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Article

Scattering of Electromagnetic Waves by Many Nano-Wires

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Abstract: Electromagnetic wave scattering by many parallel to the $z-$axis, thin, impedance, parallel, infinite cylinders is studied asymptotically as $a \to 0$. Let D_m be the cross-section of the $m-$th cylinder, a be its radius and $\hat{x}_m = (x_{m1}, x_{m2})$ be its center, $1 \leq m \leq M$, $M = M(a)$. It is assumed that the points, $\hat{x}_m$, are distributed, so that $\mathcal{N}(\Delta) = \frac{1}{2\pi a} \int_\Delta N(\hat{x}) d\hat{x}[1 + o(1)]$, where $\mathcal{N}(\Delta)$ is the number of points, $\hat{x}_m$, in an arbitrary open subset, Δ, of the plane, xoy. The function, $N(\hat{x}) \geq 0$, is a continuous function, which an experimentalist can choose. An equation for the self-consistent (effective) field is derived as $a \to 0$. A formula is derived for the refraction coefficient in the medium in which many thin impedance cylinders are distributed. These cylinders may model nano-wires embedded in the medium. One can produce a desired refraction coefficient of the new medium by choosing a suitable boundary impedance of the thin cylinders and their distribution law.

Keywords: metamaterials; refraction coefficient; EM wave scattering

Classification: PACS 03.40.Kf; 03.50 De; 41.20.Jb; 71.36.+c

1. Introduction

There is a large literature on electromagnetic (EM) wave scattering by an array of parallel cylinders (see, e.g., [1], where there are many references given, and [2]). Electromagnetic wave scattering by many parallel to the $z-$axis, thin, circular, of radius a, infinite cylinders, on the boundary of which an impedance boundary condition holds, is studied in this paper asymptotically as $a \to 0$. The cylinders are thin in the sense, $ka \ll 1$, where k is the wave number in the exterior of the cylinders.

The novel points in this paper include:

(1) The asymptotically exact, as $a \to 0$, solution of the EM wave scattering problem by one impedance cylinder is given. The solution to the EM wave scattering problem by many thin impedance cylinders is given. The limiting behavior of this solution is found in the limit $a \to 0$ when the number, $M = M(a)$, of the cylinders tends to infinity at a suitable rate. The equation for the limiting (as $a \to 0$) effective (self-consistent) field in the medium obtained by embedding into it many thin impedance cylinders is derived.

(2) This theory is a basis for a method for changing the refraction coefficient in a medium by embedding into this medium many thin impedance cylinders. The thin cylinders model nano-wires embedded in the medium. The basic physical result of this paper is the formula, (43), which shows how the embedded thin cylinders change the refraction coefficient, $n^2(x)$. This formula serves for designing a desired refraction coefficient.

The results of the paper are summarized in three theorems.

The author considered earlier scalar wave scattering by small particles of an arbitrary shape ([3–9]) and EM wave scattering by many thin perfectly conducting cylinders ([10]). The introduction of the impedance boundary condition is of principal importance, because it gives much more flexibility in creating a desired refraction coefficient; see formula (43).

Let D_m, $1 \le m \le M$, be a set of non-intersecting domains on a plane, P, which is the xoy plane. Let $\hat{x}_m \in D_m$, $\hat{x}_m = (x_{m1}, x_{m2})$, be a point inside D_m and C_m be the cylinder with the cross-section D_m, and the axis, parallel to the z-axis, passing through $\hat{x}_m$. We assume that $\hat{x}_m$ is the center of the disc D_m, if D_m is a disc of radius a.

Let us assume that on the boundary of the cylinders, an impedance boundary condition holds; see Equation (5) below. Let $a = 0.5 \text{diam} D_m$. Our basic assumptions are

$$ka \ll 1 \tag{1}$$

where k is the wave number in the region exterior to the union of the cylinders and

$$\mathcal{N}(\Delta) = \frac{1}{2\pi a} \int_{\Delta} N(\hat{x}) d\hat{x} [1 + o(1)], \quad a \to 0 \tag{2}$$

where $\mathcal{N}(\Delta) = \sum_{\hat{x}_m \in \Delta} 1$ is the number of the cylinders in an arbitrary open subset of the plane P, $N(\hat{x}) \ge 0$ is a continuous function, which can be chosen as we wish, and $2\pi a$ is the arc length of a circle of radius, a. The points, $\hat{x}_m$, are distributed in an arbitrarily large, but fixed, bounded domain on the plane P. We denote by Ω the union of domains D_m, by Ω', its complement in P, and by D' the complement of D in P. The complement in $\mathbb{R}^3$ of the union C of the cylinders C_m, we denote by C'.

