International Journal of *Geo-Information*

Article

Secant Cylinders Are Evil—A Case Study on the Standard Lines of the Universal Transverse Mercator and Universal Polar Stereographic Projections

Krisztián Kerkovits

Institute of Cartography and Geoinformatics, ELTE Eötvös Loránd University, 1117 Budapest, Hungary; kerkovits@map.elte.hu

Abstract: The literature usually calls downscaled versions of basic conformal map projections "secant", referring to conceptual developable map surfaces that intersect the reference frame. However, recent studies pointed out on the examples of various mappings of the sphere that this model may lead to incorrect conclusions. In this study, we examine the paradigm of secant surfaces for two popular map projections of the ellipsoid, the UTM (Universal Transverse Mercator) and the UPS (Universal Polar Stereographic) projections. Results will show that ellipsoidal map projections can exhibit further anomalies. To support the shift to a paradigm avoiding developable map surfaces, this study recommends the new term *reduced* map projection with a proper and simple definition to be used instead of *secant* map projections.

Keywords: UTM projection; UPS projection; secant map projection; map projections of the ellipsoid of revolution; terminology of map projections

check for updates

Citation: Kerkovits, K. Secant Cylinders Are Evil—A Case Study on the Standard Lines of the Universal Transverse Mercator and Universal Polar Stereographic Projections. *ISPRS Int. J. Geo-Inf.* **2024**, *13*, 56. https://doi.org/10.3390/ijgi13020056

Academic Editors: Wolfgang Kainz and Florian Hruby

Received: 22 December 2023
Revised: 2 February 2024
Accepted: 8 February 2024
Published: 13 February 2024

1. Introduction

There is a common belief in the field of map projections that so-called "tangent" conformal map projections become "secant" after a uniform scaling, and they become distortion-free where some *conceptual* map surface intersects the reference frame [1,2]. This is usually considered true without scientific claims. As we will see later, this concept is false and can lead us to wrong calculations.

The UTM (Universal Transverse Mercator) and the UPS (Universal Polar Stereographic) are two well-known map projections based on ellipsoid WGS84. They are commonly used for topographic maps throughout the world. Being the official coordinate system of NATO, they are defined by DMA (Defense Mapping Agency) manuals. Ref. [3] gives a textual description of these maps, while [4] is a technical document containing almost only formulae and numerical data necessary to reproduce these maps. The latter document has been revised [5], but the changes introduced are insignificant in the context of this study.

There is, however, a significant discrepancy between these two standardization documents. Refs. [4,5] introduce these maps as uniformly downscaled versions of the Gauss–Krüger and the conformal azimuthal projections. On the other hand, ref. [3] states that they are secant maps: "The cylinder of projection is modified by reducing its elliptical dimensions and making it secant to the ellipsoid, intersecting the ellipsoid along lines parallel to the central meridian. [...] Note that the scale of the projection at the lines of secancy is exact". "For use with the Universal Polar Stereographic grid, a scale factor of 0.994 is applied at the origin [the opposite pole] to lower the plane of projection to intersect the sphere [sic] at approximately 81°07′ latitude".

We should note that [3] is now obsolete and has been superseded by [6]. The new document does not classify these map projections as "secant". We could have forgotten it completely if recent papers and textbooks did not base their description on the obsolete [3]. Among others, Refs. [7–10] all depict UTM (or UPS) as map projections onto a secant cylinder (or plane) and state that the map projection is distortion-free along lines where the cylinder (or plane) intersects the ellipsoid of revolution. Ref. [11] uses the same paradigm in the opposite direction: map projection surfaces are “elevated” to local topography, hoping that the enlarged linear scale would compensate for length reduction between high terrain and the reference ellipsoid.

The usage of developable surfaces in teaching map projections is often justified by the term *conceptual* [2,12,13], but this makes scientific verification impossible. It is usually said that this makes understanding map projections easier [14]. Nevertheless, the author of the present paper doubts whether a student can understand the overly complicated derivation of the Mercator projection in [14] (projecting onto a cylinder using a toroidal light source, unrolling, and applying an arbitrary scaling function on the planar map) more easily than a correct mathematical derivation, not to mention the misconceptions detailed in the next paragraphs.

All the arguments above assume that map projections are distortion-free along secant lines. However, Ref. [1] analyzed a few azimuthal map projections and concluded that only the secant stereographic projection has a standard line along the intersection line. (Note: Ref. [1] states that this happens only if some conditions on the constants are fulfilled, but one can easily check that those conditions are always met by the stereographic projection provided that the projection plane is secant and the point of perspective is at the opposite pole.) Unfortunately, Ref. [1] examined the sphere only. It turns out in this paper that even the conformal azimuthal projection does not have this special property if the Earth is considered an ellipsoid of revolution. We should note that even secant perspective projections (e.g., gnomonic, orthographic) are not distortion-free along their secant parallel in general.

