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Quantum Effects in General Relativity: Investigating Repulsive Gravity of Black Holes at Large Distances

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Abstract: This paper proposes a theoretical study that investigates quantum effects on the gravity of black holes. This study utilizes a gravitational model that incorporates quantum mechanics derived from the classical-like quantum hydrodynamic representation. This research calculates the mass density distribution of quantum black holes, specifically in the case of central symmetry. The gravity of a quantum black hole shows contributions coming from quantum potential energy, which is also sensitive to the presence of a background of gravitational noise. The additional energy, stored in quantum potential fluctuations and constituting a form of dark energy, leads to a repulsive gravity in the weak gravity limit. This repulsive gravity overcomes the attractive classical Newtonian force at large distances of order of the intergalactic length.

Keywords: repulsive gravity; quantum black hole; dark energy; quantum mechanical general relativity

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1. Introduction

One of the biggest challenges in physics today is the unification of general relativity and quantum theory. General relativity provides a description of gravity as the curvature of spacetime caused by the presence of matter and energy, while quantum theory describes the behavior of matter and energy at the smallest scales.

The problem is that these two theories are fundamentally different in their approaches, and attempts to merge them have been unsuccessful so far. One of the key challenges is the existence of singularities, such as those found in black holes, which arise from the application of general relativity at very small scales.

Another issue is the conflict between the principles of quantum mechanics and those of general relativity, such as the principle of locality, which states that information cannot travel faster than the speed of light, and the principle of unitarity, which requires that the total probability of all possible outcomes of an experiment add up to one.

There have been various proposals for reconciling general relativity and quantum theory, such as string theory [1–3], loop quantum gravity [4–6], and causal dynamical triangulation [7–9], but none have yet been confirmed by experimental evidence, and the search for a unified theory remains an active area of research in theoretical physics.

Since the discovery of the universe's accelerated expansion, scientists have been trying to determine what is driving this acceleration. However, despite many attempts to explain it, the current observational data cannot conclusively identify a source.

In 1926, Schrödinger presented his findings on wave mechanics [1,2] by formulating a linear differential equation for a complex wave function $\psi(r,\ t) = \sqrt{n(r,\ t)}exp\ \frac{iS(r,\ t)}{\hbar}$. However, in the same year, Madelung [3] discovered an alternative formulation that consisted of two real equations, which exhibited striking similarities to equations commonly used in hydrodynamics. The first equation was a continuity equation for the squared amplitude, representing the probability density $n(\mathrm{r,\ t}) = \psi * (\mathrm{r,\ t})\psi(\mathrm{r,\ t})$, where the phase is contributed through a velocity field in a convection current. The second equation was a modified version of the Hamilton–Jacobi equation, describing the phase $S(\vec{r}, t)$ and including a contribution from the amplitude through a "quantum potential" denoted as V_{qu}. This quantum potential was seen as a characteristic quantum mechanical contribution, serving as a coupling between the two equations for amplitude and phase. Madelung regarded V_{qu} as the source of "internal forces" within the continuum.

Recently, the author demonstrated [10,11] that, by assuming the covariance of quantum field equations and utilizing their classical-like Madelung quantum hydrodynamic representation, it is possible to define the geometry of space–time through a gravity equation that incorporates quantum mechanics. This is achieved through the use of the generalized least action principle, resulting in a system of equations that describes the quantum gravitational evolution. This system couples the gravity equation with the field of quantum equation for boson or fermion fields [10,11].

The theoretical study proposed in this paper is centered on the quantum mechanical state of black holes and the resulting gravity in spacetime, where a background of gravitational noise is present. The findings of this study have the potential to confirm observational evidence, such as the existence of dark energy and the repulsive nature of gravity at large distances.

The organization of this work is as follows: Section 2.1 presents the generalized Madelung quantum hydrodynamic approach in curved spacetime, specifically addressing the relativistic case of the Klein–Gordon equation. This approach is coupled with the gravity equation derived from the covariance condition and the least action principle, which has been previously reformulated for quantum hydrodynamic formalism. In Section 2.2, the stationary mass density configuration is derived for a central-symmetric black hole mass distribution at large distances. Section 3 provides further analysis. Firstly, the gravitational field is derived for the non-punctual mass distribution of a quantum Schwarzschild black hole at large distances. Subsequently, Sections 3.1 and 3.2 calculate the characteristics of the quantum potential fluctuations of the black hole originating from the noise of a gravitational background. In Sections 3.3 and 3.4, respectively, the mean (dark) energy density of the quantum potential fluctuations of the black hole is derived, and the distance at which the repulsive gravitational force surpasses the attractive Newtonian force is determined.

2. Cosmological Scalar Boson Mass under Self-Gravity

During the collapse of a black hole, its mass distribution becomes highly concentrated, but the repulsive force of its quantum potential may become strong enough to counteract the gravitational force and prevent its collapse. This can result in the formation of stationary mass distributions. The uncertainty principle ensures that the repulsive quantum non-local potential grows sufficiently to overcome the gravitational force, thereby preventing a point-like collapse.

When the mass distribution of a scalar uncharged boson becomes extremely concentrated in space, its gravitational force can generate stable self-bonded states. These states are the quantum mechanical analogue of a black hole predicted by general relativity.

In this section, the author investigates whether quantum potential force can stop gravitational collapse when the mass distribution approaches the classical point singularity.

In order to obtain quantum mechanical stationary black hole configurations on a cosmological scale with large mass distributions, we make the assumption that the mass field can be represented as a scalar variable. This simplified model of a scalar black hole mass serves as a "macroscopic" viewpoint that is acceptable for studying the gravitational behavior of black holes on a cosmological scale.

The distribution of mass in space–time (ST) is attributed to the formation of vacuum states resulting from the quantization of spinor and massive boson fields. This mass distribution in ST is not only non-continuous, but also exhibits physical phenomena arising from the other three fundamental interactions that remain out of the description.

2.1. Stationary Scalar Mass Distribution

In the case of a scalar mass field obeying the Klein–Gordon equation, the gravitationally coupled system of motion equation reads [11]

$$R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R_{\alpha}{}^{\alpha} = \frac{8\pi G}{c^4} T_{\nu\mu} \tag{1}$$

$$\partial^{\mu} \psi_{;\mu} = \frac{1}{\sqrt{-g}} \partial^{\mu} \sqrt{-g} (g^{\mu\nu} \partial_{\nu} \psi) = -\frac{m^2 c^2}{\hbar^2} \psi \tag{2}$$

The quantum contribution in (1) is contained in the energy impulse tensor that reads [11]

$$T_{\mu\nu} = \left(T_{(k)\mu\nu_{class}} \left(1 - a_{(V_{qu(k)})} \right) - \Lambda_{Q(k)} \right) \tag{3}$$

where [11]

$$T_{(k)\mu\nu_{class}} = -\frac{\hbar^2}{m\gamma} \left(1 - \frac{V_{qu}}{mc^2} \right)^{-1} \left(\partial_{\mu} ln \frac{\psi}{\psi*} \right) \left(\partial_{\nu} ln \frac{\psi}{\psi*} \right) \tag{4}$$

$$a_{(V_{qu(k)})} = \left(1 - \sqrt{1 - \frac{V_{qu(k)}}{mc^2}} \right) \tag{5}$$

$$\Lambda_{Q(k)} = -a_{(V_{qu(k)})} |\psi_k|^2 \frac{mc^2}{\gamma_{(k)}} \frac{\Delta_{\lambda\lambda}}{4} \tag{6}$$

where $\Delta_{\lambda\lambda}$ is given in [11],

$$\gamma = \frac{1}{\sqrt{\frac{g_{\mu\nu}\dot{q}^{\nu}\dot{q}^{\mu}}{c^2}}} = \frac{1}{c\sqrt{\partial_0 S \; g_{\mu\nu}\partial^{\nu} S \partial^{\mu} S}} = \frac{2}{\hbar^{3/2} c \sqrt{i\partial_0 ln[\frac{\psi}{\psi*}] \; g_{\mu\nu}\partial^{\nu} ln[\frac{\psi}{\psi*}] \partial^{\mu} ln[\frac{\psi}{\psi*}]}} \tag{7}$$

and where the quantum potential reads

$$V_{qu} = -\frac{\hbar^2}{m} \frac{1}{|\psi|\sqrt{-g}} \partial_{\mu} \sqrt{-g} (g^{\mu\nu} \partial_{\nu} |\psi|) \tag{8}$$