The EM wave scattering problem consists of finding the solution to Maxwell's equations:

$$\nabla \times E = i\omega\mu H \tag{3}$$

$$\nabla \times H = -i\omega\epsilon E \tag{4}$$

in C', such that:

$$E_t = \zeta [n, H] \text{ on } \partial C \tag{5}$$

where ∂C is the union of the surfaces of the cylinders C_m, E_t is the tangential component of E on the boundary of C, n is the unit normal to ∂C directed out of the cylinders, μ and ϵ are constants in C', ω is the frequency, $k^2 = \omega^2\epsilon\mu$ and k is the wave number. The ζ in Equation (5) is the boundary impedance. The boundary impedance satisfies the physical restriction Re$\zeta \geq 0$ (see [1]).

The usual impedance boundary condition is $E_t = \zeta[n, H_t]$, because we use the unit normal pointing out of C, but $[n, H] = [n, H_t]$ if $H_3 = 0$, as we assume in this paper.

Denote by $n_0^2 = \epsilon\mu$, so $k^2 = \omega^2 n_0^2$. The solution to Equations (3)–(5) must have the following form:

$$E(x) = E_0(x) + v(x), \quad x = (x_1, x_2, x_3) = (x, y, z) = (\hat{x}, z) \tag{6}$$

where $E_0(x)$ is the incident field and v is the scattered field satisfying the radiation condition:

$$\sqrt{r}\left(\frac{\partial v}{\partial r} - ikv\right) = o(1), \quad r = (x_1^2 + x_2^2)^{1/2} \tag{7}$$

We assume that

$$E_0(x) = k^{-1}e^{i\kappa y + ik_3 z}(-k_3 e_2 + \kappa e_3), \quad \kappa^2 + k_3^2 = k^2 \tag{8}$$

$\{e_j\}$, $j = 1, 2, 3$ are the unit vectors along the Cartesian coordinate axes, x, y, z. We consider EM waves with $H_3 := H_z = 0$, that is, E-waves, or TH waves,

$$E = \sum_{j=1}^{3} E_j e_j, \quad H = H_1 e_1 + H_2 e_2 = \frac{\nabla \times E}{i\omega\mu} \tag{9}$$

. One can prove (see Appendix) that the components of E can be expressed by the formulas:

$$E_j = \frac{ik_3}{\kappa^2} U_{x_j} e^{ik_3 z}, \quad j = 1, 2, \quad E_3 = U e^{ik_3 z}, \quad U = \frac{\kappa}{k} u \tag{10}$$

where $u_{x_j} := \frac{\partial u}{\partial x_j}$, $u = u(x, y)$ solves the problem:

$$(\Delta_2 + \kappa^2)u = 0 \text{ in } \Omega' \tag{11}$$

$$(u_n + i\xi u)|_{\partial\Omega} = 0, \quad u_n := \nabla u \cdot n, \quad \xi := \frac{\omega\mu\kappa^2}{\zeta k^2} \tag{12}$$

$$u = e^{i\kappa y} + w \tag{13}$$

and w satisfies the radiation condition, (7).

If $k_3 = 0$, then $E_0 = e_3 e^{iky}$,

$$E_1 = E_2 = 0, \; E_3 = u; \; H_1 = \frac{u_y}{i\omega\mu}, \; H_2 = -\frac{u_x}{i\omega\mu}, \; H_3 = 0; \; \xi = \frac{\omega\mu}{\zeta}; \; \kappa^2 = k^2$$

This is the case of the EM wave incident perpendicular to the axis of the cylinders with the E-field parallel to this axis

If $k_3 \neq 0$, then one takes $\kappa = 0$, $k_3 = k$ and $E_0 = e^{ikz}e_2$. This is the case of the EM wave incident along the axis of the cylinders with the E-field parallel to the y-axis.

If $k_3 \neq 0$, then E_t in the condition (5) does not lead to Equation (12). If E_t is approximately replaced by E_3 in condition (5), then Equation (12) holds.

