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Contribution of Lienard-Wiechert Potential to the Electron Broadening of Spectral Lines in Plasmas

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Received: 28 December 2017; Accepted: 4 February 2018; Published: 8 February 2018

Abstract: Lienard-Wiechert or retarded electric and magnetic fields are produced by moving electric charges with respect to a rest frame. In hot plasmas, such fields may be created by high velocity free electrons. The resulting electric field has a relativistic expression that depends on the ratio of the free electron velocity to the speed of light in vacuum c. In this work, we consider the semi-classical dipole interaction between the emitter ions and the Lienard-Wiechert electric field of the free electrons and compute its contribution to the broadening of the spectral line shape in hot and dense plasmas.

Keywords: Stark effect; electron broadening; Lienard-Wiechert; retarded interaction; relativistic collision operator

1. Introduction

Line profiles and shifts are used to determine plasma parameters, especially in astrophysics where alternative methods (such as interferometry or Thomson scattering) are not possible. Doppler and pressure broadening (Stark broadening) are typically the two dominant mechanisms and we focus on the latter. In a number of hot astrophysical plasmas, electrons may be energetic enough that their thermal energy $K_B T$ (K_B is the Boltzmann constant and T the temperature) can be comparable to the rest mass. For the extreme densities encountered in some astrophysical objects, pressure broadening could dominate; however, for such objects the electrons may become relativistic due to the extreme temperatures and hence it makes sense to check the modifications to the pressure broadening by relativistic effects. Similarly, the laser-produced plasmas may be achieved in both very high densities and very high temperatures: the first leads to the dominance of Stark over Doppler broadening whereas the second leads to relativistic electron velocities. More specifically, plasma spectroscopy is used in a wide range of electron density from 10 particles per cm^3 (interstellar space) to 10^{25} particles per cm^3 (star interiors, inertial confinement fusion) and for temperatures between 10^7 K and 10^{10} K. More precisely, as the Doppler effect is constant for a fixed temperature (about 10^8 K in our case), we can neglect it for densities higher than 10^{20} cm^{-3}. In the present work, we investigate the region corresponding to the particular conditions of plasma: high density and high temperature. Under these conditions, (electron-ion) collisions will be, throughout this work, assumed binary and the dynamics of the electrons will be treated relativistically. Furthermore, in this work we shall use the statistical classical mechanics (not the quantum statistical mechanics as in the Fermi-Dirac distribution) because the condition $\lambda_{th} = h/\sqrt{2\pi m_e K_B T} < N_e^{-1/3}$ (m_e is the electron mass, λ_{th} is the De Broglie thermal length and N_e is the electrons density) is fulfilled in the stated density and temperatures ranges. For example, if $N_e = 10^{24}$ cm^{-3} and $T > 10^7$ K, it is easy to verify that this inequality is correct. This condition means that the wave function extent (λ_{th}) associated with the electron is smaller than the mean distance ($\sim N_e^{-1/3}$) between two free electrons. In the present work, we focus on electron broadening in the impact approximation [1,2]. We thus, reformulate the standard semiclassical collision operator by taking into account the relativistic effects of the Lienard-Wiechert retarded electric field [3] due to the free electron movement. In addition, we assume that the plasma is optically thin (the opacity phenomenon is neglected), for this raison the spectral line shape is not influenced by the absorption process. Furthermore, by neglecting the electrons recombinaison, we assume that the decoupling of the free electrons from the radiation field is satisfied. The units system used here (unless specified) is the CGS system. In many cases in line broadening, fast particles (typically electrons) are described by a collisional approach, while particles of which field have a weak variation during the inverse half width half maximum (HWHM) time scale are considered static and treated via a quasistatic microfield. For many applications, isolated lines have great importance. So, the calculations of the broadening of such a line in a plasma are normally made by using the impact approximation for electrons [1] in the semi-classical version [4], as the ionic contribution is typically negligible.

2. Theoretical Basis of the Electron Broadening

The Stark effect is important in plasmas of high degree of ionization and high temperatures. In all cases presented in the remainder of our work, the Stark effect is dominant. The two most popular approximations in the computations of the electronic collision operator Φ are the dipole approximation and the approximation of the classical path which considers the perturbing electrons in the impact approximation. Our departure point is the expression of the intensity of the spectral line shape [2,5]:

$$I(\omega) = -\frac{1}{\pi}\,\mathrm{Re}\left\{ \boldsymbol{d}_{\alpha\beta}\langle\langle\alpha\beta \left| \left(i\omega - \frac{i\,(H_g - H_e)}{\hbar} + \Phi \right)^{-1} \right| \alpha'\beta'\rangle\rangle \boldsymbol{d}^*_{\alpha'\beta'} \right\} \tag{1}$$