It is also claimed without proof that secant perspective projections are merely downscaled versions of tangent map projections. This assumption hides behind confusing downscaled maps with secant maps. It is indeed true for far-side perspectives onto a plane, but secant near-side perspectives are rather *upscaled* compared to their tangent versions, and the orthographic projection does not change when making it secant. For cylindrical map projections, the situation is even more complicated: Ref. [15] derived a series of a secant perspective cylindricals in which standard and secant parallels coincide. Although the author of that study did not highlight that this is an anomaly, his perspective cylindrical map changed the shape of its frame after making it secant. This is definitely not a uniform scaling. Thus, downscaled maps and secant maps seem to be completely different concepts.

There are other anomalies of secant developable surfaces, e.g., a conic projection with more than two standard parallels [16]. We should note that these anomalies do not contribute to errors in positioning because the mapping equations are fortunately not derived using such misleading secant developable surfaces.

This paper shows further anomalies in the current paradigm indicating that the UTM and the UPS map projections are not secant. We will see that assuming them to be secant will result in false locations for their standard lines. The arguments and the literature cited in the introduction showed that developable surfaces are not good explanations for distortions found in map projections of the sphere. The present paper extends this scope to spheroidal map projections.

2. Materials and Methods

To show that the usual visualizations of the UTM and UPS are misleading, we will calculate their distortion-free locations in two different ways. First, we will believe figures found in textbooks and calculate the position of the line where the projection surface intersects the ellipsoid of revolution. Next, we repeat all calculations the hard way: we will check exact values of linear scale and solve equations to see where it reaches unity. As the map projections examined in this study are conformal, linear scale is assumed to be independent of direction [17].

Whenever ellipsoidal quantities are needed, the WGS84 ellipsoid is used. Its semi-major axis (equatorial radius) is $a = 6{,}378{,}137$ m, and its flattening is $f = 1/298.257223563$. Its eccentricity is calculated as $e = \sqrt{2f - f^2}$. For convenience, we will use the following notation for the radii of curvature:

$$\nu = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \tag{1}$$

$$\mu = \frac{a\left(1 - e^2\right)}{\left(1 - e^2 \sin^2 \varphi\right)^{3/2}} \tag{2}$$

Whenever the prime-vertical radius of curvature ν is subscripted, it indicates substitution for φ. i.e., ν_t indicates that φ_t should be substituted for φ.

To avoid errors coming from series truncation, UTM coordinates are calculated by the closed formulae found in [18] (these equations were first derived in [19]). This is a multi-step process. First, we determine the complex number w from the following equation (Jacobi elliptic functions and elliptic integrals are expressed in terms of the elliptic modulus):

$$\operatorname{artanh} \sin \varphi - e \operatorname{artanh}(e \sin \varphi) + \mathrm{i}\lambda = \operatorname{artanh}[\operatorname{sn}(w; e)] - e \operatorname{artanh}[e \operatorname{sn}(w; e)] \tag{3}$$

Then, we can calculate the Gauss–Krüger coordinates as a complex number z by ($\mathcal{E}$ is the Jacobi epsilon function):

$$z = a\left(1 - e^2\right) \int_0^w \operatorname{nd}^2(u; e)\, \mathrm{d}u = a\left[\mathcal{E}(w; e) - e^2 \operatorname{cd}(w; e) \operatorname{sn}(w; e)\right] \tag{4}$$

UTM coordinates are Gauss–Krüger coordinates rescaled and translated:

$$y_{UTM} + \mathrm{i}x_{UTM} = 0.9996z + \mathrm{i}\,500{,}000 \tag{5}$$

The special functions above were evaluated by Wolfram Mathematica 13.3 (notebook available as Supplementary Material).

For the UPS projection, we will not need actual calculation of the coordinates, only the distance ϱ_{UPS} from the origin is needed. From [20]:

$$\varrho_{UPS} = 0.994 \frac{2a}{\sqrt{1 - e^2}} \left(\frac{1 - e}{1 + e}\right)^{e/2} \sqrt{\frac{1 - \sin \varphi}{1 + \sin \varphi}} \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi}\right)^{e/2} \tag{6}$$

3. Results

3.1. The UTM Projection with a Secant Cylinder

The exact shape of the UTM cylinder is somewhat ambiguous. The literature agrees that it is not a circular cylinder. However, some European textbooks like [7] think that it intersects the ellipsoid along two meridians (Figure 1). While it is possible to construct such an elliptical cylinder, it introduces a serious problem: in the so-called "tangent" version, the central meridian is a standard line. The North Pole is on this standard line, so it is also distortion-free. The reduction factor for the so-called "secant" variant is 0.9996, so this should be the new linear scale here. However, Ref. [7] states that the secant meridians are standard lines, which means that linear scale in the North Pole is unit. Unfortunately, $0.9996 \neq 1$, so we have a contradiction.