Our main results come from the analysis of problem, (11)–(13).

The unique solution to Equations (3)–(8) is given by the formulas:

$$E_1 = \frac{ik_3}{\kappa^2} U_x e^{ik_3 z}, \quad E_2 = \frac{ik_3}{\kappa^2} U_y e^{ik_3 z}, \quad E_3 = U e^{ik_3 z} \tag{14}$$

$$H_1 = \frac{k^2}{i\omega\mu\kappa^2} U_y e^{ik_3 z}, \quad H_2 = -\frac{k^2}{i\omega\mu\kappa^2} U_x e^{ik_3 z}, \quad H_3 = 0 \tag{15}$$

where $U_x := \frac{\partial U}{\partial x}$, $U_y = \frac{\partial U}{\partial y}$ and $u = u(\hat{x}) = u(x,y)$ solves the scalar two-dimensional problem, (11)–(13), if $k_3 = 0$. These formulas are derived in the Appendix.

Problem (11)–(13) has a unique solution (see, for example, [11]), provided that $\mathrm{Re}\zeta \geq 0$ or, equivalently, that $\mathrm{Im}\xi \geq 0$. This corresponds to the assumption that the material inside the cylinders is passive, that is, the energy absorption is non-negative. Our goal is to derive an asymptotic formula for this solution as $a \to 0$. Our results include formulas for the solution to the scattering problem, the derivation of the equation for the effective field in the medium obtained by embedding many thin perfectly conducting cylinders and a formula for the refraction coefficient in this limiting medium. This formula shows that by choosing a suitable distribution of the cylinders, one can change the refraction coefficient, for example, one can make it smaller than the original one.

The paper is organized as follows.

In Section 2 we derive an asymptotic formula for the solution to Equations (11)–(13) when $M = 1$, that is, for scattering by one cylinder. The result is formulated in Theorem 1.

In Section 3 we derive a linear algebraic system for finding some numbers that define the solution to problem (11)–(13) with $M > 1$. This gives a numerical method for solving the EM wave scattering problem by many thin impedance cylinders.

Furthermore, in Section 3 we derive an integral equation for the effective (self-consistent) field in the medium with $M(a) \to \infty$ cylinders as $a \to 0$. The result is formulated in Theorem 2.

At the end of Section 3 this result is applied to the problem of changing the refraction coefficient of a given material by embedding many thin perfectly conducting cylinders into it. An analytic formula for the refraction coefficient is derived. The result is formulated in Theorem 3.

In Section 4 conclusions are formulated.

In Appendix formulas (14) and (15) are derived.

2. EM Wave Scattering by One Thin Perfectly Conducting Cylinder

Consider problem (11)–(13) with $\Omega = D_1$, Ω' being the complement to D_1 in $\mathbb{R}^2$. Our theory and the results formulated in Theorems 1, 2 and 3 are valid for the cylinders with an arbitrary cross-section. We assume that the point $\hat{x}_1$ is a point inside D_1.

Let us look for a solution of the form:

$$u(\hat{x}) = e^{i\kappa y} + \int_{S_1} g(\hat{x},t)\sigma(t)dt, \quad g(\hat{x},t) := g(\kappa|\hat{x}-t|) = \frac{i}{4} H_0^{(1)}(\kappa|\hat{x}-t|) \tag{16}$$

where S_1 is the boundary of D_1, $H_0^{(1)}$ is the Hankel function of the order of one, with index, zero, and σ is to be found from the boundary condition, (12). Let $r = |\hat{x}-t|$. It is known that as $r \to 0$, one has:

$$g(\kappa r) = \alpha(\kappa) + \frac{1}{2\pi}\ln\frac{1}{r} + o(1), \quad \alpha := \alpha(\kappa) := \frac{i}{4} + \frac{1}{2\pi}\ln\frac{2}{\kappa} \tag{17}$$

and:

$$g(\kappa r) = \frac{i}{4} H_0^{(1)}(kr) = \frac{i}{4} \sqrt{\frac{2}{\pi k r}} e^{i(kr - \frac{\pi}{4})} \big(1 + o(1)\big), \quad r \to \infty \tag{18}$$

Thus:

$$u = u_0 + g(\hat{x}, 0) Q + o(\frac{1}{\sqrt{r}}), \quad |\hat{x}| = r \to \infty; \quad Q := \int_{S_1} \sigma(t) dt \tag{19}$$

and u_0 here is the same plane wave as in Equation (16).