The KGE (2) in hydrodynamic notation, as a function of the real variables $|\psi|^2$ and S, with momentum $\partial_{\mu} S = -p_{\mu}$, leads to a couple of real variable equations,

$$\begin{aligned} \tfrac{1}{m} g_{\mu\nu} \partial^{\nu} S \, \partial^{\mu} S = \tfrac{\hbar^2}{m} \tfrac{1}{|\psi|\sqrt{-g}} \partial_{\mu} \sqrt{-g} (g^{\mu\nu} \partial_{\nu} |\psi|) + mc^2 \\ = mc^2 - V_{qu} \end{aligned} \tag{9}$$

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} (g^{\mu\nu} |\psi|_2 \partial_{\nu} S) = 0 \tag{10}$$

Following the same method used by Landau e Lifšits in general relativity [12] from the generalized Hamilton–Jacobi Equation (9), it is possible to derive (see Appendix A) the motion equation in curved spacetime [11] that reads (see Equation (A6) in Appendix A)

$$\begin{aligned} & D_{q_0} \left(u_{(k)_{\mu}} \sqrt{1 - \tfrac{V_{qu(k)}}{mc^2}} \right) \\ & = \sqrt{1 - \tfrac{V_{qu(k)}}{mc^2}} D_{q_0} u_{(k)_{\mu}} + u_{(k)_{\mu}} D_{q_0} \sqrt{1 - \tfrac{V_{qu(k)}}{mc^2}} = \tfrac{1}{\gamma_{(k)}} \tfrac{\partial}{\partial q^{\mu}} \sqrt{1 - \tfrac{V_{qu(k)}}{mc^2}} \end{aligned} \tag{11}$$

where $u_{(k)_{\mu}} = \frac{\gamma_{(k)} \dot{q}_{(k)_{\mu}}}{c} = \frac{p_{(k)_{\mu}}}{mc\sqrt{1 - \frac{V_{qu(k)}}{mc^2}}} = -\frac{\partial_{\mu} S_{(k)}}{mc\sqrt{1 - \frac{V_{qu(k)}}{mc^2}}}$ (see Appendix A) and cD_{q_0} is the curvilinear covariant total time derivative. Moreover, by utilizing the relation $D_{q_0}u_\mu = \frac{du_\mu}{dq_0} - \frac{1}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu$, it follows that

$$\begin{aligned}\frac{du_\mu}{dt} - \frac{c}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu &= -u_\mu \frac{d}{dt}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) + \frac{1}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \\ &= -u_\mu \frac{d}{dt}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2 \frac{1}{|\psi|\sqrt{-g}}\partial_\mu\sqrt{-g}(g^{\mu\nu}\partial_\nu|\psi|)}\right) \\ &+ \frac{c}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2 \frac{1}{|\psi|\sqrt{-g}}\partial_\mu\sqrt{-g}(g^{\mu\nu}\partial_\nu|\psi|)}\right)\end{aligned} \tag{12}$$

Since we are interested in the stationary mass density distribution of a black hole, we have to impose the stationary condition $\frac{du_\mu}{dt} = 0$, which results in

$$\frac{c}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu = u_\mu \frac{d}{dt}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) + \frac{1}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \tag{13}$$

where the force generated by the gravitational force $\frac{c}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu$ is balanced by the quantum potential force

$$u_\mu \frac{d}{dt}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) + \frac{1}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \tag{14}$$

In the classical case where $\hbar = 0$, and thence $V_{qu} = 0$, the counterbalancing expansive quantum force is null, and thus, the collapse proceeds with the generation of point singularity.

2.2. The Mass Distribution in a Central Symmetric Scalar Uncharged Black Hole

In classical general relativity, the collapse of a central gravitational field results in a final point-like mass density being approached with increasing velocity. However, in the quantum case, the quantum potential generates an expansive force

$$\frac{\partial}{\partial q^\mu}\ln\sqrt{1-\frac{V_{qu}}{mc^2}}, \tag{15}$$

that counteracts gravity, leading to deceleration and potentially halting the collapse. As a result, stable stationary configurations may exist at an equilibrium point, eliminating the classical point singularity. This suggests that the interplay between quantum effects and gravity can lead to different outcomes than those predicted by classical general relativity.

From a general standpoint, the stationary mass distribution, as described by Equation (12), depends on the metric tensor defined by the quantum Einstein gravity (QGE) Equation (1) and vice versa. Although the general solution of these coupled equations is quite complex, the simplifying assumption of central symmetry can be introduced to extract useful information. This assumption leads to the quantum analogue of the Schwarzschild black hole, where the metric tensor satisfies a particular condition [12].

$$ds^2 = e^\nu c^2 dt^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^\lambda dr^2 \tag{16}$$

where $q_\mu = (ct, r, \theta, \phi)$ and

$$g_{00} = e^\nu; g_{11} = -e^\lambda; g_{22} = -r^2; g_{33} = -r^2\sin^2\theta;\ \sqrt{-g} = |e^{\frac{\lambda+\nu}{2}}r^2\sin^2\theta|^{-1}; \tag{17}$$

which, inserted into the gravity equation, leads to the relations [12]

$$\frac{8\pi G}{c^4}T_1{}^1 = -e^{-\lambda}\left(\frac{\nu\prime}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{18}$$

$$\frac{8\pi G}{c^4}T_0{}^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda\prime}{r}\right) + \frac{1}{r^2} \tag{19}$$

$$\frac{8\pi G}{c^4}T_0{}^1 = -e^{-\lambda}\frac{\dot{\lambda}}{r} \tag{20}$$

with

$$\gamma = \frac{1}{\sqrt{\frac{g_{\mu\nu}\dot{q}^\nu\dot{q}^\mu}{c^2}}} \tag{21}$$

and

$$V_{qu} = -\frac{\hbar^2}{m}\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right) \tag{22}$$

where the apex and the dot over the letters indicate derivation with respect to r and ct, respectively.

Assuming that, in stationary distributions, the mass is enclosed in a sphere of the radius R_0, for $r > R_0$, we can use the approximated gravitational relations [12]

$$-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \cong 0 \tag{23}$$

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \cong 0 \tag{24}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r} \cong 0 \tag{25}$$

whose solutions read

$$\lambda + \nu = 0 \tag{26}$$

$$lim_{\frac{r}{R_g}\to\infty}g_{11} = im_{\frac{r}{R_g}\to\infty} - e^{\lambda} = im_{\frac{r}{R_g}\to\infty} - e^{-\nu} \cong -\left(1 - \frac{R_g}{r}\right)^{-1} \tag{27}$$

$$g = -\frac{1}{r^4\sin^4\theta} \tag{28}$$

$$e^{\lambda} = \frac{r}{r - R_g} = \frac{1}{g_{00}} \tag{29}$$

from which the quantum potential reads [13]

$$V_{qu} = -\frac{\hbar^2}{m}\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right) \tag{30}$$

By introducing the relations (26)–(30) into the motion equation, it follows that

$$\begin{aligned}
\frac{du_\mu}{cdt} - \frac{1}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu &= -u_\mu\frac{1}{c}\frac{d}{dt}\left(\ln\sqrt{1 + \left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\
&+\frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1 + \left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\
&= -u_\mu\frac{1}{c}\frac{\partial}{\partial q^\nu}\left(\ln\sqrt{1 + \left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\dot{q}^\nu\\
&+\frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1 + \left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\
&= 0 + \frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1 + \left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)
\end{aligned} \tag{31}$$

and, by the stationary condition in the BH system of reference at a large distance,

$$u_\mu = \lim_{\frac{r}{R_g}\to\infty}(\gamma, 0, 0, 0) = \lim_{\frac{r}{R_g}\to\infty}\left(\frac{1}{\sqrt{g^{00}}}, 0, 0, 0\right) = (1, 0, 0, 0) \tag{32}$$

and

$$u^\mu = \lim_{\frac{r}{R_g}\to\infty}\left(g^{00}\gamma, 0, 0, 0\right) = \lim_{\frac{r}{R_g}\to\infty}\left(\sqrt{g^{00}}, 0, 0, 0\right) \cong (1, 0, 0, 0) \tag{33}$$