Another possible shape is described by the DMA manual [3]: "The cylinder of projection is modified by reducing its elliptical dimensions and making it secant to the ellipsoid, intersecting the ellipsoid along lines parallel to the central meridian. For the Universal Transverse Mercator grid this condition establishes, in one 6° zone, two lines of secancy approximately 180,000 m east and west of the central meridian. These lines of secancy, in effect, allow a more congruous relationship between ellipsoid and map distances than that of the central meridian tangency." This means that the base of a cylinder is a downscaled bimeridian. We will see, however, that the standard lines of UTM are not exactly parallel to the central meridian. Ref. [3] states: "By projecting the surface of the ellipsoid onto the cylinder [. . .] in the same manner as for the Mercator Projection, the Transverse Mercator Projection is developed on the surface of the cylinder, which is then opened and flattened". It would make the decision for the proper shape of this *conceptual* cylinder easier if we could unambiguously roll the image of the UTM or Gauss–Krüger projections to form such a cylinder. Unfortunately, this is not possible at all. Unrolled cylindrical surfaces are always rectangles (or stripes), but not the shape of the UTM/GK projections. Figure 2 shows the Gauss–Krüger projection of the full ellipsoid. To create this figure, coordinates were transformed by GeographicLib 2.0 [21], which uses closed formulae equivalent to Equations (3) and (4), so the image of areas far from the central meridian does not suffer from truncation error.

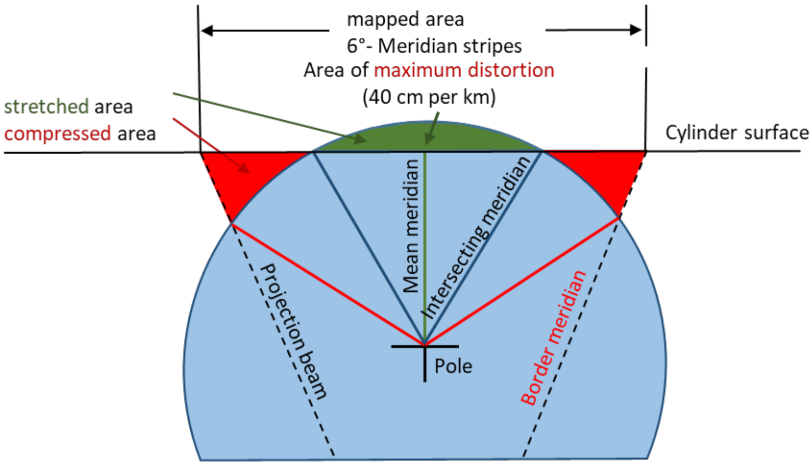

Figure 1. This figure from [7] is the poster child of all false stereotypes regarding the UTM: it shows the map surface as a secant cylinder, which intersects the ellipsoid along meridians rather than small ellipses. The figure assumes that secant lines are standard lines. Labels "stretched" and "compressed" are apparently swapped. Furthermore, the reference to "projection beam" erroneously suggests that it is a perspective projection.

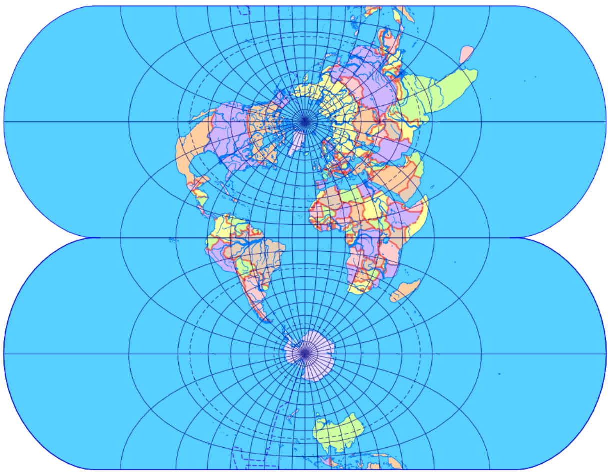

Figure 2. The Gauss–Krüger map projection of the whole ellipsoid WGS84 cannot be rolled to form a cylindrical surface.