One sees from formula (19) *that it is sufficient to find one number, Q, in order to solve the scattering problem,* (11)–(13)*, for one thin cylinder.*

This is the central idea of our work. Below, we derive an explicit analytical formula for the number, Q. This formula (formula (24)) is asymptotically exact.

Let us explain how to derive formula (19). The asymptotics of the integral term in Equation (16) is:

$$g(\hat{x}, 0) \int_{S_1} e^{-i\kappa \beta \cdot t} \sigma(t) dt, \quad \beta := \hat{x}/|\hat{x}|$$

where $0 \in D_1$ is the origin. Since $\kappa a \ll 1$, it follows that:

$$e^{-i\kappa\beta\cdot t} = 1 + O(a) \quad a \to 0$$

The remainder, $o(\frac{1}{\sqrt{r}})$, in Equation (19) comes from the radiation condition for the function, g. When one studies the scattering problem by a single thin cylinder, one can take an arbitrary point inside D_1 as an origin.

In the usual approach to the scattering problem, one has to find an unknown function, $\sigma(t)$, rather than one number, Q, in order to solve the scattering problem, (11)–(13). The function, (16), satisfies Equations (11) and (13) for any σ, and if σ is such that function (16) satisfies boundary condition (12), then u solves problem, (11)–(13). We assume σ to be sufficiently smooth (Hölder-continuous is sufficient).

The solution to problem (11)–(13) is known to be unique (see, e.g., [11]). The exact boundary condition (12) yields:

$$-u_{0n}(s) - i\xi u_0 = i\xi\alpha Q + i\xi \int_{S_1} g_0(s,t)\sigma(t)dt + (A\sigma - \sigma)/2 \tag{20}$$

where $A\sigma := \int_{S_1} \frac{\partial g_0(s,t)}{\partial n_s} dt$, and the formula for the limiting value on S_1 of the exterior normal derivative of the simple layer potential, $\int_{S_1} g_0(x,t)\sigma(t)dt$, was used and

$$u_0(s) := e^{i\kappa s_2}, \quad s \in S_1; \quad g_0(s,t) := \frac{1}{2\pi} \ln \frac{1}{r_{st}}, \quad r_{st} := |s - t| \tag{21}$$

If $ka \ll 1$ and $k^2 = \kappa^2 + k_3^2$, then

$$u_0(s) = 1 + O(\kappa a), \quad u_{0n} = i\kappa n_2 + O(\kappa a) \tag{22}$$

Equation (20) is uniquely solvable for σ if a is sufficiently small (see [11,12]).

We are interested in finding the asymptotic formula for Q as $a \to 0$, because $u(\hat{x})$ in Equation (16) can be well approximated in the region, $|\hat{x}| \gg a$, by the formula,

$$u(\hat{x}) = u_0(\hat{x}) + g(\hat{x}, 0)Q + o(1), \quad a \to 0 \tag{23}$$

. To find the asymptotic of Q as $a \to 0$, let us integrate Equation (20) over S_1; keep the main terms of the asymptotic as $a \to 0$, take into account that

$$\int_{S_1} dt n_2(t) = 0, \quad \int_{S_1} g_0(s,t)ds = O(a|\log a|) \qquad a \to 0$$

use formulas (22) and obtain

$$Q = i\xi u_0(\hat{x}_1)|S_1|(1+o(1)), \quad a \to 0 \tag{24}$$

where $\hat{x}_1$ is a point inside D_1, $|S_1|$ is the length of S_1 and $r_{st} = |s-t|$. If S_1 is the circle of radius, a, then $|S_1| = 2\pi a$.

If S_1 is an arbitrary curve, we assume that the length $|S_1| = ca$, where a is the same as in Equation (1), and $c > 0$ is a constant. The reader can check the estimate, $\int_{S_1} g_0(s,t)ds = O(a|\log a|)$, as $a \to 0$, where $s, t \in S_1$. From formulas (24) and (19) the asymptotic solution to the scattering problem (11)–(13) in the case of one circular cylinder of radius a, as $a \to 0$, is

$$u(\hat{x}) \sim u_0(\hat{x}) + i2\pi a\xi u_0(\hat{x}_1)g(\hat{x}, \hat{x}_1), \quad a \to 0, \quad |\hat{x} - \hat{x}_1| > a \tag{25}$$

Here, the sign, $\sim$, stands for the asymptotic equivalence as $a \to 0$, the point; $\hat{x}$ is an arbitrary point on the plane that cannot be at distances less than a from the point $\hat{x}_1 \in D_1$.