We obtain

$$\begin{aligned}\Gamma^\alpha_{\mu\nu} u_\alpha u^\nu = \Gamma^0_{10} u_0 u^0 &= \tfrac{1}{2} u_0 u^0 g^{00} \partial_1 g_{00} = u^0 u^0 \partial_1 g_{00} \cong \partial_1 g_{00} \\ &= -\partial_1\left(\ln\left(1 + \left(\tfrac{\hbar}{mc}\right)^2 \tfrac{1}{|\psi|\sqrt{-g}} \partial^1 \sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)\right)\right)\end{aligned} \tag{34}$$

$$\begin{aligned}-\partial_1 \tfrac{r-R_g}{r} &= -\partial_1\left(\ln\left(1 - \left(\tfrac{\hbar}{mc}\right)^2 \tfrac{1}{|\psi|\sqrt{-g}} \partial^1 \sqrt{-g}\left(\tfrac{r-R_g}{r}\partial_1|\psi|\right)\right)\right) \\ \partial_1 \tfrac{r-R_g}{r} &= \partial_1\left(\ln\left(1 - \left(\tfrac{\hbar}{mc}\right)^2 \tfrac{1}{|\psi| r^{-2}} \partial^1 r^{-2}\left(\tfrac{r-R_g}{r}\partial_1|\psi|\right)\right)\right)\end{aligned} \tag{35}$$

where R_g is the gravitational radius of BH and R_c is the Compton's length, leading to the BH mass density at large distances (see Appendix B), which follows the law

$$\lim_{r\to\infty}|\psi| = G_0 e^{-\varsigma\frac{r}{R_c}} \tag{36}$$

The constant, G_0, is defined by the normalization condition.

2.3. The Mass Distribution near the Center of a Schwarzschild Black Hole

In the classical case, a BH mass collapses into a point, whereas in the quantum case, for the uncertainty principle (see (46)), the maximum concentration is inside a sphere whose radius is in the order of magnitude of the Compton's length R_c. Thence, for a macroscopically massive BH with the condition $R_g \gg R_c$ (for a BH with a mass $m \sim 10^{35}$ kg $\frac{R_g}{R_c} \sim 10^{85}$), we can assume with good approximation that, in the limit $\frac{r}{R_g} \to 0$ (at least $\frac{r}{R_g} \gg 10^{-85}$), there is no mass for $R_c \ll r < R_g$. Thus, by observing that

$$\lim_{\frac{r}{R_g}\to 0}\lambda = \lim_{\frac{r}{R_g}\to 0}\ln\frac{r}{r-R_g} \cong -\frac{r}{R_g} = 0 \tag{37}$$

and that

$$\frac{8\pi G}{c^4} T_1{}^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \cong -\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \cong 0 \tag{38}$$

$$\frac{8\pi G}{c^4} T_0{}^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \cong -\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \cong 0 \tag{39}$$

$$e^\lambda \frac{8\pi G}{c^4} T_0{}^1 = -\frac{\dot{\lambda}}{r} = 0 \tag{40}$$

Equation (A14), in Appendix B, can be also retained for $x = \frac{r}{R_g} \to 0$.

Thus, providing that at small distance from the BH center, it holds

$$\lim_{x\to 0}\left|\widetilde{y}^2{}_{(x)}\right| << \left|\widetilde{y}_{(x)}\right| \tag{41}$$

we obtain the differential equation

$$\lim_{x\to 0}\widetilde{y}'_{(x)} = R_g y'_{(r)} = \widetilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) + \frac{R_g{}^2}{R_c{}^2}\frac{x}{x-1}\left(1 - C_0 e^{\ln\left(\frac{x-1}{x}\right)}\right) \tag{42}$$

whose solution, given in Appendix B, reads

$$\lim_{\frac{r}{R_g}\to 0}|\psi| \cong G_0 e^{-\frac{1-\varsigma^2}{e}\frac{R_g}{R_c}z}. \quad r \gg R_c,\ R_g \gg R_c \tag{43}$$

The output (43) shows that, for large mass BHs (e.g., $m \sim 10^{35}$ kg) the mass density is practically null outside the sphere of the Compton's radius at the center of the BH (for instance, at a distance $r = 10R_c$, the mass density $|\psi|^2 \sim G_0 e^{-\,2\frac{1-\varsigma^2}{e}10^{86}} \sim G_0 e^{-\,10^{86}}$(see Appendix B).

This output, in agreement with the uncertainty principle, leads to a piece of information about the minimum mass for the formation of BHs. In fact, in cases where the BH energy does not exceed the value mc^2 due to its localization (otherwise a new BH is formed), by the uncertainty principle, it follows that

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{v\hbar}{2\Delta x} \leq mc^2 \tag{44}$$

where

$$v \geq \Delta v = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x}, \tag{45}$$

which leads to $mc^2 > \frac{\hbar^2}{4m\Delta x^2}$ and, finally, to

$$\Delta x > \frac{\hbar}{2mc} = \frac{R_c}{2} \tag{46}$$

Additionally, since in order to form a BH, all the mass must be inside the gravitational radius, we must have that

$$R_g = \frac{2Gm}{c^2} > \frac{\Delta x}{2} = r_{\min} = \frac{R_c}{4} \tag{47}$$

and, thence, that

$$\frac{R_c}{4R_g} = \frac{\hbar}{8mcR_g} = \frac{\hbar c}{8m^2 G} = \pi\frac{{m_p}^2}{m^2} < 1 \tag{48}$$

leading to the following condition for the black hole mass m:

$$m > \pi^{1/2} m_p \tag{49}$$

where $m_p = \sqrt{\frac{\hbar c}{8\pi G}}$ is the reduced Planck mass.

For small masses when $m \to 0$ (quantum case), the gravitational radius R_g tends to be zero while the Compton's radius R_c goes to infinity so that, in order to have all the black hole mass inside its gravitational radius, for $R_c = R_g$, we have the minimum mass (49) for the formation of a black hole ($R_g = \frac{R_c}{4}$). This condition is safe for our universe since low energy elementary particles cannot form BHs.

On the other hand, it is noteworthy to observe that, for a very large mass $m \to \infty$, $V_{qu} \propto \frac{1}{m} \to 0$ (classical limit), the BH Compton's radius R_c goes to zero and the point singularity of the classical general relativity is asymptotically approached.

Additionally, as black holes with a Planck mass cannot be divided into two smaller black holes, they represent the lightest possible configuration of scalar uncharged mass density that can be achieved solely through gravitational interaction. Moreover, since the condition expressed in Equation (49) also applies to quantized fields, the fundamental lowest state of a quantum black hole is heavier than $\pi^{1/2} m_p$.

3. Gravitational Field of Black Holes at Large Distance in Spacetime with Background Fluctuations

In this section, we derive the weak gravitational force of black holes over long distances. The large distance approximation is used because the gravitational radius of a black hole is much smaller than the cosmological physical scale, allowing us to treat the mass distribution of the black hole as point-like.

In fact, the mass distribution of a BH (36) arranged in the form

$$|\psi|^2 = |\psi_0|^2 \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 B_{(r)} \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \tag{50}$$

at a large distance reads

$$\begin{aligned} &\lim_{\frac{r}{R_c} \to \infty} |\psi_0|^2 \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 B_{(r)} \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \\ &= \lim_{\frac{r}{R_c} \to \infty} = \frac{B_{(r)}}{\int B_{(r)} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} d^3\Omega} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \\ &= \lim_{R_c \to 0} \frac{B_{(r)}}{\int B_{(r)} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^3 e^{-\varsigma \frac{r^2}{4R_c{}^2}} d^3\Omega} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^3 e^{-\varsigma \frac{r^2}{4R_c{}^2}} \\ &= \frac{B_{(r)}}{\int B_{(r)} \delta^3{}_{(r)} d^3\Omega} \delta^3{}_{(r)} = \frac{B_{(r=0)}}{B_{(r=0)}} \delta^3{}_{(r)} = \delta^3{}_{(r)} \end{aligned} \tag{51}$$

where the normalization condition $\int |\psi|^2 d^3\Omega = 1$ has been used.

Since the BH quantum potential in space–time undergoes fluctuation $\delta \overline{E}_{qu}$ with an additional mass density $|\delta\psi_0|^2 \propto \frac{\delta \overline{E}_{qu}}{c^2}$ in the presence of background fluctuations [14–17], for the cosmological length scale (i.e., $R_c \to 0$), we assume that the total mass density of a BH field in a fluctuating space–time background reads

$$\lim_{r \to \infty} |\psi|^2 \simeq \delta^3{}_{(r)} + |\delta\psi_0|^2 \tag{52}$$

Furthermore, as black holes are quantum objects with significant quantum potential energy (as described in Appendix B), we anticipate that their gravity over long distances may result in quantum effects contributing to Newtonian law.

