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Jordan Canonical Form for Solving the Fault Diagnosis and Estimation Problems

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Abstract: The suggested methods for solving fault diagnosis and estimation problems are based on the use of the Jordan canonical form. The diagnostic observer, virtual sensor, interval, and sliding mode observer design problems are considered. Algorithms have been developed to solve these problems for both linear and nonlinear systems, considering the presence of external disturbances and measurement noise. It has been shown that the Jordan canonical form allows reducing the dimensions of interval observers and virtual sensors, thus simplifying the design process in comparison to the identification canonical form. The theoretical results are illustrated through examples.

Keywords: dynamic systems; Jordan canonical form; faults; diagnosis; estimation



Citation: Sergiyenko, O.; Zhirabok, A.; Mercorelli, P.; Zuev, A.; Filaretov, V.; Tyrsa, V. Jordan Canonical Form for Solving the Fault Diagnosis and Estimation Problems. *Technologies* **2023**, *11*, 72. <https://doi.org/10.3390/technologies11030072>

Academic Editor: Yiqi Liu

Received: 16 April 2023

Revised: 27 May 2023

Accepted: 31 May 2023

Published: 3 June 2023



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1. Introduction

Different canonical forms of dynamic systems play an important role in solving different theoretical and practical problems, see, for example, refs. [1–4]. They facilitate the simplification of solution processes and enable simple algorithms. In particular, to solve fault diagnosis and estimation problems, an identification canonical form (ICF) is used [1,5].

Another popular canonical form is the Jordan canonical form (JCF); it uses design interval observers [2,6–12] and analyzes error correction properties in discrete-time systems [13]. The matrix of the JCF, under an appropriate choice of the eigenvalues, is Hurwitz and Metzler, which means that its non-diagonal elements are non-negative. Such properties guarantee that the interval observer generates lower and upper bounds of the state vector for systems with uncertainties. An analysis of the JCF has shown its capability to facilitate stability and simplify the construction of disturbance-insensitive observers, reducing their dimensionality.

The main contribution of this paper lies in the development of methods that apply the JCF to solve the problems related to diagnostic and sliding mode observers, virtual sensors, and interval observers for linear and nonlinear systems; these methods are presented in forms that are more general than those found in [1,2,7,14] and similar papers. Unlike the known methods, the suggested approach is based on the reduced-order model derived from the original system, which is insensitive or has minimal sensitivity to the disturbance. This allows obtaining the observers and sensors of less dimensions, reducing the impact of disturbances on the accuracy of diagnosis and estimation results.

Such problems will be solved for systems described by nonlinear models

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + G\Psi(x(t), u(t)) + Ff(t) + Dd(t), \\ y(t) &= Cx(t) + w(t),\end{aligned}\quad (1)$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^l$ are the vectors representing the state, control, and output; A , B , C , G , and D are the known constant matrices; matrix F and function $f(t) \in R$ describe faults. If faults are absent, $f(t) = 0$, if a fault occurs, then $f(t)$ becomes an unknown bounded function of time; $d(t) \in R^p$ is the disturbance, it is assumed that $d(t)$ is an unknown bounded function of time, $\|d(t)\| \leq d_*$; $w(t)$ is the measurement noise; it is assumed that $w(t) \in R^l$ is an unknown bounded function of time: $\|w(t)\| \leq w_*$; $\Psi(x, u)$ is the nonlinear term:

$$\Psi(x, u) = \begin{pmatrix} \varphi_1(P_1x, u) \\ \vdots \\ \varphi_q(P_qx, u) \end{pmatrix},$$

P_1, \dots, P_q are the known constant matrices, $\varphi_1, \dots, \varphi_q$ are nonlinear functions. It is assumed that the function $G\Psi(x, u)$ is bounded for all $x \in X$ and $u \in U$ and it satisfies the Lipschitz condition with respect to x uniformly for t and u :

$$\|G(\Psi(x, u) - \Psi(x', u))\| \leq N\|x - x'\|,$$

where $N > 0$ is a constant.

Consider the initially linear systems when $G = 0$.

2. Diagnostic Observer Design

All of the considered problems are based on the reduced order model of the system (1), which has a minimal dimension and is insensitive to the disturbance, as described by the equations

$$\begin{aligned}\dot{x}_*(t) &= A_*x_*(t) + B_*u(t) + J_*y(t), \\ y_*(t) &= C_*x_*(t),\end{aligned}\quad (2)$$

where $x_* \in R^k$, with $k < n$, is the vector of state, $y_* \in R$, A_* , B_* , J_* , and C_* are matrices to be determined. One may say that model (2) is a simplified version of the original system.

The diagnostic observer is based on this model and generates the residual $r(t) = R_*y(t) - y_*(t)$ used to make a decision about the faults. We assume in this section that $w(t) = 0$; if $w(t) \neq 0$, an adaptive threshold for $r(t)$ can be used [1].

As usual, we assume that the relation $x_*(t) = \Phi x(t)$ is true, where Φ is a constant matrix to be determined. It is known [5,15] that they satisfy the following equations

$$\Phi A = A_*\Phi + J_*C, \quad R_*C = C_*\Phi, \quad \Phi B = B_*. \quad (3)$$

As usual, to construct model (2), the matrices A_* and C_* are sought in ICF

$$A_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad C_* = (1 \ 0 \ 0 \ \dots \ 0). \quad (4)$$

This form enables obtaining simple equations for the matrices that describe the model (2) [5]; it also ensures the stability of the observer through feedback $Kr(t) = (k_1 \ k_2 \ \dots \ k_k)^T r(t)$ and desirable eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, which are assumed to be negative and different.