Let us formulate the result.

Theorem 1. *Electromagnetic wave, scattered by a single cylinder, is calculated by formulas* (14) *and* (15) *in which* $u = u(\hat{x}) := u(x_1, x_2)$ *is given by formula* (25).

If the cylinder is not circular, then formula (25) takes the form:

$$u(\hat{x}) \sim u_0(\hat{x}) + i|S_1|\xi u_0(\hat{x}_1)g(\hat{x}, \hat{x}_1), \quad a \to 0, \quad |\hat{x} - \hat{x}_1| > a \tag{26}$$

where $|S_1|$ is the length of S_1.

3. Wave Scattering by Many Thin Cylinders

Problem (11)–(13) should be solved when Ω is a union of many small domains, D_m, $\Omega = \cup_{m=1}^M D_m$. We assume that D_m is a circle of radius a centered at the point, $\hat{x}_m$.

Let us look for u of the form

$$u(\hat{x}) = u_0(\hat{x}) + \sum_{m=1}^M \int_{S_m} g(\hat{x}, t)\sigma_m(t)dt \tag{27}$$

We assume that the points, $\hat{x}_m$, are distributed in a bounded domain D on the plane $P = xoy$ by formula (2). The field $u_0(\hat{x})$ is the same as in Section 2, $u_0(\hat{x}) = e^{i\kappa y}$, and Green's function, g, is the same as in formulas (16)–(18). It follows from Equation (2) that $M = M(a) = O(\ln \frac{1}{a})$. We define the effective field, acting on the D_j by the formula

$$u_e = u_e^{(j)} = u(\hat{x}) - \int_{S_j} g(\hat{x}, t)\sigma_j(t)dt, \quad |\hat{x} - \hat{x}_j| > a \tag{28}$$

which can also be written as

$$u_e(\hat{x}) = u_0(\hat{x}) + \sum_{m=1, m\neq j}^{M} \int_{S_m} g(\hat{x}, t)\sigma_m(t)dt$$

We assume that the distance $d = d(a)$ between neighboring cylinders is much greater than a:

$$d \gg a, \quad \lim_{a\to 0} \frac{a}{d(a)} = 0 \tag{29}$$

Let us rewrite Equation (27) as

$$u = u_0 + \sum_{m=1}^{M} g(\hat{x}, \hat{x}_m)Q_m + \sum_{m=1}^{M} \int_{S_m} [g(\hat{x}, t) - g(\hat{x}, \hat{x}_m)]\sigma_m(t)dt \tag{30}$$

where

$$Q_m := \int_{S_m} \sigma_m(t)dt \tag{31}$$

As $a \to 0$, the second sum in Equation (30) (let us denote it Σ_2) is negligible compared with the first sum in Equation (30), denoted Σ_1:

$$|\Sigma_2| \ll |\Sigma_1|, \quad a \to 0 \tag{32}$$

The idea of the proof of this is similar to the one given in [13] for a quite different problem of scalar wave scattering in $\mathbb{R}^3$. Let us sketch this proof.

Let us check that

$$|g(\hat{x}, \hat{x}_m)Q_m| \gg |\int_{S_m} [g(\hat{x}, t) - g(\hat{x}, x_m)]\sigma_m(t)dt|, \quad a \to 0 \tag{33}$$

If $k|\hat{x} - \hat{x}_m| \gg 1$, and $k > 0$ is fixed, then

$$|g(\hat{x}, \hat{x}_m)| = O(\frac{1}{|\hat{x} - \hat{x}_m|^{1/2}}), \qquad |g(\hat{x}, t) - g(\hat{x}, x_m)| = O(\frac{a}{|\hat{x} - \hat{x}_m|^{1/2}})$$

and $Q_m \neq 0$; so, estimate Equation (33) holds.