The contribution coming from the quantum potential, contained in the energy density tensor of the QGE, reads

$$\begin{aligned} R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R_\alpha{}^\alpha &= \frac{8\pi G}{c^4} \frac{mc^2 |\psi_\pm|^2}{\gamma} \left(\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} - 1 \right) g_{\mu\nu} + \sqrt{1 - \frac{V_{qu}}{mc^2}}^{-1} \left(\frac{\hbar}{2mc} \right)^2 \frac{\partial \ln[\frac{\psi}{\psi*}]}{\partial q^\mu} \frac{\partial \ln[\frac{\psi}{\psi*}]}{\partial q^\nu} \right) \\ &= \frac{8\pi G}{c^4} \frac{mc^2 |\psi_\pm|^2}{\gamma} \left(\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} - 1 \right) g_{\mu\nu} + \sqrt{1 - \frac{V_{qu}}{mc^2}} u_\mu u_\nu \right) \end{aligned} \tag{53}$$

where it has used the identity $\left(\frac{1}{mc} \right)^2 p_\mu p^\lambda = u_\mu u_\nu \left(1 - \frac{V_{qu}}{mc^2} \right)$. Moreover, given that

$$V_{qu} = -\frac{\hbar^2}{m} \frac{1}{|\psi| \sqrt{-g}} \partial^1 \sqrt{-g} \left(e^{-\lambda} \partial_1 |\psi| \right) \tag{54}$$

$$g = -\frac{1}{r^4 \sin^4 \theta} \tag{55}$$

$$e^\lambda = \frac{r}{r - R_g} = \frac{1}{g_{00}} = -g_{11} \tag{56}$$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} = \frac{1}{\sqrt{\frac{g^{\mu\nu} \dot{q}_\nu \dot{q}_\mu}{c^2}}} = \frac{1}{\sqrt{g^{00}}} \\ \lim_{r \to \infty} |\psi| &= G_0 e^{-\varsigma \frac{r}{R_c}} \end{aligned} \tag{57}$$

it follows that

$$\begin{aligned}\lim_{r\gg R_g} V_{qu} &= \lim_{r\gg R_g} -\tfrac{\hbar^2}{m}\tfrac{1}{e^{-\varsigma x}} r^2 \partial^r r^{-2}\left(e^{-\lambda}\partial_r e^{-\varsigma z}\right)\\ &= \lim_{r\gg R_g} \tfrac{\hbar^2}{m}\left(\tfrac{\varsigma}{R_c}\right)\tfrac{1}{e^{-\varsigma x}} r^2 \partial^r r^{-2}\left(\left(1-\tfrac{R_g}{r}\right)e^{-\varsigma x}\right)\\ &= \lim_{r\gg R_g} \tfrac{\hbar^2}{m}\tfrac{\varsigma}{R_c}\left(-2r^{-1}\left(1-\tfrac{R_g}{r}\right)+\left(\tfrac{\varsigma}{R_c}\left(1-\tfrac{R_g}{r}\right)+\tfrac{R_g}{r^2}\right)\right)\\ &= \lim_{r\gg R_g} -\tfrac{\hbar^2}{m}\tfrac{\varsigma}{R_c}\left(2r^{-1}-\tfrac{\varsigma}{R_c}\right) = \tfrac{\hbar^2}{m}\tfrac{\varsigma^2}{{R_c}^2} = mc^2\varsigma^2\end{aligned} \tag{58}$$

Thence, the gravitational Equation (53) in a mixed form reads

$$R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R_\alpha{}^\alpha = \frac{8\pi G}{c^4}\frac{mc^2|\psi_\pm|^2}{\gamma}\begin{pmatrix}\left(\sqrt{1+\varsigma^2}-1\right)\delta_\mu{}^\nu\\ +\sqrt{1+\varsigma^2}u_\mu u^\nu\end{pmatrix} \tag{59}$$

leading to the equation for the trace of the Ricci tensor

$$R_\alpha{}^\alpha = -\frac{8\pi G}{c^4}\frac{mc^2|\psi_\pm|^2}{\gamma}\begin{pmatrix}4\left(\sqrt{1+\varsigma^2}-1\right)\\ +\sqrt{1+\varsigma^2}u_\alpha u^\alpha\end{pmatrix} \tag{60}$$

and to

$$R_0{}^0 = \frac{8\pi G}{c^4}\frac{mc^2\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)}{\gamma}\begin{pmatrix}\left(\sqrt{1+\varsigma^2}-1\right)\left(\delta_0{}^0-2\right)\\ +\frac{\sqrt{1+\varsigma^2}}{2}u_0 u^0\end{pmatrix} \tag{61}$$

Given that, at a large distance, we can use the approximations

$$lim_{r\to\infty} u_0 = 1 \tag{62}$$

$$g_{00} = \frac{r-R_g}{r} = \left(1-\frac{R_g}{r}\right), \tag{63}$$

(61) reads

$$\begin{aligned} R_0{}^0 &= \tfrac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)}{\gamma}\lim_{r\to\infty}\begin{pmatrix}\left(\sqrt{1+\varsigma^2}-1\right)\left(\delta_0{}^0-2\right)\\ +\frac{\sqrt{1+\varsigma^2}}{2g_{00}}\end{pmatrix}\\ &= \tfrac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)}{\gamma}\lim_{r\to\infty}\left(1+\left(\tfrac{1}{g_{00}}-2\right)\tfrac{\sqrt{1+\varsigma^2}}{2}\right)\\ &= \tfrac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)}{\gamma}\lim_{r\to\infty}\left(1+\left(\frac{\frac{r}{R_g}+2}{1-\frac{r}{R_g}}\right)\tfrac{\sqrt{1+\varsigma^2}}{2}\right)\\ &\cong \tfrac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)}{\gamma}\left(1-\tfrac{\sqrt{1+\varsigma^2}}{2}\right)\end{aligned} \tag{64}$$

leading to the identity

$$R_0{}^0 = R_{00} = \frac{1}{c^2}\frac{\partial}{\partial q^\alpha}\frac{\partial\phi}{\partial q^\alpha} \cong \frac{4\pi G}{c^2} m\left(\delta_{(r-r_{BH})}+\left|\psi_0\right|^2\right)\left(1-\frac{\sqrt{1+\varsigma^2}}{2}\right). \tag{65}$$

By integrating the flux of the gravitational force $\frac{\partial\phi}{\partial q^\alpha}$ with a sphere with the radius $r - r_{BH}$, it follows that

$$\iiint \frac{\partial}{\partial q^\alpha}\frac{\partial\phi}{\partial q^\alpha} d\mathrm{V} = \oiint \frac{\partial\phi}{\partial q^\alpha}\cdot dS^\alpha = \frac{\partial\phi}{\partial r}4\pi(r-r_{BH})^2 = 4\pi Gm\iiint\left(\delta_{(r-r_{BH})}+\left|\delta\psi_0\right|^2\right)\left(1-\frac{\sqrt{1+\varsigma^2}}{2}\right)d\mathrm{V}. \tag{66}$$

By the Dirac δ-shape approximation of the BH mass distribution, it can be posed that $V_{qu_{(r-r_{BH}=0)}} = 0$, so that the BH gravitational field at a large distance reads

$$\begin{aligned}\frac{\partial\phi}{\partial r} &= Gm\left(\frac{1-\pi\sqrt{1+\varsigma^2}\left|\delta\psi_0\right|2\iiint(r-r_{BH})^2\frac{dr}{R_c}}{(r-r_{BH})^2}\right)\\ &= Gm\left(\frac{1}{(r-r_{BH})^2} - \frac{1}{3}\frac{\pi\sqrt{1+\varsigma^2}\left|\delta\psi_0\right|^2}{R_c}(r-r_{BH})\right)\end{aligned} \tag{67}$$

where the repulsive force

$$-Gm\frac{1}{3}\frac{\pi\sqrt{1+\varsigma^2}\left|\delta\psi_0\right|^2}{R_c}(r-r_{BH}) \tag{68}$$

overcomes the attractive one when

$$(r-r_{BH})^3 \geq \frac{3R_c}{\pi\sqrt{1+\varsigma^2}\left|\delta\psi_0\right|^2} \tag{69}$$

From (59) we can observe that the cosmological pressure density originating from a BH at a large distance is constant and reads

$$\lim_{r\to\infty}\mathbf{\Lambda}_Q = \left(\sqrt{1+\varsigma^2}-1\right) \cong \frac{\varsigma^2}{2}, \tag{70}$$

and that the repulsive gravity is generated by the presence of the dark energy/mass density $|\delta\psi_0|$ of the background fluctuations.