We suggest specifying the matrix A_* as purely diagonal JCF

$$A_* = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix}. \quad (5)$$

It is known that the model with A_* and C_* in the form (4) and feedback $Kr(t)$ can be transformed into the model with A_* (5) with $\lambda_i < 0$ to ensure stability.

Instead of applying such a transformation, we will use the JCF form of the matrix F_* in (2). In this case, the equation $\Phi A = A_*\Phi + J_*C$ is presented in the form of k -independent equations:

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} C, \quad i = 1, 2, \dots, k, \quad (6)$$

where Φ_i and J_{*i} are i -th rows of the matrices Φ and J_* , respectively. An additional condition $\Phi_i D = 0$ (insensitivity to the disturbance) can be taken into account as follows. Introduce the matrix D_0 of maximal rank, such that $D_0 D = 0$. Then, $\Phi = N D_0$ for matrix N . As a result, (6) can be rewritten as

$$(N_i - J_{*i}) \begin{pmatrix} D_0(A - \lambda_i I_n) \\ C \end{pmatrix} = 0, \quad i = 1, 2, \dots, k, \quad (7)$$

where I_n is the identical $n \times n$ -matrix.

Matrices R_* and C_* can be obtained from $R_* C = C_* \Phi$, and rewritten in the form

$$(R_* - C_*) \begin{pmatrix} C \\ \Phi \end{pmatrix} = 0. \quad (8)$$

This equation has a solution if and only if

$$\text{rank} \begin{pmatrix} \Phi \\ C \end{pmatrix} < \text{rank}(\Phi) + \text{rank}(C). \quad (9)$$

To design the diagnostic observer, one chooses $\lambda_1 < 0$ and finds from (7) the row $\Phi_1 = N_1 D_0$ of the matrix Φ for which condition $\Phi_1 F \neq 0$ of sensitivity to the fault is satisfied. Then one has to find from (7) the minimum number of rows Φ_i corresponding to $\lambda_i < 0$ for which condition (9) is satisfied with Φ_1 . The matrices B_* , J_* , R_* , and C_* can be determined using (3), (7), and (8), respectively. Since the matrix (5) with $\lambda_i < 0$ is stable, there is no need to use feedback.

Remark 1. If (7) or (8) has no solution, the model insensitive to the disturbance cannot be designed; in this case, one has to use the robust method described in Section 7.

3. Virtual Sensor Design

Different sensors are an integral part of modern complex technical systems. They are used, in particular, to measure the components of the state vector in order to address control and fault diagnosis problems. Clearly, the greater the number of components that are measured, the easier it becomes to obtain simpler solutions. The use of additional physical sensors may result in extra expenses and cannot always be realized in practice. In addition, such sensors are not highly reliable. In this case, virtual sensors are of interest; moreover, they can be used for replacing faulty physical sensors. In practice, virtual sensors are used to solve different problems, particularly for fault detection, isolation, data recovery, and fault-tolerant control [16,17]. In [1,14,16–18], virtual sensors were constructed using the Luenberger observer; in [1,14], and similar papers, the authors considered full dimensions. In [19], the problem of designing virtual sensors with minimal dimensions, capable of

estimating a prescribed linear function of a nonlinear system, has been solved using the ICF approach.

The use of the JCF enables a further reduction in dimensions compared to the ICF because the JCF ensures stability itself. As a result, the virtual sensor becomes simpler when compared to papers such as [1,14], and similar papers. Assuming $w(t) = 0$ and $d(t) = 0$, we consider the general problem of estimating the variable $z(t) = Mx(t)$, where the known matrix is M . This problem can be viewed as the design of a virtual sensor that estimates the variable $z(t)$. We assume that $\Phi D = 0$ and describe such a sensor by

$$\begin{aligned}\dot{x}_*(t) &= A_*x_*(t) + B_*u(t) + J_*y(t), \\ z(t) &= C_zx_*(t) + Qy(t),\end{aligned}\quad (10)$$

where C_z and Q are matrices to be determined. It follows from $z(t) = Mx(t)$ and (10)

$$M = C_z\Phi + QC = (C_z \ Q) \begin{pmatrix} \Phi \\ C \end{pmatrix}. \quad (11)$$

This equation has a solution if and only if

$$\text{rank} \begin{pmatrix} \Phi \\ C \end{pmatrix} = \text{rank} \begin{pmatrix} \Phi \\ C \\ M \end{pmatrix}. \quad (12)$$

To design the virtual sensor, the value $\lambda_i < 0$ and rows $\Phi_i = N_iD_0$ in (7) must be such that the matrix Φ with the minimum number of rows satisfies condition (12), then matrices B_* , C_z , and Q can be determined using (3) and (11), respectively.

Remark 2. If $w(t) \neq 0$ or the solution of (7) does not satisfy condition (12), the accuracy of the estimation diminishes. In these cases, lower and upper bounds for the variable $z(t)$ can be generated by the interval observer.

4. Interval Observer Design

In recent years, different kinds of interval observers have been presented for many types of models, including linear and non-linear continuous-time [10,20–22], discrete-time [9,23,24], time delay [2,6], and algebraic differential [6]. They have also been successfully applied to solve many real-time life problems [11,16]. Exhaustive reviews can be found in [2,7].