If

$$|\hat{x} - \hat{x}_m| \sim d \gg a$$

then

$$|g(\hat{x}, \hat{x}_m)| = O(\frac{1}{\ln \frac{1}{a}}), \qquad |g(\hat{x}, t) - g(\hat{x}, x_m)| = O(\frac{a}{d})$$

as follows from the asymptotic of $H_0^1(r) = O(\ln \frac{1}{r})$ as $r \to 0$ and from the formulas $\frac{dH_0^1(r)}{dr} = -H_1^1(r) = O(\frac{1}{r})$ as $r \to 0$. Thus, Equation (33) holds for $|\hat{x} - \hat{x}_m| \gg d \gg a$.

Consequently, the scattering problem is reduced to finding the numbers Q_m, $1 \leq m \leq M$.

Let us estimate Q_m asymptotically as $a \to 0$. To do this, we use the exact boundary condition on S_m and an argument similar to the one given in the case of wave scattering by one cylinder. The role of the incident field, u_0, is played now by the effective field, u_e. The result is a formula, similar to Equation (24):

$$Q_j = i2\pi a\xi_j u_e(\hat{x}_j), \quad a \to 0 \tag{34}$$

The formula, similar to Equation (25), is

$$u(\hat{x}) \sim u_0(\hat{x}) + i2\pi a \sum_{m=1}^{M} g(\hat{x}, \hat{x}_m)\xi_m u_e(\hat{x}_m), \quad a \to 0 \tag{35}$$

The numbers, $u_e(\hat{x}_m)$, $1 \le m \le M$, in Equation (35) are not known. Setting $\hat{x} = \hat{x}_j$ in Equation (35), neglecting the $o(1)$ term and using the definition (28) of the effective field, one gets a linear algebraic system for finding numbers $u_e(\hat{x}_m)$:

$$u_e(\hat{x}_j) = u_0(\hat{x}_j) + i2\pi a \sum_{m \ne j} g(\hat{x}_j, \hat{x}_m)\xi_m u_e(\hat{x}_m), \quad 1 \le j \le M \tag{36}$$

This system can be solved numerically if the number M is not very large, say $M \le O(10^3)$.

If M is very large, $M = M(a) \to \infty$, $a \to 0$, then we derive a linear integral equation for the limiting effective field in the medium obtained by embedding many cylinders.

Passing to the limit $a \to 0$ in system Equation (36) is done by the method used in [9] in the problem of wave scattering by many small bodies. Consider a partition of the domain D into a union of $\mathbf{P}$ small squares Δ_p of size $b = b(a)$, $b \gg d \gg a$. For example, one may choose $b = O(a^{1/4})$, $d = O(a^{1/2})$, so that there are many discs D_m in the square Δ_p. We assume that squares Δ_p and Δ_q do not have common interior points if $p \ne q$. Let $\hat{y}_p$ be the center of Δ_p. One can also choose as $\hat{y}_p$ any point $\hat{x}_m$ in a domain $D_m \subset \Delta_p$. Since u_e is a continuous function, one may approximate $u_e(\hat{x}_m)$ by $u_e(\hat{y}_p)$, provided that $\hat{x}_m \subset \Delta_p$. The error of this approximation is $o(1)$ as $a \to 0$. Let $\xi(\hat{x})$ be a continuous function in D, such that $\xi(\hat{x}_m) = \xi_m$. Let us rewrite the sum in Equation (36) as follows:

$$2\pi a \sum_{m \ne j} g(\hat{x}_j, \hat{x}_m)\xi_m u_e(\hat{x}_m) = \sum_{\substack{p=1 \\ x_j \notin \Delta_p}}^{\mathbf{P}} g(\hat{x}_j, \hat{y}_p)\xi(\hat{y}_p)u_e(\hat{y}_p)2\pi a \sum_{x_m \in \Delta_p} 1 \tag{37}$$

and use formula (2) in the form

$$2\pi a \sum_{x_m \in \Delta_p} 1 = N(\hat{y}_p)|\Delta_p|[1 + o(1)], \quad a \to 0 \tag{38}$$

Here, $|\Delta_p|$ is the area of the square Δ_p.