Nevertheless, we will continue our erroneous calculations based on the secant cylinder described in [3]. Let us first see the left part of Figure 3. This shows the ellipsoid from the top view. The base of the cylinder is assumed to be an ellipse, which is a bimeridian scaled by 0.9996. Therefore, we can draw a right triangle of hypotenuse a (the equatorial radius of the ellipsoid). The leg adjacent to the secant longitude λ_s is the equatorial radius scaled by 0.9996. From this:

$$\cos\lambda_s = \frac{0.9996 \cdot a}{a} \tag{7}$$

$$\lambda_s = 1°37'14.244'' \tag{8}$$

The ellipsoidal distance of the secant line from the central meridian at the Equator is:

$$d_{sE} = a\lambda_s = 180{,}406.97 \text{ m} \tag{9}$$

Substituting $\varphi = 0°$ and $\lambda = \lambda_s$ into the formulae of the UTM projection, we obtain the same distance on the projected plane:

$$s_{sE} = x_{UTM} - 500{,}000 = 180{,}159.02 \text{ m} \tag{10}$$

This is quite close to the 180 km value usually found in the literature. We should, however, note that our intentionally wrong calculation brought us to the first contradiction, because according to the figure, the following equation should hold, but actually it fails:

$$s_{sE} \neq a\sin\lambda_s = 180{,}382.92 \text{ m} \tag{11}$$

Although the discrepancy is only ca. 200 m, such a large difference cannot be a result of a round-off error; it indicates that using a secant cylinder is not a correct approach.

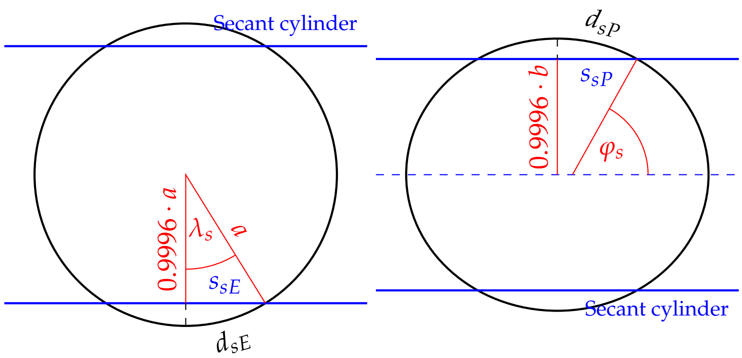

Figure 3. Incorrect calculation of the location where UTM should be distortion-free.

Now, we switch to the right part of Figure 3 (side view). We can calculate the distance z from the equatorial plane to the secant latitude φ_s (this is the same as the coordinate z of φ_s in a geocentric Cartesian coordinate system) as $z = (1-e^2)\nu_s \sin\varphi_s$ [22]. This is equal to the polar radius $b = a\sqrt{1-e^2}$ scaled by 0.9996:

$$0.9996 \cdot a\sqrt{1-e^2} = \frac{a\left(1-e^2\right)}{\sqrt{1-e^2\sin^2\varphi_s}}\sin\varphi_s \tag{12}$$

$$\varphi_s = 88°23'5.307'' \tag{13}$$

The distance along the meridian ellipse between the pole and the secant cylinder is:

$$d_{sP} = \int_{\varphi_s}^{90°} \mu \, \mathrm{d}\varphi = 180{,}406.81 \text{ m} \tag{14}$$

The fact that d_{sP} is 16 cm shorter than d_{sE} is not an inaccuracy. The reason is that meridians are a bit shorter than the Equator. We can also calculate the map distance by substituting $\varphi = \varphi_s$ and $\lambda = 90°$ into the mapping equations of the UTM:

$$s_{sP} = x_{UTM} - 500{,}000 = 180{,}158.54 \text{ m} \tag{15}$$

This shows that the intersection ellipse of the cylinder and the ellipsoid is not mapped to an exactly straight line, although the difference is subtle: the mapped image of the ellipse is only ca. 0.5 m closer to the vertical axis near the pole compared to its equatorial distance.

3.2. The UTM Projection with Exact Formulae

Instead of the incorrect derivation above, one should calculate the distortion-free locations of the UTM projection by closed formulae. According to [18], the local linear scale of the Gauss–Krüger projection along the Equator is $\mathrm{nd}(v; \sqrt{1-e^2})$, where $v = \Im(w)$ and w is the complex number introduced by (3). The linear scale of the UTM is this multiplied by 0.9996. On the standard line, it is exactly one:

$$0.9996\,\mathrm{nd}\left(v; \sqrt{1-e^2}\right) = 1 \tag{16}$$

This equation determines the value of v. Knowing that $\Re(w) = 0$ on the Equator, we can substitute $w = \mathrm{i}v$ into (4) to get the true distance s_{tE} of the standard line from the central meridian:

$$s_{tE} = x_{UTM} - 500{,}000 = 179{,}553.99 \text{ m} \tag{17}$$

This shows that the error of the method using the secant cylinder is more than 600 m. We can also calculate the true longitude of the intersection point between the standard line and the Equator by substituting the same w into (3):

$$\lambda_t = 1°36'54.678'' \tag{18}$$

To calculate the true location of the standard line near the poles, we must check the linear scale of the Gauss–Krüger projection where $\lambda = 90°$ (the standard line reaches its Northernmost point). Ref. [18] states that it is $a/(\nu\cos\varphi)\,\mathrm{sc}(v; \sqrt{1-e^2})$. From this, at the standard line of the UTM:

$$0.9996\frac{\sqrt{1-e^2\sin^2\varphi_t}}{\cos\varphi_t}\,\mathrm{sc}\left(v; \sqrt{1-e^2}\right) = 1 \tag{19}$$