In this paper, the interval observers were designed to estimate the prescribed linear function $z(t) = Mx(t)$ of the state vector $x(t)$. Such observers are based on the JCF-reduced order model of the original system of minimal dimensions and are insensitive or minimally sensitive to disturbances. This allows for reducing the interval width and the dimensions of the observer when compared to papers such as [2,7], and in similar works, where the full vector $x(t)$ is estimated.

From the above, it follows that interval observers can be considered as generalized virtual sensors when $w(t) \neq 0$ or $\Phi D \neq 0$ for Φ satisfying condition (12).

Given the variable $z(t) = Mx(t)$, we construct an interval observer with minimal dimensions generating lower $\underline{z}(t)$ and upper $\bar{z}(t)$ bounds, such that $\underline{z}(t) \leq z(t) \leq \bar{z}(t)$ for all $t \geq 0$, where, by using an approach similar to [2], for two vectors $x^{(1)}, x^{(2)} \in R^n$ or matrices $P_1, P_2 \in R^{n \times n}$, the relations $x^{(1)} \leq x^{(2)}$ and $P_1 \leq P_2$ are understood element-wise. In [8], the interval observer used for estimating the vector $x(t)$ is based on the stable observer, which is then transformed into the JCF. In contrast, the matrix A_* in our approach is sought in the JCF.

When the requirement for insensitivity to the disturbance is not present, Equation (7) can be simplified as follows:

$$(\Phi_i - J_{*i}) \begin{pmatrix} A - \lambda_i I_n \\ C \end{pmatrix} = 0, \quad i = 1, 2, \dots, k, \quad (13)$$

and model (2) takes the form

$$\begin{aligned} \dot{x}_*(t) &= A_* x_*(t) + B_* u(t) + J_* C x(t) + D_* d(t), \\ z(t) &= C_z x_*(t) + Q y(t), \end{aligned} \quad (14)$$

where $D_* = \Phi D$. The interval observer is given by

$$\begin{aligned} \dot{\underline{x}}_*(t) &= A_* \underline{x}_*(t) + B_* u(t) + J_* y(t) - |J_*| E_k w_* - |D_*| E_k d_*, \\ \dot{\bar{x}}_*(t) &= A_* \bar{x}_*(t) + B_* u(t) + J_* y(t) + |J_*| E_k w_* + |D_*| E_k d_*, \\ \underline{z}(t) &= C_z \underline{x}_*(t) + Q y(t), \\ \bar{z}(t) &= C_z \bar{x}_*(t) + Q y(t), \\ \underline{x}_*(0) &= \underline{x}_{*0}, \quad \bar{x}_*(0) = \bar{x}_{*0}, \end{aligned} \quad (15)$$

where $\underline{x}_{*0} \leq x_*(0) \leq \bar{x}_{*0}$ for the known $\underline{x}_{*0}, \bar{x}_{*0}$; the elements of the matrix $|A|$ are absolute values of the corresponding elements of A ; $E_k = (1 \ 1 \ \dots \ 1)^T$.

Theorem 1. If $C_z \geq 0$ and $\underline{x}_*(0) \leq x_*(0) \leq \bar{x}_*(0)$, then for the interval observer (15), $\underline{z}(t) \leq z(t) \leq \bar{z}(t)$ holds.

Proof of Theorem 1. Using an approach similar to [2], we introduce the estimation errors

$$\begin{aligned} \underline{e}_*(t) &= x_*(t) - \underline{x}_*(t), \quad \bar{e}_*(t) = \bar{x}_*(t) - x_*(t), \\ \underline{e}_z(t) &= z(t) - \underline{z}(t), \quad \bar{e}_z(t) = \bar{z}(t) - z(t). \end{aligned} \quad (16)$$

It follows from $\underline{x}_*(0) \leq x_*(0) \leq \bar{x}_*(0)$ that $\underline{e}_*(0) \geq 0$ and $\bar{e}_*(0) \geq 0$. Taking into account (14) and (15), one obtains:

$$\begin{aligned} \dot{\underline{e}}_*(t) &= A_* \underline{e}_*(t) + J_* (C x(t) - y(t)) + D_* d(t) + |J_*| E_k w_* + |D_*| E_k d_* \\ &= A_* \underline{e}_*(t) - J_* w(t) + D_* d(t) + |J_*| E_k w_* + |D_*| E_k d_*, \\ \dot{\bar{e}}_*(t) &= A_* \bar{e}_*(t) - J_* (C x(t) - y(t)) - D_* d(t) + |J_*| E_k w_* + |D_*| E_k d_* \\ &= A_* \bar{e}_*(t) + J_* w(t) - D_* d(t) + |J_*| E_k w_* + |D_*| E_k d_*. \end{aligned} \quad (17)$$