where Φ is the relativistic collision operator (in Hz) which is independent from time and electric micro-field and has matrix elements given by

$$\begin{aligned} \langle\langle\alpha\beta\ |\Phi|\ \alpha'\beta'\rangle\rangle = \sum_{\alpha''} \boldsymbol{r}_{\alpha\alpha''}\boldsymbol{r}_{\alpha''\alpha'}\Phi_d\left(\omega_{\alpha\alpha''},\omega_{\alpha''\alpha'}\right) \\ +\sum_{\beta''}\boldsymbol{r}_{\beta\beta''}\boldsymbol{r}_{\beta''\beta'}\Phi_d\left(\omega_{\beta\beta''},\omega_{\beta''\beta'}\right) - \boldsymbol{r}_{\alpha\alpha'}\,\boldsymbol{r}_{\beta'\beta}\Phi_{int}\left(\omega_{\alpha\alpha'},\omega_{\beta'\beta}\right) \end{aligned} \tag{2}$$

where α, β are the upper and lower levels respectively and $\boldsymbol{r}_{ab}$ ($\boldsymbol{d}_{ab}$ is dipole operator) is the matrix element of position operator of the bounded electron. We now aim to calculate the direct relativistic term Φ_d (in Hz/cm^2) and the relativistic term of interference $\Phi_{int} = 2\Phi_d$ [5]. Before starting this task, let us remind that the criterion of validity of the impact theory, according to Voslamber [6], is not applicable for any pair (ω_1, ω_2). However, for the isolated lines ($\omega_1 = -\omega_2$), this theoretical problem does not arise, and impact theory is valid. Specifically, the study of Φ_d is performed in the case of isolated lines for an ionic emitter and hyperbolic paths for free electrons. This treatment is based on the results obtained under the same conditions in the non-relativistic case. The relativistic collision operator is then given by

$$\Phi_d(\omega_1,\omega_2) = -\tfrac{2\pi N_e\, e^2}{3\hbar^2}\int_0^c v\, f(\beta)\, d\beta \int_{\rho_{\min}}^{\rho_{\max}} \rho_0 d\rho_0 \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, e^{(i\omega_1 t_1 + i\omega_2 t_2)} \left[\boldsymbol{E}_{LW}(t_1)\cdot \boldsymbol{E}_{LW}(t_2)\right] \tag{3}$$

where c is the speed of light in vacuum, v is the initial velocity of the colliding electron and $f(\beta)\, d\beta$ is the distribution of the velocities of Juttner-Maxwell (generalized Maxwell distribution of the velocities when the movement of particles is relativistic) given by

$$f(\beta)\,d\beta = \frac{\gamma^5 \beta^2 d\beta}{\theta K_2(1/\theta)} \exp(-\gamma/\theta) \tag{4}$$

where

$$\theta = \frac{K_B T}{m_e c^2}, \quad \gamma = 1/\sqrt{(1-\beta^2)}, \quad \beta = v/c \tag{5}$$

and $K_2(1/\theta)$ is a Bessel function, T is the temperature, m_e is the electron mass and ρ_0 is the impact parameter, whereas $\rho_{\min}$ and $\rho_{\max}$ are the limits of the last integral (Formula (3)) that will be chosen later. The electric field of Lienard-Wiechert is given by [3]

$$\boldsymbol{E}_{LW}(\boldsymbol{R}, \boldsymbol{\alpha}, t) = -e\frac{(\boldsymbol{\eta}-\boldsymbol{\alpha})(1-\alpha^2)}{k^3 R^2(t')} - \frac{e}{c^2}\frac{\boldsymbol{\eta}}{k^3 R(t')} \times \left\{(\boldsymbol{\eta}-\boldsymbol{\alpha}) \times \frac{d\boldsymbol{v}(t')}{dt'}\right\} \tag{6}$$

then

$$\boldsymbol{\alpha} = \frac{\boldsymbol{v}(t')}{c}, \boldsymbol{\eta} = \frac{\boldsymbol{R(t')}}{R(t')}, t' = t - R(t')/c \tag{7}$$

where the retarded time is given by: $t' = t - \frac{R(t')}{c}$, e is the charge of the electron, $\boldsymbol{R}(t')$ is the electron position vector, and $\boldsymbol{\eta} = \frac{\boldsymbol{R}(t')}{R(t')}$ is a unit vector directed from the position of the moving charge (electron) towards the observation point (where the emitter is located), and k is given by

$$k = \frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 + \boldsymbol{\eta\alpha} \tag{8}$$