From Equations (37) and (38), one obtains:

$$2\pi a \sum_{m \ne j} g(\hat{x}_j, \hat{x}_m)\xi_m u_e(\hat{x}_m) = \sum_{\substack{p=1 \\ \hat{x}_j \notin \Delta_p}}^{\mathbf{P}} g(\hat{x}_j, \hat{y}_p)N(\hat{y}_p)\xi(\hat{y}_p)u_e(\hat{y}_p)|\Delta_p|[1 + o(1)] \tag{39}$$

The sum in the right-hand side of formula (39) is the Riemannian sum for the integral,

$$\lim_{a \to 0} \sum_{p=1}^{\mathbf{P}} g(\hat{x}_j, \hat{y}_p)N(\hat{y}_p)\xi(\hat{y}_p)u_e(\hat{y}_p)|\Delta_p| = \int_D g(\hat{x}, \hat{y})N(\hat{y})\xi(\hat{y})u(\hat{y})dy \tag{40}$$

where $u(\hat{x}) = \lim_{a \to 0} u_e(\hat{x})$. Therefore, system (36) in the limit, $a \to 0$, yields the integral equation for the limiting effective field,

$$u(\hat{x}) = u_0(\hat{x}) + i\xi \int_D g(\hat{x}, \hat{y})N(\hat{y})\xi(\hat{y})u(\hat{y})d\hat{y} \tag{41}$$

Note that the function, $\xi(\hat{y})$, can be chosen by the experimenter as he wishes. This is an advantage of having impedance cylinders with the impedance that can be chosen as one wishes.

Let us formulate this result.

Theorem 2. *The effective field in the limiting medium satisfies Equation* (41).

One obtains system (36) if one solves Equation (41) by a collocation method. Convergence of this method to the unique solution of Equation (41) is proved in [6]. Existence and uniqueness of the solution to Equation (41) are proved as in [13], where a three-dimensional analog of this equation was studied.

One has $(\Delta_2 + \kappa^2)g(\hat{x}, \hat{y}) = -\delta(\hat{x} - \hat{y})$. Using this relation and applying the operator $\Delta_2 + \kappa^2$ to Equation (41) yields the following differential equation for $u(\hat{x})$:

$$\Delta_2 u(\hat{x}) + \kappa^2 u(\hat{x}) + i\xi(\hat{x})N(\hat{x})u(\hat{x}) = 0 \quad \hat{x} \in \mathbb{R}^2 \tag{42}$$

This is a Schrödinger-type equation, and $u(\hat{x})$ is its scattering solution corresponding to the incident wave, $u_0 = e^{i\kappa y}$.

Let us assume that $N(\hat{x}) = N$ and $\xi(\hat{x})$ are constants. One concludes from Equation (42) that the limiting medium, obtained by embedding many perfectly conducting circular cylinders, has new parameter, $\kappa_N^2 := \kappa^2 + i\xi N$. This means that $k^2 = \kappa^2 + k_3^2$ is replaced by $\tilde{k}^2 := k^2 + i\xi N$. The quantity, k_3^2, is not changed. One has $\tilde{k}^2 = \omega^2 n^2$, $k^2 = \omega^2 n_0^2$. Consequently, $n^2/n_0^2 = (k^2 + i\xi N)/k^2$. Therefore, the new refraction coefficient, n^2, is

$$n^2 = n_0^2(1 + i\xi N k^{-2}), \quad \xi = \frac{\omega\mu\kappa^2}{\zeta k^2} \tag{43}$$

Let us formulate this result.

Theorem 3. *If $N(\hat{x}) = N$ and $\xi(\hat{x}) = \xi$ are constants in D, then the refraction coefficient in the limiting medium is given by formula* (43).

Since the number, $N > 0$, and the impedance, ζ, are at our disposal, Equation (43) shows that, choosing suitable N, one can create a medium with a desired refraction coefficient. It is of interest that if $\xi(\hat{x})$ is not a constant, one can create a refraction coefficient depending on $\hat{x}$ in a desired way and having a desired absorption as a function of the position, $\hat{x}$.

In practice, one does not go to the limit, $a \to 0$, but chooses a sufficiently small a. As a result, one obtains a medium with a refraction coefficient n_a^2, which differs from Equation (43) a little, $\lim_{a\to 0} n_a^2 = n^2$.

4. Conclusions

Asymptotic, as $a \to 0$, solution is given for the EM wave scattering problem by many perfectly conducting parallel cylinders of radius, a. The equation for the effective field in the limiting medium obtained when $a \to 0$ and the distribution of the embedded cylinders is given by formula (2). The presented theory gives formula (43) for the refraction coefficient in the limiting medium. This formula shows how the distribution of the cylinders influences the refraction coefficient.