Note that in (17) $\pm J_* w(t) + |J_*| E_k w_* \geq 0$ and $\pm D_* d(t) + |D_*| E_k d_* \geq 0$ hold for all $t \geq 0$, and the non-diagonal elements of the matrix A_* are non-negative. Solutions of this system under $\underline{e}_*(0) \geq 0$ and $\bar{e}_*(0) \geq 0$ are non-negative element-wise; that is, $\underline{e}_*(t) \geq 0$ and $\bar{e}_*(t) \geq 0$ for all $t \geq 0$ [2]. This with (16) gives $\underline{x}_*(t) \leq x_*(t) \leq \bar{x}_*(t)$. Since $z(t) = C_z x_*(t) + Q y(t)$, it follows from (16)

$$\begin{aligned} \underline{e}_z(t) &= C_z x_*(t) + Q y(t) - (C_z \underline{x}_*(t) + Q y(t)) = C_z \underline{e}_*(t), \\ \bar{e}_z(t) &= C_z \bar{x}_*(t) + Q y(t) - (C_z x_*(t) + Q y(t)) = C_z \bar{e}_*(t). \end{aligned}$$

As a result, under $\underline{e}_*(t) \geq 0, \bar{e}_*(t) \geq 0$, and $C_z \geq 0$, one obtains $\underline{e}_z(t) \geq 0, \bar{e}_z(t) \geq 0$, which is equivalent to $\underline{z}(t) \leq z(t) \leq \bar{z}(t)$. \square

To construct the interval observer that estimates the variable $z(t) = Mx(t)$, one has to find a minimum number of solutions of (13) with $\lambda_i < 0$, which form the matrix Φ that satisfies condition (12); moreover, matrices J_*, B_* , and D_* need to be calculated.

Remark 3. To reduce the width of the interval $(\underline{z}(t), \bar{z}(t))$, one has to find the matrices Φ and J_* from (7), which ensures $D_* = 0$ or uses the robust solution in Section 7.

Remark 4. It can be seen that if $C_z \leq 0$, bounds should be calculated as

$$\bar{z}(t) = C_z \bar{x}_*(t) + Qy(t), \quad \underline{z}(t) = C_z \bar{x}_*(t) + Qy(t).$$

The suggested approach to the interval estimation of the variable $z(t) = Mx(t)$ can be used for a similar estimation of the vector $x(t)$, as follows. Assume that the matrix C is of maximal rank and

$$C = (C_0 \ 0), \quad y(t) = C_0 x^{(1)}(t) + w(t), \quad x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix},$$

C_0 is a nonsingular matrix. We introduce

$$\begin{aligned} y(t) &= y(t) - E_l w_*, & \bar{y}(t) &= y(t) + E_l w_*, \\ \underline{x}^{(1)}(t) &= C_0^{-1} y(t), & \bar{x}^{(1)}(t) &= C_0^{-1} \bar{y}(t). \end{aligned} \quad (18)$$

Then

$$\begin{aligned} \underline{e}^{(1)}(t) &= x^{(1)}(t) - \underline{x}^{(1)}(t) = C_0^{-1}(y(t) - w(t)) - C_0^{-1} y(t) = C_0^{-1}(E_l w_* - w(t)), \\ \bar{e}^{(1)}(t) &= \bar{x}^{(1)}(t) - x^{(1)}(t) = C_0^{-1} \bar{y}(t) - C_0^{-1}(y(t) - w(t)) = C_0^{-1}(E_l w_* + w(t)). \end{aligned}$$

Assuming that $C_0^{-1} \geq 0$, one obtains from $E_l w_* \pm w(t) \geq 0$ that $\underline{e}^{(1)}(t) \geq 0$ and $\bar{e}^{(1)}(t) \geq 0$; as a result, $\underline{x}^{(1)}(t) \leq x^{(1)}(t) \leq \bar{x}^{(1)}(t)$.

Thus, the variable $x^{(1)}(t)$ under $C_0^{-1} \geq 0$ is estimated by (18); the variable $x^{(2)}(t)$ can be estimated by using an approach similar to the observer (15). Note that the disturbance $d(t)$ does not affect the estimation (18).

Remark 5. Condition $C_0^{-1} \geq 0$ is satisfied in practical important cases when components of the vector $x^{(1)}(t)$ are measured by sensors and $C_0 = C_0^{-1} = I_l$.

5. Sliding Mode Observer Design

Sliding mode observers (SMOs) provide a solution for the problem of state and fault estimation in dynamic systems. The design methods for such observers have been developed in various works, including [25–33] for different classes of systems and fault-tolerant control [34]. A distinguishing feature of these and similar papers is that when constructing SMOs, some limitations are imposed on the original system; for example, in references [26,35], and similar papers, the system should be a minimum phase and satisfy the matching condition. In [30], this condition is relaxed and only requires detectability. Moreover, SMOs are constructed based on the original system. As a result, sliding mode observers are of full order. The slightest conditions were obtained in [36] based on the reduced-order model of the original system with different sensitivities to faults and disturbances. Such a model in [36] is realized in the ICF.

The suggested approach below is a modification of what was presented in [36], and is based on the JCF. Assume that $w(t) = 0$. Since the JCF is stable, the additional requirements, including the minimum phase or detectability [26,30], are not imposed upon the original system in the suggested approach.

As noted in Section 2, by solving Equation (7), one can construct a minimal-dimension model that is insensitive to disturbances:

$$\begin{aligned} \dot{x}_*(t) &= A_* x_*(t) + B_* u(t) + J_* y(t) + F_* f(t), \\ y_*(t) &= C_* x_*(t), \end{aligned} \quad (19)$$

where $F_* = \Phi F$. Since one-dimensional subsystems in the JCF are independent of each other, the sliding mode observer is one-dimensional as well. One has to choose $\lambda_1 < 0$ and find a row $\Phi_1 = N_1 D_0$ from (7) that satisfies the condition for sensitivity to the fault,

as well as $\Phi_1 = R_*C$ for some matrix R_* . Note that these conditions are equivalent to $\text{rank}(CF) = \text{rank}(F)$ [26]. Matrix J_* can be determined using (7); finally, matrix $B_* = \Phi_1 B$ is calculated.