The first term of the field (6), the velocity field, goes to the known Coulomb field when $v \to 0$ whereas the second term of the field, is the acceleration field or the radiation field. As the ratio of second term (the radiation field) of the field $\boldsymbol{E}_{LW}$ on the first term is less than v^2/c^2, we can therefore neglect the second and use only the first part of the field in the subsequent development

$$\boldsymbol{E}_{LW}(\boldsymbol{R}, t) \simeq e\left[\frac{(\boldsymbol{\eta}-\boldsymbol{\alpha})(1-\alpha^2)}{k^3 R^2}\right] \tag{9}$$

By using the approximation $1-\alpha^2 \simeq 1$, which is justified in our subsequent studies ($T = 8 \times 10^8$ K, the probable α is about 0.22), therefore the electric field becomes

$$\boldsymbol{E}_{LW}(\boldsymbol{R}, t) = e\left[\frac{(\boldsymbol{\eta}-\boldsymbol{\alpha})}{k^3 R^2}\right] \tag{10}$$

If we neglect the fine structure ($\omega_1 = \omega_2 = 0$), we can write the collision operator as

$$\Phi_d(0,0) = -\frac{\pi N_e e^2}{3\hbar^2}\int_0^c v f(\beta) d\beta \int_{\rho_{\min}}^{\rho_{\max}} \rho_0 d\rho_0 \int_{-\infty}^{+\infty} \boldsymbol{E_{LW}}(t_1) dt_1 \int_{-\infty}^{+\infty} \boldsymbol{E_{LW}}(t_2)\, dt_2 \tag{11}$$

or equivalently

$$\Phi_d(0,0) = -\frac{\pi N_e e^2}{3\hbar^2}\int_0^c v f(\beta) d\beta \int_{\rho_{\min}}^{\rho_{\max}} \rho_0 d\rho_0 \boldsymbol{G}^2 \tag{12}$$

such as

$$\boldsymbol{G} = -e\int_{-\infty}^{+\infty} \frac{\left(\frac{\boldsymbol{R}(t')}{R(t')} - \frac{\boldsymbol{v}(t')}{c}\right)}{\left(\frac{dt}{dt'}\right) k^2 R^2(t')} dt \tag{13}$$

or, by integrating over t'

$$\boldsymbol{G} = -e\int_{-\infty}^{+\infty} \frac{\left(\frac{\boldsymbol{R}(t')}{R(t')} - \frac{d\boldsymbol{R}(t')}{c\,dt'}\right)}{\left(1 + \frac{1}{c}\frac{dR(t')}{dt'}\right)^2 \; R^2(t')} dt' \tag{14}$$

$$\boldsymbol{G}=-e\int_{-\infty}^{+\infty}\left(\frac{\boldsymbol{R}\left(t'\right)}{\left(1+\frac{1}{c}\frac{dR(t')}{dt'}\right)^2\ R^3\left(t'\right)}-\frac{d\boldsymbol{R}\left(t'\right)}{\left(1+\frac{1}{c}\frac{dR(t')}{dt'}\right)^2\ cdt'\ R^2\left(t'\right)}\right)dt' \tag{15}$$

In the following, we use the notations and the variable change

$$\begin{aligned}
\epsilon &= (1+\frac{m^2v^4\rho_0^2}{Z_{em}^2e^4})^{1/2}\\
t' &= \frac{\rho_0}{v}\left(\epsilon\sinh\left(x\right)-x\right)\\
dt' &= \frac{\rho_0}{v}\left(\epsilon\cosh\left(x\right)-1\right)dx\\
R\left(t'\right) &= \rho_0\ \left(\epsilon\cosh\left(x\right)-1\right)\\
X &= \rho_0\ \left(\epsilon-\cosh\left(x\right)\right)\\
\frac{dX}{dt'} &= \frac{\frac{dX}{dx}}{\frac{dt'}{dx}}=-v\frac{\sinh(x)}{\left(\epsilon\cosh\left(x\right)-1\right)}\\
Y &= \ \rho_0\ \sqrt{\epsilon^2-1}\sinh\left(x\right)\\
\frac{dY}{dt'} &= \frac{\frac{dY}{dx}}{\frac{dt'}{dx}}=v\frac{\sqrt{\epsilon^2-1}\cosh(x)}{\left(\epsilon\cosh\left(x\right)-1\right)}\\
\boldsymbol{R}\left(t'\right) &= X\boldsymbol{i}+Y\boldsymbol{j}\\
\boldsymbol{v}\left(t'\right) &= \frac{dX}{dt'}\boldsymbol{i}+\frac{dY}{dt'}\boldsymbol{j}
\end{aligned}$$