As both φ_t and v are unknown, we need another equation to solve them. At $\lambda = 90°$, $\Re(w) = K(e)$, where K is the complete elliptic integral of the first kind. Substituting $w = K(e) + \mathrm{i}v$ into (3):

$$\operatorname{artanh}\sin\varphi_t - e\operatorname{artanh}(e\sin\varphi_t) = \Re(\operatorname{artanh}[\mathrm{sn}(K[e] + \mathrm{i}v; e)] - e\operatorname{artanh}[e\,\mathrm{sn}(K[e] + \mathrm{i}v; e)]) \tag{20}$$

Simultaneous solution of (19) and (20) yields:

$$\varphi_t = 88°22'45.740'' \tag{21}$$

The solution of the system also gave a value for v, which can be substituted into (4) to gain the true distance s_{tP} of the standard line from the pole:

$$s_{tP} = x_{UTM} - 500{,}000 = 180{,}765.60 \text{ m} \tag{22}$$

This brings us to interesting conclusions. The first is that the error of the calculation based on a secant cylinder is again more than 600 m, but now in the opposite direction. Apparently, the distance of the distortion-free line (s_{tE} and s_{tP}) varies significantly more (ca. 1.2 km) than was expected after calculating on a secant cylinder (ca. 0.5 m). Furthermore, the trend of the line is exactly opposite to what was expected based on the secant cylinder: in fact, it is closest to the central meridian at the Equator and the farthest near the pole.

It is interesting that this unexpected phenomenon is acknowledged only by [23] (p. 144), but that work reports the location of the standard line 200 m too far from the central meridian due to the inaccurate truncated series that was used for calculating the linear scale. We should point out that [23] completely avoided the usage of cylinders for ellipsoidal projections: this was a key for drawing correct conclusions.

3.3. The UPS Projection Based on a Secant Plane

To calculate the position of the UPS projection plane, we fall into a contradiction at the very beginning. Unlike the cylinder, the plane does not have a unique axis that could serve as the invariant point or line of the similarity transform (reduction). In secant perspective azimuthal projections, the point of perspective is invariant. However, the conformal azimuthal projection of the ellipsoid is not a perspective projection [24], so we must determine some *conceptual* point of perspective. Thus, the UPS is only *conceptually* secant: given its reduction factor (0.994), one can basically pick such a suitable invariant point of the scaling so that the new position of an originally tangent plane can be at any desired location, even completely outside the ellipsoid of revolution.

Putting this not-so-subtle problem aside, we will assume that the point of perspective is in the opposite pole of the ellipsoid (Figure 4). This choice was made because this is where most textbooks or even the DMA manual [3] would place the point of perspective. We must warn the reader that one must not confuse the conformal azimuthal projection of the ellipsoid (which is not perspective) with the stereographic projection of the sphere (which is definitely conformal and perspective).

As this point is invariant under the scaling and the original distance to the tangent plane is $2b$, the new distance is reduced to $0.994 \cdot (2b)$. This means that the distance from the equatorial plane is $z = (2 \cdot 0.994 - 1)b = 0.988a\sqrt{1-e^2}$. On the other hand, $z = (1-e^2)\nu_s \sin\varphi_s$.

$$0.988 \cdot a\sqrt{1-e^2} = \frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\varphi_s}} \sin\varphi_s \tag{23}$$

$$\varphi_s = 81°8'39.094'' \tag{24}$$

This result is quite surprising, as the DMA manual states that such a plane should intersect the ellipsoid WGS84 at 81°7′ [3].

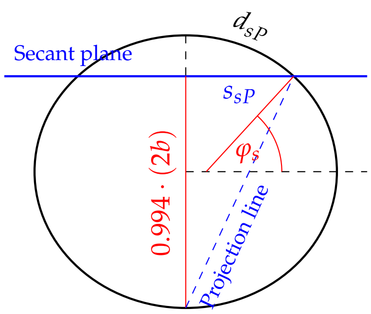

Figure 4. Incorrect calculation of the location where UPS should be distortion-free.