As a result, model (19) becomes

$$\begin{aligned}\dot{x}_*(t) &= \lambda_1 x_*(t) + B_* u(t) + J_* y(t) + F_* f(t), \\ y_*(t) &= x_*(t) = R_* y(t),\end{aligned}\quad (20)$$

where $x_* = \Phi_1 x$. The sliding mode observer is of the form

$$\begin{aligned}\dot{\hat{x}}_*(t) &= \lambda_1 \hat{x}_*(t) + B_* u(t) + J_* y(t) - k_1 v(t), \\ \hat{y}_*(t) &= \hat{x}_*(t),\end{aligned}\quad (21)$$

where $v(t) = \text{sign}(e(t))$, $e(t) = \hat{y}_*(t) - R_* y(t)$, $k_1 > 0$.

The estimation error $e(t)$ is described by

$$\dot{e}(t) = \lambda_1 e(t) - k_1 v(t) - F_* f(t). \quad (22)$$

Since $f(t)$ is the bounded function and $\|v(t)\| = 1$, then $\|k_1 v(t) + F_* f(t)\| \leq g_0$ for some $g_0 > 0$. It is known that $e(t)$ is bounded as well and $\|e(t)\| \leq \delta$ for some $\delta > 0$.

Theorem 2. The observer (21) estimates the function $f(t)$ as

$$\hat{d}(t) = -F_*^{-1} k_1 v_{eq}(t), \quad (23)$$

where $v_{eq}(t)$ is the so-called equivalent output injection signal representing the average behavior of the discontinuous function $v(t)$. According to [26], we use as $v_{eq}(t)$ the continuous approximation $v_{eq}(t) = e(t)/(|e(t)| + \varepsilon)$, where ε is a small positive scalar.

Proof of Theorem 2. We can prove that by selecting a suitable observer gain k_1 , $e = 0$ in finite time and sliding motion are achieved. We consider the Lyapunov function $V(t) = e^2(t)$ and find its derivative with respect to time, taking into account (22):

$$\dot{V}(t) = 2e(t)\dot{e}(t) = 2e(t)(\lambda_1 e(t) - k_1 v(t) - F_* f(t)).$$

Since $v = \text{sign}(e)$, then $ek_1 v = k_1 |e|$ and

$$\dot{V} \leq 2|e|(-k_1 + \lambda_1 \delta + f_* \|F_*\|).$$

If $k_1 > \lambda_1 \delta + f_* \|F_*\|$, then $\dot{V} < 0$, and the sliding motion is achieved, which is $e = \dot{e} = 0$ in finite time. Then it follows from (22) that the fault is estimated by (23). \square

When the measurement noise $w(t) \neq 0$, the main result remains the same, but the requirement for the coefficient k_1 becomes more rigorous. In this case, Equation (22) for the error $e(t)$ is supplemented by $J_* w(t)$:

$$\dot{e}(t) = \lambda_1 e(t) - k_1 v(t) - F_* f(t) + J_* w(t).$$

As a result, the additional term appears in the derivative of the function \dot{V} :

$$\dot{V} \leq 2|e|(-k_1 + \lambda_1 \delta + f_* \|F_*\| + w_* \|J_*\|),$$

and the formula for k_1 changes: $k_1 > \lambda_1 \delta + f_* \|F_*\| + w_* \|J_*\|$. The existence of the measurement noise means that the estimation (23) becomes approximate:

$$\hat{d}(t) \approx -F_*^{-1} k_1 v_{eq}(t).$$

6. Nonlinear Systems

If the original system is nonlinear, the nonlinear term supplements the right-hand side of model (2)

$$G_* \Psi_*(x_*, y, u) = \begin{pmatrix} \varphi_{i_1}(P_{*1,i_1}x_* + P_{*2,i_1}y, u) \\ \dots \\ \varphi_{i_k}(P_{*1,i_k}x_* + P_{*2,i_k}y, u) \end{pmatrix},$$

where $P_{*1,i_1}, P_{*2,i_1}, \dots, P_{*1,i_k}, P_{*2,i_k}$ are matrices to be determined, $G_* = \Phi G$; $G_* \Psi_*$ is a function $G_* \Psi$ in which the vector x is replaced by x_* and y according to $P_i x = P_{*1,i}x_* + P_{*2,i}y, i = i_1, \dots, i_k$; the numbers i_1, \dots, i_k are nonzero columns of the matrix G_* .

The relations (3) are supplemented by

$$\Phi G = G_*, \quad P_i = (P_{*1,i} \ P_{*2,i}) \begin{pmatrix} \Phi \\ C \end{pmatrix}, \quad i = i_1, \dots, i_k. \quad (24)$$

The second one has a solution if and only if

$$\text{rank} \begin{pmatrix} \Phi \\ C \end{pmatrix} = \text{rank} \begin{pmatrix} \Phi \\ C \\ P_i \end{pmatrix}, \quad i = i_1, \dots, i_k. \quad (25)$$

To construct the nonlinear term, one finds from (7) the minimum number of the matrix Φ rows with $\lambda_i < 0$; set $G_* := \Phi G$, calculate the product $G_* \Psi(x, u)$, and check (25). If it is satisfied, find the matrices $P_{*1,i}$ and $P_{*2,i}$, and $i = i_1, \dots, i_k$ from (24). If (25) is not satisfied, find another solution of (7) with the former or incremented value k . If (25) is not satisfied for all k , the model insensitive to the disturbance does not exist. In this case, one may use a robust approach with minimal sensitivity to the disturbance; see [33] and Section 7.

