that every finitely supported subset of Fix_φ has a least upper bound. Now, we have to prove that V has a greatest lower bound, where $V \in \wp_{fs}(Fix_\varphi)$. For $f \in Fix_\varphi$, let $\downarrow f = \{g \in Fix_\varphi \,|\, g \leq f\}$. Let $\pi \in Fix(supp(X) \cup supp(\varphi) \cup supp(f))$. We have that π fixes $supp(Fix_\varphi)$ and $supp(\leq)$ pointwise. Let $g \in \downarrow f$. We have that $\pi \widetilde{\star} g \in Fix_\varphi$ and $\pi \widetilde{\star} g \leq \pi \widetilde{\star} f = f$. Thus, $\pi \widetilde{\star} g \in \downarrow f$, meaning that $\downarrow f$ is finitely supported. Let $M_V = \cap\{\downarrow h \,|\, h \in V\}$, $\pi \in Fix(supp(V) \cup supp(X))$, and $f \in M_V$. It follows that $f \leq h$ for all $h \in V$. Let $g \in V$ be an arbitrary element. Since π fixes $supp(V)$ pointwise, we have $\pi^{-1} \widetilde{\star} g \in V$. However, $f \leq \pi^{-1} \widetilde{\star} g$. Since $supp(\leq) \subseteq supp(X)$, we have that π fixes $supp(\leq)$ pointwise. Thus, $\pi \widetilde{\star} f \leq \pi \widetilde{\star} (\pi^{-1} \widetilde{\star} g) = g$, and so $\pi \widetilde{\star} f \in M_V$, meaning that M_V is finitely supported. Therefore, $\underset{s\in M_V}{\vee} s$ is the least upper bound of M_V. We prove that $\underset{s\in M_V}{\vee} s = \underset{t\in V}{\wedge} t$. For $t \in V$, we have that t is an upper bound of M_V, and so $\underset{s\in M_V}{\vee} s \leq t$. Since t has been arbitrarily chosen from V, we get that $\underset{s\in M_V}{\vee} s \in M_V$. Since $\underset{s\in M_V}{\vee} s$ is maximal among the lower bounds of V (and it is a lower bound of V), we conclude that it is the greatest lower bound of V. □

Theorem 4. *Let X be a finitely supported set such that $\wp_{fs}(X)$ is un-finite. Then every finitely supported chain in $(Soft(X), \leq, \widetilde{\star})$ is finite.*

Proof. According to Theorem 1, the set $Soft(X)$ is un-finite. Let T be a finitely supported chain in $Soft(X)$, i.e., a finitely supported, totally ordered subset of $(Soft(X), \leq, \widetilde{\star})$. Let f be an arbitrary element from T, and $\pi \in Fix(supp(T) \cup supp(X))$. We have $\pi \widetilde{\star} f \in T$. Since T is totally ordered (i.e., any two elements of T are comparable with respect to $\leq$), we must have either $f < \pi \widetilde{\star} f$ or $f = \pi \widetilde{\star} f$, or $\pi \widetilde{\star} f < f$. Assume $f < \pi \widetilde{\star} f$. By induction on m, we prove that $f < \pi^m \widetilde{\star} f$ for all $m \in \mathbb{N}^*$. For $m = 1$, this is clear from our assumption. Assume $f < \pi^n \widetilde{\star} f$ for some positive integer n. Since the order relation $\leq$ is supported by