From (70) it is also interesting to note that the large distance mass density of a BH (36) acquires the form

$$\lim_{r\to\infty}|\psi|^2 = G_0 e^{-\sqrt{8\mathbf{\Lambda}_Q}(r/R_c)}. \tag{71}$$

Generally speaking, beyond the centrally symmetric case, the pressure density tensor, denoted as $\mathbf{\Lambda}_Q$, is a function of mass fields and exhibits point dependence similar to the quintessence model. The key distinction lies in its dependence on the quantum properties of spacetime rather than an obscure physical field. Additionally, $\mathbf{\Lambda}_Q$ can yield a cosmological constant, representing the mean value in the universe, with magnitudes consistent with observed values (see reference [18]). Furthermore, the definition of the minimum radius of a black hole mass distribution, which solves the case of the classical general relativity point singularity and, consequently, the determination of a minimum mass for black hole formation, represents the primary large-scale manifestation of quantum effects on the space–time curvature within the theory. The establishment of a minimum mass for black hole formation holds significant importance, ensuring a secure condition for our universe, as elementary particles cannot generate black holes, and the quantum instability of vacuum does not lead to the massive production of micro black holes.

3.1. Quantum Potential Fluctuations Generated by Background Fluctuations

To determine the parameter $|\delta\psi_0|^2$, we must move beyond the static vacuum solution and consider that the vacuum is filled with stochastic gravitational waves. These waves originate from various sources, including relic gravitational waves from the Big Bang and other sources [19].

Considering the vacuum fluctuations in the background, it becomes possible to define the stochastic generalization of the quantum hydrodynamic equations [20] so that the wave function $\psi = |\psi| e^{-\frac{iS}{\hbar}}$, in the low velocity limit, is given by the equations

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial}{\partial q_i}(|\psi|^2 \dot{q}_i) = 0. \tag{72}$$

$$\dot{q}_i = \frac{p_i}{m} = \frac{1}{m}\frac{\partial S_{(q,t)}}{\partial q_i}, \tag{73}$$

$$\dot{p}_i = -\frac{\partial (H + V_{qu})}{\partial q_i}, \tag{74}$$

where $S_{(q,t)} = -\frac{\hbar}{2} ln \frac{\psi}{\psi *}$, H is the Hamiltonian of the system, and V_{qu} is given by the low velocity limit of (8).

The ripples of the vacuum curvature are assumed to manifest themselves through an additional fluctuating mass density δn_{vac} into the vacuum so that

$$n_{tot} \equiv \overline{n} + \delta n_{vac} \tag{75}$$

where $\overline{n}$ is linked to n by the relation $lim_{\delta n_{vac} \to 0} \overline{n} = n$, that, introduced into the quantum potential

$$V_{qu_{(n_{tot})}} = -\frac{\hbar^2}{2m} n_{tot}{}^{-1/2} \frac{\partial^2 n_{tot}{}^{1/2}}{\partial q_i \partial q_i}, \tag{76}$$

leads to the quantum fluctuating force [20] $-\frac{\partial V_{qu_{(n)}}}{\partial q_i}$ that we are going to determine.

Given that the energy/mass density δn_{vac} is defined as positive, the mean vacuum fluctuations give rise to an additional non-zero (dark) energy density in the vacuum.

Given that the energy/mass density δn_{vac} is defined as positive, this paragraph describes the assumption that the mean vacuum fluctuations $< \delta n_{vac} >$ give rise to an additional dark energy density in the vacuum. The assumption is made that this vacuum of dark energy/matter does not interact with the physical system under consideration, and therefore, the gravity interaction is disregarded in the Hamiltonian H in (74). The evolution of the total dark energy is assumed to depend on cosmological dynamics and to have reached an equilibrium configuration.

Thence, we assume $< \delta n_{vac} >$ is locally, uniformly distributed with zero mean fluctuations $\delta n_{(q,t)}$, such as

$$\delta n_{vac} \cong < \delta n_{vac} > + \delta n_{(q,t)} \tag{77}$$

3.2. Spectrum and Correlation Function of Mass Density Noise in Quantum Spacetime with Curvature Fluctuations

When determining the features of the fluctuations of quantum potential, which consequently produce force noise, we employ the postulate that the fluctuations of the vacuum curvature are described by the wave function ψ_{vac} with the density $\delta n_{vac} = |\psi_{vac}|^2$ and that they do not have a Hamiltonian interaction with the physical system (gravitational interaction is disregarded).

In this case, the wave function of the overall system ψ_{tot} reads

$$\psi_{tot} \cong \psi \psi_{vac} \tag{78}$$

Moreover, by assuming that the equivalent mass of dark energy is much smaller than the mass of the system (i.e., $m_{tot} = m_{dark} + m \cong m$), the overall quantum potential (8) reads

$$\begin{aligned} V_{qu_{(n_{tot})}} &= -\frac{\hbar^2}{2m_{tot}} |\psi|^{-1} |\psi_{vac}|^{-1} \frac{\partial^2 |\psi||\psi_{vac}|}{\partial q_i \partial q_i} = \\ &= -\frac{\hbar^2}{2m}\left(|\psi|^{-1} \frac{\partial^2 |\psi|}{\partial q_i \partial q_i} + |\psi_{vac}|^{-1} \frac{\partial^2 |\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1} |\psi_{vac}|^{-1} \frac{\partial |\psi_{vac}|}{\partial q_i} \frac{\partial |\psi|}{\partial q_i} \right). \end{aligned} \tag{79}$$

Moreover, given the vacuum mass density noise of wave-length λ,

$$\delta n_{vac(\lambda)} = |\psi_{vac(\lambda)}|^2 \propto cos^2 \frac{2\pi}{\lambda} q \tag{80}$$

associated with the fluctuation wave function

$$\psi_{vac} \propto \pm cos \frac{2\pi}{\lambda} q \tag{81}$$

it follows that the quantum potential energy fluctuations read

$$\delta \overline{E}_{qu} = \int_V n_{tot\,(q,t)} \delta V_{qu(q,t)} dV, \tag{82}$$

where

$$\begin{aligned} \delta V_{qu(q,t)} &= -\frac{\hbar^2}{2m}\left(|\psi_{vac}|^{-1}\frac{\partial^2|\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1}|\psi_{vac}|^{-1}\frac{\partial|\psi_{vac}|}{\partial q_i}\frac{\partial|\psi|}{\partial q_i}\right) \\ &= \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial|\psi|}{\partial q_i}\left(\pm cos\frac{2\pi}{\lambda}q\right)^{-1}\left(\pm sin\frac{2\pi}{\lambda}q\right)\right) \\ &= \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial|\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right) \end{aligned} \tag{83}$$

For $V \to \infty$, the unidimensional case leads to

$$\begin{aligned} &\delta \overline{E}_{qu(\lambda)} = \frac{1}{\overline{n}_{tot} V}\frac{\hbar^2}{2m}\int_V n_{tot\,(q,t)}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial|\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq \\ &= \frac{1}{\overline{n}_{tot} V}\frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2\int_V n_{tot\,(q,t)}dq + \int_V n_{tot\,(q,t)}\left(|\psi|^{-1}\frac{\partial|\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq\right) \cong \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2. \end{aligned} \tag{84}$$

In (84), the normalization condition $\int_V n_{tot\,(q,t)}dq = \overline{n}_{tot} V$ has been used, and for a large volume ($V \gg {\lambda_c}^3$ see (87)), the following approximation has been used:

$$lim_{\lambda \to 0} \int_{-\infty}^{\infty} n_{tot\,(q,t)}\left(|\psi|^{-1}\frac{\partial|\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq \ll \overline{n}_{tot} V\left(\frac{2\pi}{\lambda}\right)^2.$$

For the three-dimensional case, (84) leads to

$$\delta \overline{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m}\sum_i (k_i)^2 = \frac{\hbar^2}{2m}|k|^2 \tag{85}$$

Equation (85) reveals that the energy arising from the mass density fluctuations of the vacuum becomes greater as the square of the inverse of λ. Thus, the corresponding fluctuations in quantum potential produce extremely large energy fluctuations $\delta \overline{E}_{qu}$, even for very small noise amplitudes (i.e., $T \to 0$ when λ approaches zero) at very short distances.