The main problem in the nonlinear case involves the stability of the observer. Consider only the case where the nonlinear term does not affect stability, ensured by the JCF of the matrix A_* . Introduce the error $e_*(t) = \Phi x(t) - x_*(t)$. It follows from (1) and model (2) with the nonlinear term

$$\begin{aligned} \dot{e}_* &= A_* e_* + G_* \Psi(x, u) - G_* \Psi_*(x_*, y, u) \\ &= A_* e_* + G_* \Psi_*(\Phi x, y, u) - G_* \Psi_*(x_*, y, u). \end{aligned}$$

Since the function $G\Psi(x, u)$ is satisfied the Lipschitz condition, then $G_* \Psi_*(\Phi x, y, u)$ is satisfied in such a condition as well,

$$\|G_*(\Psi_*(\Phi x, y, u) - \Psi_*(x_*, y, u))\| \leq N_* \|e_*\|,$$

where $N_* > 0$. Since the matrix A_* is stable, symmetric positive definite matrices exist, L_* and W_* , such that $A_*^T L_* + L_* A_* = -W_*$. In [4], the Lyapunov function $V(t) = e_*^T(t) L_* e_*(t)$ was considered, showing that $\dot{V}(t) < 0$, which is that the observer is stable, if

$$2N_* \lambda_{\max}(L_*) < \lambda_{\min}(W_*), \quad (26)$$

where $\lambda_{\max}(L_*)$ and $\lambda_{\min}(W_*)$ are the maximal and minimal eigenvalues of matrices L_* and W_* , respectively. Assume that condition (26) is satisfied; therefore, the stability of the observer is ensured by the JCF matrix A_* .

The demand for stability is important for nonlinear diagnostic observers and virtual sensors. For sliding mode observers, the nonlinear term can be taken into account by using an approach similar to the measurement noise. The coefficient k_1 must satisfy the conditions

$$k_1 > \lambda_1 \delta + f_* \|F_*\| + w_* \|J_*\| + \delta N_*,$$

where N_* represents the Lipschitz constant.

For the interval observer, in addition to the demand for stability, the function $G_* \Psi_*(x_*, y, u)$ should exhibit monotonicity with respect to x , uniformly for y and u , in the sense of the relation " \leq ":

$$x_* \leq x'_* \Rightarrow G_* \Psi_*(x_*, y, u) \leq G_* \Psi_*(x'_*, y, u).$$

This is necessary to prove $\underline{e}_*(t) \geq 0, \bar{e}_*(t) \geq 0$ for all $t \geq 0$.

Since the variable $y(t)$ is subject to measurement noise and appears in the nonlinear term, the right-hand sides of (15) should be supplemented by $\pm k_* w_*$, where the coefficient k_* can be determined experimentally.

7. Robust Solution

If conditions (9) and (12) for virtual sensors and interval observers) are not satisfied, the model invariant with respect to the disturbance cannot be constructed, and one has to use robust methods. For the ICF, such a method is described in [33]. It involves minimizing the Frobenius norm $\|\Phi L\|_F$, which described the contribution of the disturbance in the model, and is realized through the singular value decomposition of a matrix [33,37].

This approach cannot be used for the JCF since rows of the matrix Φ determined by (13) are independent of each other. To solve the problems of robust diagnostics and sliding mode observer design, one needs to choose a certain $\lambda_1 < 0$ and find from (13) a row $\Phi_1 = N_1 D_0$ where condition $\Phi_1 F \neq 0$ of the sensitivity to the fault is satisfied. Then one has to find from (13) the minimum number of rows Φ_i for which condition (9) is satisfied with Φ_1 . If different solutions of (13) are possible, a choice has to be made to minimize the norm $\|\Phi_i D\|_F$. To construct virtual sensors and interval observers, the condition (12) should be satisfied for the minimum number of Equation (13) solutions.

As a result, it can be concluded that condition (7) of invariance with respect to the disturbance is simple; on the other hand, when condition (9) is not satisfied, one has to use more complex rules to minimize the contribution of the disturbance in the model. Moreover, the JCF restricts the possibility of such minimization. An analysis has shown that, in this case, the ICF is preferable, it enables minimizing the contribution of the disturbance more effectively. This is not true for the interval observer since the transformation of the ICF model, designed on the basis of the singular value decomposition into the JCF model, may increase the contribution of the disturbance.