$supp(X)$ and π fixes $supp(X)$ pointwise, we get that $\pi \star f \leq \pi \star (\pi^n \star f) = (\pi \circ \pi^n) \star f = \pi^{n+1} \star f$. However, since $\star$ is a group action and permutations of atoms are bijective, we get that $\pi \star f \neq \pi \star (\pi^n \star f)$; otherwise, if $\pi \star f = \pi \star (\pi^n \star f)$, we would get that $\pi^{-1} \star (\pi \star f) = \pi^{-1} \star (\pi \star (\pi^n \star f))$, that is $f = \pi^n \star f$, contradicting the inductive hypothesis. Thus, $\pi \widetilde{\star} f < \pi^{n+1} \widetilde{\star} f$. Since $f < \pi \widetilde{\star} f$, we get that $f < \pi^{n+1} \widetilde{\star} f$, and so $f < \pi^m \widetilde{\star} f$ for all $m \in \mathbb{N}^*$. However, since any permutation of atoms is a finite composition of transpositions, it must have a finite order in the permutation group S_A, i.e., there is $k \in \mathbb{N}$ such that $\pi^k = Id_A$. We have that $f < f$, a contradiction. Similarly, if $\pi \widetilde{\star} f < f$, we would get that $\pi^m \widetilde{\star} f < \ldots < \pi \widetilde{\star} f < f$ for all $m \in \mathbb{N}^*$, and so $f < f$, a contradiction. Therefore, $\sigma \widetilde{\star} f = f$ for all $\sigma \in Fix(supp(T) \cup supp(X))$, meaning that $supp(f) \subseteq supp(T) \cup supp(X)$. Since f has been arbitrarily chosen from T, we have that T is uniformly supported by $supp(T) \cup supp(X)$. However, $Soft(X)$ is un-finite, and so T should be finite. $\square$

6. Conclusions

A soft set with atoms on a finitely supported set X is defined as a finitely supported function $f : X \to \wp_{fs}(A)$. Examples of such soft sets with atoms are the mapping that associates to each $x \in X$ its support and the mapping that associates to each $x \in X$ the complement of its support, respectively (Proposition 2). The un-finite sets are those finitely supported sets which do not contain infinite uniformly supported subsets. A closure un-finiteness property states that whenever the finitely supported powerset of a finitely supported set X is un-finite, we have that the set of all soft sets with atoms on X and the set of all soft sets with atoms on $X \times A^n$ with $n \in \mathbb{N}^*$ are also un-finite (Theorem 1 and Corollary 2). If the finitely supported powerset of X is un-finite, then any finitely supported injective self-mapping on the set of all soft sets with atoms over $X \times A^n$ is also surjective (Corollary 3).

Furthermore, according to Theorem 2, the family $Soft(X)$ of all soft sets with atoms on a finitely supported set X can be organized as a finitely supported complete lattice, i.e., there may be defined a finitely supported order relation on $Soft(X)$ such that any finitely supported subset of $Soft(X)$ has a least upper bound and a greatest lower bound. A Tarski-like fixed point result presented as Theorem 3 states that the set of all fixed points of a finitely supported order preserving self-function on $Soft(X)$ form a non-empty, finitely supported complete lattice. According to Theorem 4, when the additional condition of un-finiteness for the finitely supported powerset of X is imposed, we have that every finitely supported totally ordered subset of $Soft(X)$ is finite.

The concept of a soft set with atoms could be extended by replacing the universal set A with a higher-order atomic set. If Y is the finitely supported universal set (that replaces A) and X is a finitely supported set, a *finitely supported soft set of higher-order* on X is a pair (X, f, Y), where $f : X \to \wp_{fs}(Y)$ is a finitely supported function. We denote $Soft^Y(X) = \{f : X \to \wp_{fs}(Y) \,|\, f \text{ finitely supported}\}$. In fact, $Soft^A(X) = Soft(X)$ with the notations used in this article. The proofs of Lemma 2, Theorem 2 and Theorem 3 do not use the particular form of the canonical S_A-action on A but only the properties of a group action. Thus, involving similar proving methods and the S-finite support principle for constructing higher-order supports, we additionally obtain the following results:

- Let $(X, \cdot)$ and $(Y, \diamond)$ be finitely supported sets. On $Soft^Y(X)$, we define the relation $f \leq g$ if and only if $f(x) \subseteq g(x)$ for all $x \in X$. Then $(Soft^Y(X), \leq, \widetilde{\star})$ is a finitely supported complete lattice, supported by $supp(X) \cup supp(Y)$.
- Let $(X, \cdot)$ and $(Y, \diamond)$ be non-empty finitely supported sets.
  Let $\varphi : Soft^Y(X) \to Soft^Y(X)$ be a finitely supported, order-preserving function. The set $Fix_\varphi = \{f \in Soft^Y(X) \,|\, \varphi(f) = f\}$ is itself a non-empty, finitely supported complete lattice, supported by $supp(X) \cup supp(\varphi) \cup supp(Y)$.

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