A convergence to the deterministic limit of quantum mechanics (72–74) (for $T \to 0$) is warranted by the fact that the higher the energy-to-noise amplitude ratio $\frac{\delta \overline{E}_{qu}}{kT}$ is, the smaller the probability is of a convergence happening. This brings a condition on the spatial correlation function of the quantum potential noise as $\lambda \to 0$ or for $T \to 0$.

One way to obtain the shape of the spatial correlation function $G(\lambda)$ is through a stochastic calculation, which can be quite complex [20]. However, a simpler approach for obtaining $G(\lambda)$ can be achieved by considering the spectrum of the fluctuations, as described in [16].

Since each component of spatial frequency, $k = \frac{2\pi}{\lambda}$, brings the quantum potential energy contribution (84), its probability of happening reads

$$\begin{aligned} p_{(\lambda)} &\propto exp\left[-\frac{\delta \overline{E}_{qu}}{kT}\right] \\ &= exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] \end{aligned} \tag{86}$$

where

$$\lambda_c = 2\frac{\hbar}{(2mkT)^{1/2}} \tag{87}$$

is the De Broglie length.

From (86), it comes out that the spectrum $S(k)$ of the spatial frequency

$$S(k) \propto p(\frac{2\pi}{\lambda}) = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] = exp\left[-\left(\frac{k\lambda_c}{2}\right)^2\right] \tag{88}$$

is not white, and the components with a wave-length, λ, smaller than λ_c go quickly to zero.

Additionally, from (88), the spatial shape $G(\lambda)$ reads

$$\begin{aligned} G_{(\lambda)} &\propto \int_{-\infty}^{+\infty} exp[ik\lambda]S_{(k)}dk \propto \int_{-\infty}^{+\infty} exp[ik\lambda]exp\left[-\left(k\frac{\lambda_c}{2}\right)^2\right]dk \\ &\propto \frac{\pi^{1/2}}{\lambda_c}exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right] \end{aligned} \tag{89}$$

One can see from Equation (89) that the quantum potential progressively suppresses uncorrelated mass density fluctuations at shorter and shorter distances, which in turn allows for the realization of deterministic quantum mechanics in systems whose physical length is much smaller than the De Broglie length.

The assumption for a sufficiently general case is that the mass density noise correlation function is Gaussian with zero correlation time, isotropic in space, and independent among different coordinates. Under these assumptions, it can be expressed as

$$< \delta n_{(q_\alpha,t)}, \delta n_{(q_\beta+\lambda,t+\tau)} >=< \delta n_{(q_\alpha)}, \delta n_{(q_\beta)} >_{(T)} \; G(\lambda)\delta(\tau)\delta_{\alpha\beta} \tag{90}$$

3.3. The (Dark) Energy Density of Quantum Potential Fluctuations

The energy associated with the quantum potential noise of a body with a mass m can be evaluated using the probability energy fluctuation function

$$p_{(E_{(\lambda)})} = Aexp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] \tag{91}$$

where

$$\begin{aligned} A &= \frac{1}{\int_0^{\lambda_{max}} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda} = \frac{1}{\int_0^{N\lambda_c} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda + \int_{N\lambda_c}^{\lambda_{max}} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda} \\ &= \frac{1}{\int_0^{N\lambda_c} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda + \int_{N\lambda_c}^{\lambda_{max}} d\lambda} = \frac{1}{\int_0^{N\lambda_c} exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right]d\lambda + (\lambda_{max} - N\lambda_c)} \end{aligned} \tag{92}$$

where $\lambda_c = \sqrt{2}\frac{\hbar}{\sqrt{mkT}}$ and $N \gg 1$.

In this case, the energy density of the quantum potential fluctuation reads

$$
\begin{aligned}
mc^2\left|\delta\psi_0\right|^2{}_{(r)} &= \frac{\int \delta\overline{E}_{qu(\lambda)}p_{(\lambda)}d\lambda}{\int p_{(\lambda)}d\lambda} = \frac{\int\limits_0^{\lambda_{max}} \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 p_{(\lambda)}d\lambda}{\int\limits_0^{\lambda_{max}} p_{(\lambda)}d\lambda} \cong \int\limits_0^{\lambda_{max}} \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 p_{(\lambda)}d\lambda \\
&= \frac{1}{\lambda_{max}}\int\limits_0^{\lambda\max} \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda
\end{aligned} \tag{93}
$$

where, for (85) in the three-dimensional case $\lambda = |k|$, $m|\delta\psi_0|^2$ is the additional mass density in the vacuum that the black hole mass m acquires due to the background fluctuations, and where

$$\lambda_{max} = l_u \approx 10^{27}\ \text{m}. \tag{94}$$

where l_u is the diameter of the universe.

Moreover, for SMBHs (in the order of Sagittarius A* with a mass of about of 10^{38} kg) at

$$1^\circ K\lambda_c = 2\frac{\hbar}{(2mkT)^{1/2}} \approx \frac{1.41\times 10^{-34}}{(3\times 10^{38}10^{-23})^{1/2}} \approx 3\times 10^{-41}\ \text{m} \approx 0, \tag{95}$$

it follows that

$$A = \frac{1}{l_u} \tag{96}$$

and that

$$
\begin{aligned}
mc^2|\delta\psi_0|^2 &= \frac{1}{l_u}\frac{\hbar^2}{2m}\int\limits_0^{l_u}\left(\frac{2\pi}{\lambda}\right)^2 exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda \\
&= \frac{1}{l_u}\frac{\hbar^2}{2m}\int\limits_{\infty}^{2\pi l_u{}^{-1}} exp\left[-\frac{\frac{\hbar^2}{2m}}{kT}|k|^2\right]d|k| = \frac{\hbar}{2l_u}\sqrt{\frac{\pi kT}{2m}}
\end{aligned} \tag{97}
$$

leading to

$$|\delta\psi_0|^2{}_{(r)} \approx \frac{1}{l_u}\frac{\hbar}{2mc}\sqrt{\frac{\pi kT}{2mc^2}} \tag{98}$$

3.4. Repulsive Gravity at Large Distance

By introducing (98) into (69), the repulsive Newtonian gravity overcomes the attractive one at the distance

$$
\begin{aligned}
(r - r_{BH})_{rep} &\approx \left(\frac{3R_c}{\pi\sqrt{1+\varsigma^2}\left|\delta\psi_0\right|^2}\right)^{1/3} \\
&= \left(\frac{3R_c}{\pi\left(\mathbf{1}+\mathbf{\Lambda}_Q\right)\left|\delta\psi_0\right|^2}\right)^{1/3} \sim \left(l_u\sqrt{\frac{2mc^2}{\pi kT}}\right)^{1/3}
\end{aligned} \tag{99}
$$

so that for SMBHs with a mass in the order of $10^{38\div 41}$ kg, $\varsigma \ll 1$, the equation reads

$$(r - r_{BH})_{rep} > \approx 10^{21}\ \text{m}. \tag{100}$$

The gravitational force between galaxies becomes repulsive at intergalactic distances, which is on the same order of magnitude as the typical radius of galaxies ($\sim 10^{20}m$). This may affect the external part of the galactic disc. However, since the energy density of BH quantum potential fluctuations decreases with the expansion of the radius l_u of the universe according to $lim_{t\to\infty}|\delta\psi_0|^2{}_{(r)} \sim \frac{1}{l_u}\frac{\hbar}{2mc}\sqrt{\frac{\pi kT}{2mc^2}} \to 0$), the BH repulsive force asymptotically approaches zero, leading to a final static universe. This effect can furnish the empirical confirmation of the theory.

Since, as shown in [21], the quantum potential of macroscopic low-density mass bodies is practically null and does not contribute to the expansive gravity of the universe, the repulsive gravitational force in (100), causing the repulsion of the galaxies, is mainly attributed to black holes and supermassive black holes due to the quantum nature of space–time with fluctuating background metrics.

The correction to Newtonian gravity that arises from the fluctuations of quantum potential in massive bodies such as BHs and SMBHs complies with the concept of modified Newtonian dynamics (MOND) [22], which suggests a modification in Newtonian gravity for very low accelerations in order to account for the observed motion of the galaxies.