8. Example

Consider the nonlinear control system

$$\begin{aligned} \dot{x}_1 &= a_1 u_1 / \vartheta_1 - a_2 a_4 \sqrt{x_1 - x_2}, \\ \dot{x}_2 &= a_3 u_2 / \vartheta_2 + a_2 a_4 \sqrt{x_1 - x_2} - a_5 \sqrt{x_2 - x_3} + \rho, \\ \dot{x}_3 &= a_5 \sqrt{x_2 - x_3} - a_6 \sqrt{x_3 - \vartheta_7}, \\ y_1 &= x_2 + w_1, \quad y_2 = x_3 + w_2, \end{aligned} \quad (27)$$

where $a_4 = \vartheta_4 \sqrt{2\vartheta_8} / \vartheta_1$, $a_5 = \vartheta_5 \sqrt{2\vartheta_8} / \vartheta_2$, and $a_6 = \vartheta_6 \sqrt{2\vartheta_8} / \vartheta_3$. Equation (27) represents the model of the well-known example of a three-tank system (Figure 1), where x_1 , x_2 , and x_3 correspond to the liquid levels in the tanks. The initial conditions are $x_1(0) = 3$, $x_2(0) = 1$, and $x_3(0) = 0$. It is assumed that areas of the cross-sections of tanks ϑ_1 , ϑ_2 , and ϑ_3 , areas of the cross-sections ϑ_4 , ϑ_5 , and ϑ_6 of pipes, and the controls u_1 and u_2 are such that $x_1(t) \geq x_2(t) \geq x_3(t) \geq 0$ for all $t \geq 0$. This assumption is made to simplify model (27) since the main purpose of the example is to show how the JCF can be used to solve the interval observer design problems.

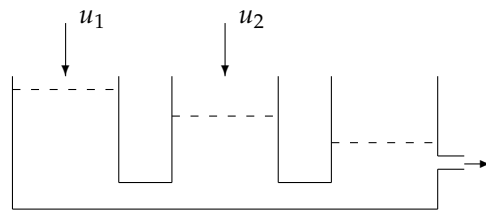


Figure 1. Three-tank system.

Clearly, model (27) is described by matrices, where $F = 0$. To overcome this difficulty, we can transform (27) by introducing formal addends $-(x_1 - x_2) + (x_1 - x_2)$, $((x_1 - x_2) - (x_2 - x_3)) - ((x_1 - x_2) - (x_2 - x_3))$, and $(x_2 - x_3 - x_3) - (x_2 - x_3 - x_3)$ in the first, second, and third equations, respectively. Then $-(x_1 - x_2)$ is added to the linear part and $(x_1 - x_2)$ to the nonlinear part; other addends are considered analogously. As a result, the system is described by matrices and nonlinearities as follows:

$$\begin{aligned} A &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ G &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} -\sqrt{P_1 x} + P_1 x \\ -\sqrt{P_2 x} + P_2 x \\ -\sqrt{P_3 x} + P_3 x \end{pmatrix}, \\ P_1 &= (1 \ -1 \ 0), \quad P_2 = (0 \ 1 \ -1), \quad P_3 = (0 \ 0 \ 1). \end{aligned}$$

Calculate interval estimates for $x(t)$.

It follows from Section 4 that $x^{(1)} = (x_2, x_3)^T$, $x^{(2)} = x_1$. Since $C_0 = I_2$, we obtain

$$\begin{aligned} \underline{x}_2(t) &= y_1(t) - w_{*1}(t), & \bar{x}_2(t) &= y_1(t) + w_{*1}(t), \\ \underline{x}_3(t) &= y_2(t) - w_{*2}(t), & \bar{x}_3(t) &= y_2(t) + w_{*2}(t). \end{aligned}$$

To estimate $z(t) = x_1(t)$, set $M := (1 \ 0 \ 0)$, $D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Equation (7) becomes

$$(N_i - J_{*i}) \begin{pmatrix} -1 - \lambda & 1 & 0 \\ 0 & 1 & -2 - \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Set $\lambda := -1$ and obtain $k = 1$ and $N = J_* = (1 \ 0)$; as a result, $\Phi = (1 \ 0 \ 0)$, $B_* = (1 \ 0)$, $D_* = 0$, $C_z = 1$, $Q = 0$, $G_* = (1 \ 0 \ 0)$. Clearly, the condition (25) is satisfied, and $P_{*1} = (1 \ -1 \ 0)$. Model (14) is of the form

$$\begin{aligned} \dot{x}_*(t) &= u_1(t) - \sqrt{x_*(t) - y_1(t)}, \\ z(t) &= x_*(t). \end{aligned}$$

Clearly, the model is stable; the function “ $\sqrt{\cdot}$ ” is monotonic. The interval observer estimating the variable $x_1(t)$ is given by

$$\begin{aligned} \dot{\underline{x}}_1(t) &= u_1(t) - \sqrt{\underline{x}_1(t) - y_1(t)} - k_* w_{*1}(t), \\ \dot{\bar{x}}_1(t) &= u_1(t) - \sqrt{\bar{x}_1(t) - y_1(t)} + k_* w_{*1}(t). \end{aligned}$$

Note that the approach suggested in [2] gives the observer of dimension 6, and its estimates contain the disturbance $\rho(t)$, which is absent in our approach.