Following our previous calculations regarding the UTM, we will calculate the distance of the secant latitude both on the ellipsoid and the projection plane.

$$d_{sP} = \int_{\varphi_s}^{90°} \mu \, \mathrm{d}\varphi = 989{,}061.10 \text{ m} \tag{25}$$

$$s_{sP} = \varrho_{UPS} = 985{,}088.42 \text{ m} \tag{26}$$

According to the figure, s_{sP} is also the radius of parallel φ_s, so it should be equal to $\nu_s \cos\varphi_s$ [22]. In fact:

$$s_{sP} \neq \frac{a\cos\varphi_s}{\sqrt{1-e^2\sin^2\varphi_s}} = 985{,}127.99 \text{ m} \tag{27}$$

The difference is again not too large, but it indicates that UPS is not stereographic (not a perspective projection).

3.4. The UPS Projection with Exact Formulae

The linear scale of an azimuthal projection in east–west direction is $\varrho/(\nu\cos\varphi)$. As UPS is conformal, linear scale is independent of direction. Thus, we need to solve:

$$\varrho_{UPS}\frac{\sqrt{1-e^2\sin^2\varphi_t}}{a\cos\varphi_t} = 1 \tag{28}$$

$$\varphi_t = 81°6'52.264'' \tag{29}$$

This latitude is significantly closer to the value reported by [3] and is equal to the value found in [6]. Ref. [3] apparently ignored the fact that the distortion-free latitude and the secant latitude differ almost by two minutes. This shows that secant and standard parallels do not coincide even for the conformal azimuthal projection of the ellipsoid, only on the conformal azimuthal (stereographic) projection of the sphere. To express this difference in metres, we calculate:

$$d_{tP} = \int_{\varphi_t}^{90°} \mu\,\mathrm{d}\varphi = 992{,}374.82 \text{ m} \tag{30}$$

$$s_{tP} = \varrho_{UPS} = 988{,}402.08 \text{ m} \tag{31}$$

This shows that the *conceptual* secant parallel indicates the location of the standard parallel almost 3.5 km closer to the North Pole than it should be. Such a large difference is surprising provided that the stereographic projection of the sphere is perspective, and its standard parallel coincides with its secant parallel.

4. Discussion

To the best knowledge of the author, this study is the first to calculate the location of the standard line in the UTM projection using closed formulae. These results are not influenced at all by series truncation, so they can be regarded as accurate.

Furthermore, this study added a new anomaly to the current widespread paradigm of map projections: the *conceptual* secant lines do not coincide with standard lines of map projection even in the most popular conformal map projections. Conclusions based on a secant cylinder could even perfectly contradict reality: the secant cylinder made us believe that the standard line of the UTM is closer to the central meridian at high latitudes, but in fact, its closest point lies on the Equator. The difference between the *conceptual* and real location of standard lines reached the magnitude of kilometres. Furthermore, the placement of the UPS plane is completely arbitrary; we just assumed a *conceptual* point of perspective in the opposite pole, but such a point does not exist. Picking another suitable *conceptual* point of perspective, one could easily demonstrate that the UPS plane touches or is even completely outside the ellipsoid of revolution.

These, added to the anomalies already indicated in previous studies (and mentioned in the introduction), indicate the urgent need of a paradigm shift in the field of map projections, which has fortunately been started by a few recent publications. Thus, the UTM and UPS projections should never be called secant map projections.

Whoever explains these map projections as mappings onto a secant plane or cylinder implicitly agrees with all the following claims:

- If one transforms a perspective map projection from tangent placement to secant placement, it suffers only a uniform scaling. Thus, all secant map projections are merely downscaled projections (so we have a kind of justification to regard downscaled non-perspective map projections as secant).
- Standard lines and secant lines always (or at least usually) coincide (otherwise we have no hint where to pick *conceptual* secant lines of non-perspective projections).
- UTM and UPS are either perspective or their distortion pattern can be explained by the geometry of a secant surface.
- The image of the UTM projection can be easily rolled to form a cylinder.

As seen before either in this paper or in the literature cited, *none* of the above statements are true, so conclusions based on these arguments can be misleading. The author warns the reader again that despite its name, UPS is not a stereographic (perspective) projection, and conformal azimuthal projections of the ellipsoid are not perspective. Intentionally confusing the UPS and the stereographic projections led us to wrong results in this study.

We should emphasize that the new standards [5,6] can perfectly and accurately describe these map projections without referring to secant developable surfaces.

If a secant cylinder is not a proper visualization of the UTM, is it even a *cylindrical* map projection? New paradigms necessitate new terminology. A good starting point to classify map projections without using developable surfaces is found in [2]. However, those definitions are based on the image of the pseudograticule (also known as the metagraticule). Th metagraticule is defined only for a sphere, so the definitions of [2] are not usable at all for ellipsoidal map projections. A good candidate solution for the problem is given by [24]: "The trick is to take the equations of the given projection [. . .] and to set the eccentricity of the spheroid everywhere equal to zero in these equations".