4. Conclusions

This work shows that quantum black holes with a central symmetry have a mass density distribution that is not point-like but is concentrated in a sphere with a radius in the order of its Compton wavelength.

Due to significant quantum potential energy, there exists a supplemental term in the gravity equation, resulting in an additional contribution to the gravity force at large distances.

The pressure density tensor is a function of mass fields and demonstrates a point-specific behavior akin to the quintessence model. However, what sets it apart is its reliance on the quantum properties of space–time instead of an elusive physical field.

The present model shows that the alteration of Newtonian gravity over long distances is explained by the gravitational effect of the quantum potential of enormously massive entities, such as black holes and supermassive black holes, subject to background dark energy fluctuations. In the presence of fluctuations in the space–time background metric, the dark energy arising from fluctuations in the quantum potential energy of black holes results in a repulsive contribution to gravitational force. This repulsive force of SMBH dominates over the Newtonian force at distances characteristic of intergalactic space. On this basis, it has the capacity to generate a cosmological constant producing the acceleration of the universe, aligning reasonably well with the observed low value.

Furthermore, the establishment of a minimum radius for the mass distribution of black holes, which solves the issue of classical general relativity's point singularity, and subsequently, the determination of a minimum mass required for black hole formation, represents the foremost large-scale manifestation of quantum effects on the curvature of spacetime as posited by the theory. This concept holds important significance as it ensures the stability of our universe, preventing elementary particles from spontaneously generating black holes and averting the excessive production of microscopic black holes due to quantum vacuum instability.

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Appendix A

Since the theory proposed in this work is based on the covariance of the motion equation in curved space–time, we derive the motion equation in the Minkowskian case. By utilizing the following identities for stationary states [11] (i.e., eigenstates),

$$p_\mu = -\partial_\mu S \tag{A1}$$

$$\frac{dS_{(k)}}{dt} = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\dot{q}_i = L_{(k)} = -p_{(k)_\mu}\dot{q}^{\mu}_{(k)} = -\frac{mc^2}{\gamma_{(k)}}\sqrt{1 - \frac{V_{qu(k)}}{mc^2}}, \tag{A2}$$

where for eigenstates $\gamma_{(k)} = const$,

$$p_{(k)\mu} = -\frac{\partial}{\partial \dot{q}^{\mu}} \frac{dS_{(k)}}{dt} = m\gamma_{(k)} \dot{q}_{(k)_\mu} \sqrt{1 - \frac{V_{qu\,(k)}}{mc^2}} = mcu_{(k)_\mu} \sqrt{1 - \frac{V_{qu\,(k)}}{mc^2}} \tag{A3}$$

Therefore, since for systems that do not explicitly depend on time, it holds

$$-\frac{\partial}{\partial q^{\mu}} \frac{dS_{(k)}}{dt} = -\left(0, -\frac{\partial}{\partial q_i} \frac{dS_{(k)}}{dt}\right) = -\left(0, -\frac{d}{dt} \frac{\partial S_{(k)}}{\partial q_i}\right) = \frac{d}{dt}\left(-\frac{\partial S_{(k)}}{\partial q^{\mu}}\right) = \dot{p}_{(k)_\mu} \tag{A4}$$

the motion equation reads

$$\dot{p}_{(k)_\mu} = -\frac{d}{dt} \frac{\partial}{\partial \dot{q}^{\mu}} \frac{dS_{(k)}}{dt} = -\frac{\partial}{\partial q^{\mu}} \frac{dS_{(k)}}{dt} \tag{A5}$$

which, in curved space–time, reads

$$\dot{p}_{(k)_\mu} = D_{q_0} p_{(k)\mu} = -\frac{1}{c} \frac{\partial L_{(k)}}{\partial q^{\mu}} = \frac{mc}{\gamma_{(k)}} \frac{\partial}{\partial q^{\mu}} \sqrt{1 - \frac{V_{qu(k)}}{mc^2}}. \tag{A6}$$

Appendix B

Schwarzschild black hole mass distribution:

The differential Equation (35) can be solved by posing $\left(\frac{\hbar}{mc}\right)^2 \frac{r^2}{|\psi|} \partial^1 r^{-2} \left(\frac{r-R_g}{r} \partial_1 |\psi|\right) = 1 - A_0 e^{f_{(r)}}$ from which it follows that

$$\left(\partial_1 \frac{r - R_g}{r}\right) A_0 e^{f_{(r)}} = \left(\partial_1 f_{(r)}\right) A_0 e^{f_{(r)}} \tag{A7}$$

leading to the solution

$$f_{(r)} = \left(\frac{r - R_g}{r} + C\right) \tag{A8}$$

Furthermore, by posing

$$|\psi| = G_0 e^{g_{(r)}} \tag{A9}$$

it follows that

$$\left(\frac{\hbar}{mc}\right)^2 r^2 G_0{}^{-1} e^{-g_{(r)}} \partial^1 r^{-2} \left(\frac{r-R_g}{r} g'_{(r)} G_0 e^{g_{(r)}}\right) = 1 - A_0 e^{f_{(r)}} \tag{A10}$$

and thence,

$$\begin{gathered}\left(\frac{\hbar}{mc}\right)^2 r^2 G_0{}^{-1} e^{-g_{(r)}} \left(g'_{(r)} G_0 e^{g_{(r)}} \partial^1 \left(r^{-2} \frac{r-R_g}{r}\right) + r^{-2} \frac{r-R_g}{r} \partial^1 \left(g'_{(r)} G_0 e^{g_{(r)}}\right)\right) = \\ \left(\frac{\hbar}{mc}\right)^2 \left(r^2 g'_{(r)} \partial^1 r^{-3} (r - R_g) + \frac{r-R_g}{r} \left(g'^2{}_{(r)} + g''_{(r)}\right)\right) = \\ \left(\frac{\hbar}{mc}\right)^2 \left(g'_{(r)} \left(\frac{1}{r} - \frac{3(r-R_g)}{r^2}\right) + \frac{r-R_g}{r} \left(g'^2{}_{(r)} + g''_{(r)}\right)\right) = 1 - A_0 e^{f_{(r)}}\end{gathered} \tag{A11}$$

which, by posing

$$g'_{(r)} = y_{(r)} \tag{A12}$$

leads to the Riccati's differential equation

$$y'_{(r)} = y_{(r)} \left(\frac{3}{r} - \frac{1}{r - R_g}\right) - y^2{}_{(r)} + \left(\frac{mc}{\hbar}\right)^2 \frac{r}{r - R_g} \left(1 - C_0 e^{\frac{r-R_g}{r}}\right) \tag{A13}$$

where $C_0 = A_0 e^{-C}$.

Moreover, by using the adimensional variable $x = \frac{r}{R_g}$, it follows that

$$\widetilde{y}^{|}{}_{(x)} = R_g y'_{(r)} = \widetilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) - R_g \widetilde{y}^2{}_{(x)} + \frac{R_g}{R_c{}^2}\frac{x}{x-1}\left(1 - C_0 e^{\frac{x-1}{x}}\right) \tag{A14}$$

where $R_c = \frac{\hbar}{mc}$.

The condition of mass density at infinity

$$\lim_{r\to\infty}|\psi| = \lim_{r\to\infty} G_0 e^{g_{(r)}} = 0 \tag{A15}$$

$$\lim_{r\to\infty}\left|\psi\right|' = \lim_{r\to\infty} G_0 g'_{(r)} e^{g_{(r)}} = 0 \tag{A16}$$

leads to a condition on $g_{(r)}$ that

$$\lim_{r\to\infty} g_{(r)} = -\infty. \tag{A17}$$

Large distance BH mass density distribution:

On the condition that $lim_{x\to\infty}|\widetilde{y}^2{}_{(x)}| \gg \frac{|\widetilde{y}_{(x)}|}{x}$ (to be checked at the end), and by choosing $c = 1$ so that $C_0 = \frac{A_0}{e}$, Equation (A14) simplifies to

$$\lim_{x\to\infty}\widetilde{y}^{|}{}_{(x)} \cong -R_g \widetilde{y}^2{}_{(x)} + \frac{R_g}{R_c{}^2}(1 - A_0) \tag{A18}$$

which, by posing

$$\widetilde{y}_{(x)} = \frac{u'}{R_g u} \tag{A19}$$

leads to

$$\lim_{x\to\infty}\left(\frac{u'}{R_g u}\right)' = \lim_{x\to\infty}\left(\frac{u''}{R_g u} - \frac{u'^2}{R_g u^2}\right) \cong -\frac{u'^2}{R_g u^2} + \frac{R_g}{R_c{}^2}(1 - A_0) \tag{A20}$$

$$\lim_{x\to\infty} u'' \cong u\frac{R_g{}^2}{R_c{}^2}(1 - A_0) \tag{A21}$$

providing the solution

$$u = u_0 e^{\pm\frac{R_g}{R_c}(1-A_0)^{1/2}x} \tag{A22}$$

leading to

$$\widetilde{y}_{(x)} = \pm\frac{R_g}{R_c}(1 - A_0)^{1/2}. \tag{A23}$$

and to

$$\begin{aligned}\lim_{x\to\infty}|\psi| &= \lim_{x\to\infty} G_0 e^{\int g'_{(r)} dr} = G_0 e^{\int R_g \widetilde{y}_{(x)} dx} \\ &= G_0 e^{-\frac{R_g}{R_c}(1-A_0)^{1/2}x} = G_0 e^{-\varsigma\frac{r}{R_c}}\end{aligned} \tag{A24}$$

where $\varsigma = (1 - A_0)^{1/2}$ and where the condition $\lim_{r\to\infty} g_{(r)} = -\infty$ requires considering the negative solution of $\widetilde{y}_{(x)}$.