For the simulation, assume for simplicity that $a_1 = a_2 = \dots = a_6 = 1$, $\vartheta_7 = 0$; $|\rho| \leq \rho_*$, $|w_1| \leq w_{*1}$, $|w_2| \leq w_{*2}$. Then take $u_1(t) = 0.5$ and $u_2(t) = 0.2$, $x_1(0) = 3$, $\underline{x}_1(0) = 1$, and $\bar{x}_1(0) = 4$. The simulation results with $D[\rho] = 0.6$, $D[w_1] = 0.1$, $k_* w_{*1} = 0.3$ and $\rho = 0$,

$D[w_1] = 0.2$, $k_*w_{*1} = 0.6$ are shown in Figures 2 and 3, respectively, where the graphs of the functions $x_1(t)$, $\underline{x}_1(t)$, and $\bar{x}_1(t)$ are presented. Clearly, the product k_*w_{*1} is greater, and the interval $\underline{x}_1(t), \bar{x}_1(t)$ is wider. Moreover, the value of the disturbance ρ does not affect the interval width; this confirms that the model is decoupled from the disturbance.

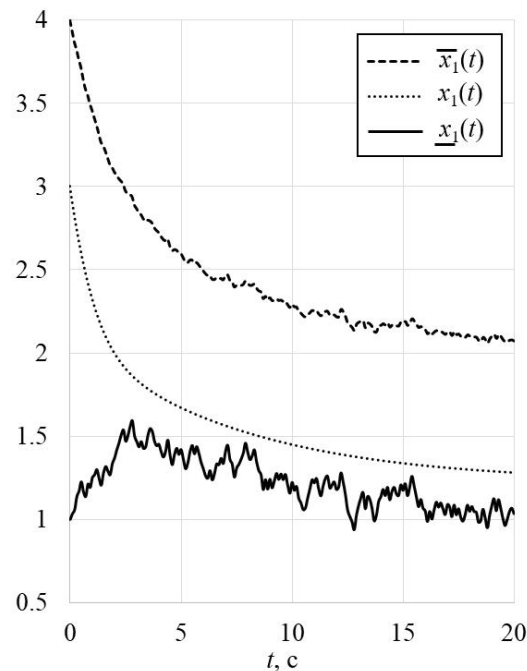


Figure 2. Graphs of the functions $x_1(t)$, $\underline{x}_1(t)$, and $\bar{x}_1(t)$ with $k_*w_{*1} = 0.3$ and $D[\rho] = 0.6$.

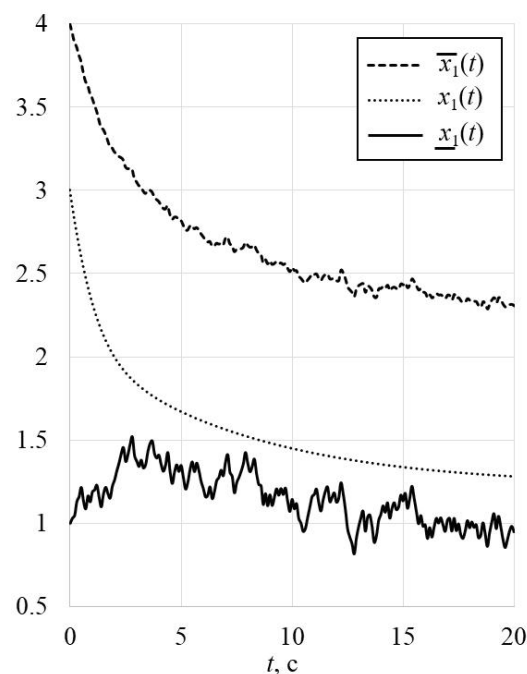


Figure 3. Graphs of the functions $x_1(t)$, $\underline{x}_1(t)$, and $\bar{x}_1(t)$ with $k_*w_{*1} = 0.6$ and $\rho = 0$.

9. Discussion

The problems of designing diagnostic observers, virtual sensors, interval observers, and sliding mode observers based on the Jordan canonical form have been addressed and resolved. The methods for solving these problems have been developed for both linear and nonlinear systems, taking into account external disturbances and measurement noise. The

observers are based on the reduced-order model of the original system; they are invariant with respect to the disturbance or have minimal sensitivity to the disturbance. It was shown that when the invariance, with respect to the disturbance, can be achieved, the JCF allows reducing the dimensions of observers and virtual sensors, making the design procedure simpler when compared with the ICF. On the other hand, when the invariance, with respect to the disturbance, is impossible, and a robust solution is used, the JCF has to use more complex rules to minimize the impact of the disturbance in the model. Moreover, the JCF restricts the possibility of this minimization. An analysis has shown that, in this case, ICF is preferable; it enables minimizing the contribution of the disturbance more effectively. This is not true for the interval observer since the transformation of the ICF model, designed on the basis of the singular value decomposition into the JCF model, may increase the contribution of the disturbance. Theoretical results are illustrated through the well-known tree-tank system. Future work will investigate the JCF stability in the system with external disturbance.

Author Contributions: Conceptualization, A.Z. (Alexey Zhirabok) and A.Z. (Alexander Zuev); methodology, O.S. and V.F.; software, V.T.; validation, P.M. and V.F.; formal analysis, V.T. and P.M.; investigation, A.Z. (Alexander Zuev); resources, O.S.; data curation, A.Z. (Alexey Zhirabok); writing—original draft preparation, A.Z. (Alexey Zhirabok); writing—review and editing, A.Z. (Alexey Zhirabok) and A.Z. (Alexander Zuev); visualization, V.T.; supervision, P.M.; project administration, O.S.; funding acquisition, P.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Russian Science Foundation, project no. 23-29-00191 (<https://rscf.ru/en/project/23-29-00191/>, accessed on 1 January 2023).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

ICF	identification canonical form
JCF	Jordan canonical form
SMO	sliding mode observers

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