References

1. Martin, P. *Multiple Scattering*; Cambridge University Press: Cambridge, UK, 2006.
2. Mei, C.; Vernescu, B. *Homogenization Methods for Multiscale Mechanics*; World Science: New Jersey, NJ, USA, 2010.
3. Ramm, A.G. Distribution of particles which produces a "smart" material. *J. Stat. Phys.* **2007**, *127*, 915–934.
4. Ramm, A.G. Electromagnetic wave scattering by small bodies. *Phys. Lett. A* **2008**, *372*, 4298–4306.
5. Ramm, A.G. Wave scattering by many small particles embedded in a medium. *Phys. Lett. A* **2008**, *372*, 3064–3070.
6. Ramm, A.G. A collocation method for solving integral equations. *Int. J. Comput. Sci. Math.* **2009**, *3*, 222–228.
7. Ramm, A.G. A method for creating materials with a desired refraction coefficient. *Int. J. Mod. Phys. B* **2010**, *24*, 5261–5268.
8. Ramm, A.G. Materials with a desired refraction coefficient can be created by embedding small particles into the given material. *Int. J. Struct. Chang. Solids* **2010**, *2*, 17–23.
9. Ramm, A.G. Wave scattering by many small bodies and creating materials with a desired refraction coefficient. *Afrika Mat.* **2011**, *22*, 33–55.
10. Ramm, A.G. Scattering of electromagnetic waves by many thin cylinders. *Results Phys.* **2011**, *1*, 13–16.
11. Ramm, A.G. *Scattering by Obstacles*; D.Reidel: Dordrecht, The Netherlands, 1986.
12. Ramm, A.G. *Wave Scattering by Small Bodies of Arbitrary Shapes*; World Science Publishers: Singapore, Singapore, 2005.
13. Ramm, A.G. Many-body wave scattering by small bodies and applications. *J. Math. Phys.* **2007**, *48*, 103511.
14. Landau, L.; Lifshitz, E. *Electrodynamics of Continuous Media*; Pergamon Press: London, UK, 1984.

Appendix

Let us derive formulas, (14) and (15). Look for the solution to Equations (3) and (4) of the form:

$$E_1 = e^{ik_3z}\tilde{E}_1(x,y), \quad E_2 = e^{ik_3z}\tilde{E}_2(x,y), \quad E_3 = e^{ik_3z}u(x,y) \tag{44}$$

$$H_1 = e^{ik_3z}\tilde{H}_1(x,y), \quad H_2 = e^{ik_3z}\tilde{H}_2(x,y), \quad H_3 = 0 \tag{45}$$

where $k_3 = const$. Equation (3) yields

$$u_y - ik_3\tilde{E}_2 = i\omega\mu\tilde{H}_1, \quad -u_x + ik_3\tilde{E}_1 = i\omega\mu\tilde{H}_2, \quad \tilde{E}_{2,x} = \tilde{E}_{1,y} \tag{46}$$

where, for example, $\tilde{E}_{j,x} := \frac{\partial \tilde{E}_j}{\partial x}$. Equation (4) yields

$$ik_3\tilde{H}_2 = i\omega\epsilon\tilde{E}_1, \quad ik_3\tilde{H}_1 = -i\omega\epsilon\tilde{E}_2, \quad \tilde{H}_{2,x} - \tilde{H}_{1,y} = -i\omega\epsilon u \tag{47}$$

Excluding $\tilde{H}_j$, $j = 1, 2$, from Equation (46) and using Equation (47), one gets

$$\tilde{E}_1 = \frac{ik_3}{\kappa^2} u_x, \quad \tilde{E}_2 = \frac{ik_3}{\kappa^2} u_y, \quad \tilde{E}_3 = u \tag{48}$$

$$\tilde{H}_1 = \frac{k^2 u_y}{i\omega\mu\kappa^2}, \quad \tilde{H}_2 = -\frac{k^2 u_x}{i\omega\mu\kappa^2}, \quad \tilde{H}_3 = 0 \tag{49}$$

Since $E_j = \tilde{E}_j e^{ik_3 z}$ and $H_j = \tilde{H}_j e^{ik_3 z}$, formulas (14) and (15) follow immediately from Equations (48) and (49).