In order to evaluate the numerical constant ζ, we observe that the ratio between the total mass of the BH and the part outside its gravitational radius due to its quantum mass distribution (with the unitary normalization $\int_0^\infty |\psi|^2 = 1$) reads

$$\frac{\Delta m_{out}}{m} = \frac{\int_{r=R_g}^{\infty} |\psi|^2}{\int_0^\infty |\psi|^2} = \int_{r=R_g}^{\infty} |\psi|^2, \tag{A25}$$

and it is vanishingly small for cosmological BHs (e.g., for BHs with a mass in the order of 10^{38}kg, assuming a unitary outside mass (1 kg), it results in $\int\limits_{r=R_g}^{\infty} |\psi|^2 \approx 10^{-38}$). Moreover, by utilizing expression (2),

$$\lim_{\frac{r}{R_c}\to\infty} |\psi|^2 = \delta^3{}_{(r)} \cong \lim_{R_c\to 0}\left(\frac{\varsigma}{2\sqrt{\pi}R_c}\right)^3 e^{-\varsigma\frac{r^2}{4R_c{}^2}} \tag{A26}$$

It follows that

$$\begin{aligned}\int\limits_{r=R_g}^{\infty} |\psi|^2 dV &= \frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 \int\limits_{r=R_g}^{\infty} \left(\frac{r}{R_c}\right)^2 e^{-\varsigma\frac{r^2}{4R_c{}^2}} d\frac{r}{R_c} \\ &= \frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 \int\limits_{t=\frac{R_g}{R_c}}^{\infty} t^2 e^{-\varsigma\frac{t^2}{4}} dt = \frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 \int\limits_{t=\frac{R_g}{R_c}}^{\frac{l_u}{R_c}} t^2 e^{-\varsigma\frac{t^2}{4}} dt = \frac{\Delta m_{out}}{m}.\end{aligned} \tag{A27}$$

By utilizing the relation (A27), for a BH of 10^{38}kg, a kg of Δm_{out} gives the contribution

$$\frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 \int\limits_{t=\frac{R_g}{R_c}}^{\infty} t^2 e^{-\varsigma\frac{t^2}{4}} dt = \frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 \int\limits_{10^{85}}^{\infty} t^2 e^{-\varsigma\frac{t^2}{4}} dt = 10^{-38} < \frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 10^{170} e^{-\varsigma\frac{10^{170}}{4}} 10^{102} \tag{A28}$$

and that

$$\frac{1}{\pi^{3/2}}\left(\frac{\varsigma}{2}\right)^3 10^{170} e^{-\varsigma\frac{10^{170}}{4}} 10^{102} > 10^{-38} \tag{A29}$$

$$\left(\frac{\varsigma}{2}\right)^3 e^{-\varsigma\frac{10^{170}}{4}} = \left(\frac{\varsigma}{2}\right)^3 10^{-\varsigma\frac{10^{170}}{4}\lg_{10}e} >\sim 10^{-310} \tag{A30}$$

which, by posing $\varsigma = 10^{-n}$, leads to

$$10^{-\frac{10^{170-3n}}{4}\lg_{10}e} >\sim 10^{-309+3n}, \tag{A31}$$

to

$$10^{170-3n} > 4\frac{309-3n}{\lg_{10}e} \sim 10^2 \tag{A32}$$

and to

$$0 < \zeta <\sim 10^{-56}. \tag{A33}$$

Even for larger values of $\frac{\Delta m_{out}}{m} \sim 10^{-9}$, the order of magnitude (A33) remains practically the same. This is because the ratio $\frac{R_c}{R_g}$ determines ζ.

Thus, BHs with a small mass, close to the Planck one with $R_c \sim R_g$, can lead to higher values of ζ.

Mass distribution at short distance ($r \ll R_g$):

Near the center for $x \ll 1$, we must use the relation for the matter

$$\Gamma^{\alpha}_{\mu\nu} u_\alpha u^\nu = u^0 u^0 \partial_1 g_{00} = g^{00}\partial_1 g_{00} = \frac{1}{g_{00}}\partial_1 g_{00} \cong \partial_1 \ln g_{00} \tag{A34}$$

which leads to the equation

$$\left(\partial_1 \ln\left(\frac{r-R_g}{r}\right)\right) = \left(\partial_1 f_{(r)}\right), \tag{A35}$$

with the solution

$$f_{(r)} = \left(\ln\left(\frac{r-R_g}{r}\right) + C\right). \tag{A36}$$

and the equation

$$\tilde{y}^{|}{}_{(x)} = R_g y'_{(r)} = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) - R_g \tilde{y}^2{}_{(x)} + \frac{R_g}{R_c{}^2}\frac{x}{x-1}\left(1 - C_0 e^{\ln\left(\frac{x-1}{x}\right)}\right) \quad (A37)$$

Furthermore, on the condition that $\lim_{x \to 0}|\tilde{y}^2{}_{(x)}| << \frac{|\tilde{y}_{(x)}|}{x}$ (to be checked at the end), Equation (A14) simplifies to

$$\tilde{y}^{|}{}_{(x)} = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) + \frac{R_g}{R_c{}^2}\frac{x}{x-1}\left(1 - C_0\frac{x-1}{x}\right) = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) - C_0\frac{R_g}{R_c{}^2} \quad (A38)$$

where $x \ll 1$ and $\tilde{y}_{(x)} = \frac{u'}{R_g u}$, leading to the solution

$$\begin{aligned} y_{(x)} &= \left(e^{\int \frac{R_g}{R_c{}^2} C_0 e^{-\int\left(\frac{3}{x} - \frac{1}{x-1}\right)dx} dr} \right) e^{\int\left(\frac{3}{x} - \frac{1}{x-1}\right)dx} \\ &= \left(e^{\int \frac{R_g c^2}{R_c{}^2} C_0 e^{-\int \frac{2}{x}dx} dx} + C \right) e^{\int \frac{2}{x}dx} \\ &\cong \left(e^{-C_0 \frac{R_g{}^2}{R_c{}^2}\frac{1}{x^2}} + C \right) x^2 = g'_{(r)} = \frac{1}{R_g} g'_{(x)} \end{aligned} \quad (A39)$$

and, thence, for $R_c \ll r \ll R_g$,

$$\begin{aligned} \lim_{x \to 0}|\psi| = \lim_{x \to 0} G_0 e^{\int\left(e^{-C_0 \frac{R_g{}^2}{R_c{}^2}\frac{1}{x^2}} + C\right)x^2 dx} &= \lim_{x \to 0} G_0 e^{\int\left(1 - C_0 \frac{R_g{}^2}{R_c{}^2}\frac{1}{x^2} + C\right)x^2 dx} \\ = \lim_{x \to 0} G_0 e^{\frac{(1+C)}{3}x^3 - C_0 \frac{R_g{}^2}{R_c{}^2}x} \cong G_0 e^{-C_0 \frac{R_g{}^2}{R_c{}^2}x} &\cong G_0 e^{-\frac{1-\varsigma^2}{e}\frac{R_g}{R_c}z} \end{aligned} \quad (A40)$$

where the identity $C_0 = \frac{1-\varsigma^2}{e} \cong e^{-1}$ has been used.

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