where Z_{em} is the charge of the ionic emitter and $(\boldsymbol{i},\boldsymbol{j})$ are the basis of the cartesian coordinates. Then the function G is given by

$$\begin{aligned}
\boldsymbol{G}=&-\frac{e}{\rho_0^2}\int_{-\infty}^{+\infty}\frac{X(t')\boldsymbol{i}+Y(t')\boldsymbol{j}}{\left[\left(\epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right)\right]^2R\left(t'\right)}dt'\\
&-\frac{e}{c\rho_0^2}\int_{-\infty}^{+\infty}\frac{\frac{dX}{dt'}\boldsymbol{i}+\frac{dY}{dt'}\boldsymbol{j}}{\left[\ \left(\epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right)\right]^2}\ dt'
\end{aligned} \tag{16}$$

Using (17), we jump to integrate over x:

$$\boldsymbol{G}=-\frac{e}{v\rho_0^2}\int_{-\infty}^{+\infty}\frac{\rho_0\ \left(\epsilon-\cosh\left(x\right)\right)\boldsymbol{i}+\rho_0\sqrt{\epsilon^2-1}\sinh\left(x\right)\boldsymbol{j}}{\left[\epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right]^2}dx$$

$$-\frac{e}{c\rho_0^2}\int_{-\infty}^{+\infty}\frac{-v\frac{\sinh(x)}{(\epsilon\cosh(x)-1)}\boldsymbol{i}+v\frac{\sqrt{\epsilon^2-1}\cosh(x)}{(\epsilon\cosh(x)-1)}\boldsymbol{j}}{\left[\ \epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right]^2}\ \frac{\rho_0}{v}\left(\epsilon\cosh\left(x\right)-1\right)dx$$

$$\begin{aligned}
=&-\frac{e}{v\rho_0}\int_{-\infty}^{+\infty}\frac{\left(\epsilon-\cosh\left(x\right)\right)\boldsymbol{i}+\sqrt{\epsilon^2-1}\sinh\left(x\right)\boldsymbol{j}}{\left[\epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right]^2}dx\\
&-\frac{e}{c\rho_0}\int_{-\infty}^{+\infty}\frac{-\sinh(x)\boldsymbol{i}+\sqrt{\epsilon^2-1}\cosh(x)\boldsymbol{j}}{\left[\ \epsilon\cosh\left(x\right)-1+\frac{v}{c}\epsilon\sinh(x)\right]^2}dx
\end{aligned} \tag{17}$$

or in a more simplified form

$$\Phi_d(0,0)=-\frac{N_ee^4}{3\pi\hbar^2}\int_0^c\frac{\gamma^5\beta d\beta}{\theta cK_2\left(1/\theta\right)}\exp\left(-\gamma/\theta\right)\int_{\rho_{\min}}^{\rho_{\max}}\frac{d\rho_0}{\rho_0}\ \left[\overline{A}^2+(\epsilon^2-1)\overline{B}^2\right] \tag{18}$$

where

$$\overline{A} = \int_{-\infty}^{+\infty} \frac{-(\epsilon - \cosh(x)) + \beta \sinh(x)}{[\epsilon \cosh(x) - 1 + \beta\epsilon \sinh(x)]^2} dx$$
$$= 2\frac{(\epsilon^2 - 1)\sqrt{1-\epsilon^2+\beta^2\epsilon^2} - \beta^2\epsilon^2 \tanh^{-1}(\sqrt{1-\epsilon^2+\beta^2\epsilon^2})}{\epsilon(1-\epsilon^2+\beta^2\epsilon^2)^{3/2}} \tag{19}$$

$$\overline{B} = -\int_{-\infty}^{+\infty} \frac{\sinh(x) + \beta \cosh(x)}{[\epsilon \cosh(x) - 1 + \beta\epsilon \sinh(x)]^2} dx = 0 \tag{20}$$

Then the relativistic collision operator caused by the Lienard-Wiechert electric field $\Phi_{d,LW}$ is given by

$$\Phi_d(0,0)(Hz/cm^2) = \Phi_{d,LW} = -\frac{4N_e e^4}{3\pi\hbar^2\theta c K_2(1/\theta)} \int_0^1 \frac{\exp(-\frac{1}{\theta\sqrt{1-\beta^2}})}{(1-\beta^2)^{5/2}} \beta d\beta \int_{\rho_{\min}}^{\rho_{\max}} \frac{[(\epsilon^2-1)\delta - \beta^2\epsilon^2 \tanh^{-1}(\delta)]^2}{\epsilon^2 \rho_0 \delta^6} d\rho_0 \tag{21}$$

where we have put

$$\delta \sim \delta(\beta, \epsilon) = \sqrt{1-\epsilon^2+\beta^2\epsilon^2} \tag{22}$$

By taking the maximum of the impact parameter $\rho_{max} = 0.68\lambda_D$ (λ_D is the Debye length) [5] and the minimum of the impact parameter equal to Bohr radius (the Wiesskopf radius is much smaller than Bohr radius in our application), and after numerical integration of (31) we find the following result in Table 1.