Answering our original question, if one sets $e = 0$ in the closed formulae of the UTM, we yield the transverse Mercator projection of the sphere [18]. Now, we have to determine if there exists such a rotation of the spherical graticule that makes formulae simpler. Such a rotation exits, so we should rotate the graticule back to the normal aspect [24]. We have now the Mercator projection, which is (according to the definitions of [2]) indeed *cylindrical*.

Let us put another question: If UTM and UPS are not secant, how should one properly call such downscaled map projections? Again, a new paradigm needs new definitions and a new way of thinking. First, we should observe that uniform scaling is not generally used for map projections. There is no point in enlarging or reducing an equal-area or an equidistant projection: they would lose their special distortion properties. This is why it is strange that, e.g., ref. [25] emphasizes that Romanian Cassini and Bonne projections "had a scale factor at origin equal to unity". Uniform scaling is a meaningful operation only to *conformal* mappings.

It is quite rare that the minimal linear scale of a conformal projection would be zero. (Such an example is the Littrow projection.) If the minimum of the linear scale is positive, then there exists a uniform scaling that transforms the minimum of the linear scale to one. This can be regarded as the "natural size" of the conformal map. Then, the *conformal* map may be called a *reduced map projection* if its minimum linear scale is between zero and one. It is an *enlarged map projection* if its minimum linear scale is greater than one. The latter term could cover "elevated" mappings found, e.g., in [11]. The *factor of reduction* equals to the minimum of the linear scale.

The definitions above should cover everyday map projections used for surveying. We should note that the term *reduced* is not meant to be used for the Albers projection: it is not *conformal*. In general, the *factor of reduction* coincides with the linear scale at the natural origin (as expected), but there is an exception: the minimum linear scale of the oblique Thomas "stereographic" projection found in [20] is not exactly at the origin of the coordinate system, and thus it is *reduced* even if the scale at the origin is unit. Fortunately, this map projection is not used in practical surveying, and the double stereographic projection used in the Netherlands and New Brunswick or the Roussilhe projection used in Poland and Romania do not have this problem.

As a closing thought, one might come up with the idea to create a UTM-like map projection on the surface of a secant cylinder. However, such an idea seems to be counterproductive even for the sphere. Ref. [26] mentions in a footnote that a rudimentary approximation of the Mercator projection is possible using a perspective map projection onto a cylinder. The result is not exactly conformal, but maximal angular deviation reaches 1° at latitude 29° N and S, and 3° only at latitude 73° N and S. Its transverse aspect is easily formulated. The problem starts when trying to make the cylinder secant: it turns out that the secant version of this perspective map projection is not simply downscaled, but some more complicated transformation occurs, similar to the one found in the secant maps of [15]. This transformation dilutes the approximate conformality of the mapping. Furthermore, the secant version will not have a standard line along its secant line (there are no distortion-free points at all in this map). This experience shows that *secant* and *reduced* map projections are unrelated concepts.

5. Conclusions

In this paper, we found that the exact location of the standard line and the secant lines of the UTM and UPS projections do not coincide. In the case of the UPS, the term *secant* turned out to be completely arbitrary, as the location of the *conceptual* projection plane depends solely on some non-existing point of perspective. The use of secant cylinders could not explain the shape of the standard line in the UTM projection.

Instead of using misleading terms, a new paradigm should be followed. The fundamentals of it have already been laid in previous studies. Based on those definitions eliminating the references to developable surfaces, the author recommends using the new term *reduced* instead of the old term *secant*. The term *reduced* has an easy-to-understand definition that expresses the point of downscaled maps and can be used only for *conformal* map projections. The term *secant* should only be used for *perspective* projections in the future.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/ijgi13020056/s1; Mathemtica source code.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are all available in the article or its supplementary materials.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Lapaine, M. Standard Parallel and Secant Parallel in Azimuthal Projections. *Kartogr. Geoinformacije* **2017**, *16*, 72–88. Available online: http://kig.kartografija.hr/index.php/kig/article/view/783/1458 (accessed on 8 January 2024).
2. Lapaine, M.; Frančula, N. Map Projections Classification. *Geographies* **2022**, *2*, 274–285.
3. Hager, J.W.; Fry, L.L.; Jacks, S.S.; Hill, D.R. *Datums, Ellipsoids, Grids, and Grid Reference Systems*; Technical Report TM8358.1; Defense Mapping Agency Hydrographic/Topographic Center: Washington, DC, USA, 1990. Available online: https://apps.dtic.mil/sti/citations/ADA247651 (accessed on 8 January 2024).
4. Hager, J.W.; Behensky, J.F.; Drew, B.W. *The Universal Grids: Universal Transverse Mercator (UTM) and Universal Polar Stereographic (UPS)*; Technical Report TM8358.2; Defense Mapping Agency Hydrographic/Topographic Center: Washington, DC, USA, 1989. Available online: https://apps.dtic.mil/sti/citations/ADA266497 (accessed on 8 January 2024).
5. GWG World Geodetic System and Geomatics Focus Group. *The Universal Grids and the Transverse Mercator and Polar Stereographic Map Projections*; Technical Report NGA.SIG.0012; National Geospatial-Intelligence Agency (NGA): Springfield, VA, USA, 2014. Available online: https://nsgreg.nga.mil/doc/view?i=4056 (accessed on 8 January 2024).
6. GWG World Geodetic System and Geomatics Focus Group. *Universal Grids and Grid Reference Systems*; Technical Report NGA.STND.0037; National Geospatial-Intelligence Agency (NGA): Springfield, VA, USA, 2014. Available online: https://nsgreg.nga.mil/doc/view?i=4057 (accessed on 8 January 2024).
7. Hey, A.; Bill, R. Tutorial: Map Projections and Coordinate Reference Systems. Universität Rostock, 2019. Available online: https://learn.opengeoedu.de/tutorials/OGE-Tutorial_KNE_Koordinaten-en.pdf (accessed on 8 January 2024).
8. Buchroithner, M.F.; Pfahlbusch, R. Geodetic grids in authoritative maps—New findings about the origin of the UTM Grid. *Cartogr. Geogr. Inf. Sci.* **2017**, *44*, 186–200. [CrossRef]