We note that we have considered the lower limit of the integration over the impact parameter equal to the Bohr radius, because in our study we only intented to compare the two collision operators corresponding to Coulomb and Lienard-Wiechert interactions. In reality, as Formula (31) shows, by decreasing the lower limit of the impact parameter, the value of the collision operator increases and by increasing the lower limit, the value of the collision operator decreases. Another reason to mention is: by regarding Formula (3) of Ref. [5], we see that the minimum of the impact parameter (Wiesskopf radius r_W), in our conditions of the high temperature, high charge number $Z = 70$ and the upper and lower levels of the transition n_a = 3 (1s3d),n_b = 2 (1s2p)(in triplet case), is much smaller than the Bohr radius a_0 that is to say $r_W << a_0$. For example, for a density 10^{18} cm^{-3}, for the lower limit r_W, the value of the collision operator $\Phi_{d,LW}$ is 0.35×10^{-2} eV (that is overestimated in our opinion) whereas it is 0.21×10^{-2} eV for the lower limit equal to the Bohr radius a_0. In fact, in the region $r < a_0$, the quantum effects must be taken into account. For these reasons, we have only considered, in our calculation, the minimum of the impact parameter equal to the Bohr radius a_0. In addition, we remark from the Table 1 that the effect of the Lienard-Wiechert field is to reduce the amplitude of the collision operator. This reduction is more pronouced for the weak electron densities. We must also note that the criteria of the isolated lines becomes not valid for densities great than 10^{21} cm^{-3} because the FWHM = 2 × HWHM of two neighboring spectral lines becomes of the same order of magnitude of the separation (about 3eV) between the line arising from 1s3d to 1s2p (triplet case that we study) transition and the neighboring line arising from 1s3p to 1s2s (singlet case) transition. Following the Table 1, and for densities less than 10^{21} cm^{-3}, the HWHM is small enough to be can considered that the studied line is isolated.

Table 1. Comparison between collision operators for Coulomb interaction $\Phi_{d,C}$ and for Lienard Wiechert interaction $\Phi_{d,LW}$ (multiplied by $\hbar$ in eV*s and by a_0^2 in cm^2 where a_0 is Bohr radius) at a temperature $T = 8 \times 10^8$ K and different densities and for Helium-like Ytterbium $Z_{em} = 68$, for the radiative transition 1s3d to 1s2p for triplet case. After this muliplication, the following results are in eV.

Ne in cm^{-3}	$\Phi_{d,C}$ in eV	$\Phi_{d,LW}$ in eV	HWHM$_C$ in eV	HWHM$_{LW}$ in eV	Percent
10^{16}	0.32×10^{-4}	0.24×10^{-4}	0.25×10^{-4}	0.17×10^{-4}	25
10^{18}	0.27×10^{-2}	0.21×10^{-2}	0.20×10^{-2}	0.15×10^{-2}	22
10^{20}	0.24	0.22	0.18	0.17	8
10^{22}	26	25	20	19	4

Note: We have defined the percent to be equal $\left(\frac{\Phi_{d,C}-\Phi_{d,LW}}{\Phi_{d,C}} \cdot 100\right)$.

3. Conclusions

In this work, we have investigated Lienard-Wiechert or retarded electric fields produced by moving electric charges with respect to a rest frame. Specifically, we have studied its contribution to the broadening of the spectral line shape of the Helium-like Ytterbium in hot and dense plasmas radiative (transition from 1s3d to 1s2p). The principal result, as the table shows is: the retarded Lienard-Wiechert interaction, narrows the line shape comparatively to the pure Coulomb interaction. The narrowing is more pronouced for the low densities.

Acknowledgments: We wish to acknowledge the support of LRPPS laboratory and Fethi Khelfaoui, by offering us the encouragement, and some technical materials for developing this work.

Author Contributions: All authors M.T.M., K.A., K.C., K.A.T. and S.D. were participated equivalently to this theoretical work.

Conflicts of Interest: The authors declare no conflict of interest.

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