9. Usery, E.L.; Finn, M.P.; Mugnier, C.J. Coordinate Systems and Map Projections. In *Manual of Geographic Information Systems*; ASPRS Publications: Washington DC, USA, 2009; Chapter 8, pp. 87–112.
10. Yu, W.; Chen, S.; Zhang, S. Implement of Overhauser magnetometer coordinate transformation software. *IOP Conf. Ser. Mater. Sci. Eng.* **2019**, *563*, 052060. [CrossRef]
11. Dennis, M.L. Ground Truth: Low Distortion Map Projections for Engineering, Surveying, and GIS. In *Pipelines 2016*; American Society of Civil Engineers: Reston, VA, USA, 2016; pp. 857–869. [CrossRef]
12. Kraak, M.J.; Roth, R.E.; Ricker, B.; Kagawa, A.; Sourd, G.L. *Mapping for a Sustainable World*; The United Nations: New York, NY, USA, 2021. Available online: https://www.un.org/geospatial/sites/www.un.org.geospatial/files/MappingforaSustainableWorld20210124.pdf (accessed on 8 January 2024).
13. Snyder, J.P. *Flattening the Earth: Two Thousand Years of Map Projections*; The University of Chicago Press: Chicago, IL, USA, 1993.
14. Heitzler, M.; Bär, H.R.; Schenkel, R.; Hurni, L. The Light Source Metaphor Revisited—Bringing an Old Concept for Teaching Map Projections to the Modern Web. *ISPRS Int. J. -Geo-Inf.* **2019**, *8*, 162. [CrossRef]
15. Lapaine, M. Gall Stereographic Projection and its Generalization. *Geod. List.* **2023**, *77*, 1–10.
16. Lapaine, M. Conic Projections with Three or More Standard Parallels. *Proc. ICA* **2021**, *4*, 64. [CrossRef]
17. Tissot, N.A. *Mémoire sur la repréSentation des Surfaces et les Projections des Cartes Géographiques*; Gauthiers-Villars: Paris, France, 1881.
18. Lee, L.P. *Conformal Projections Based on Elliptic Functions*; York University: Toronto, UK, 1976.
19. Ludwig, K. Die der transversalen Mercatorkarte der Kugel entsprechende Abbildung des Rotationsellipsoids. *J. Die Reine Angew. Math.* **1943**, *185*, 193–230. [CrossRef]
20. Snyder, J.P. *Map Projections—A Working Manual*; US Government Printing Office: Washington, DC, USA, 1987; Volume 1395. [CrossRef]
21. Karney, C.F.F. Transverse Mercator with an accuracy of a few nanometers. *J. Geod.* **2011**, *85*, 475–485. [CrossRef]
22. Smith, W.H.F. Direct conversion of latitude and height from one ellipsoid to another. *J. Geod.* **2022**, *96*, 36. [CrossRef]
23. Osborne, P. *The Mercator Projections*; Technical Report; University of Edinburgh: Edinburgh, UK, 2013. [CrossRef]
24. Wray, T. *The Seven Aspects of a General Map Projection*; York University: Toronto, ON, Canada, 1974.
25. Mugnier, C.J. Grids & Datums: România. *Photogramm. Eng. Remote Sens.* **2001**, *67*, 545–548. Available online: https://www.asprs.org/wp-content/uploads/2012/05/05-2001-romania.pdf (accessed on 8 January 2024).
26. Braun, C. Ueber zwei neue geographische Entwurfsarten. *Wochenschr. Astron. Meteorol. Geogr.* **1867**, *11*, 269–272.