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$\phi(2170)$ Decaying to $\phi\pi\pi$ and $\phi K\bar{K}$

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Abstract: Within the framework of dispersion theory, we study the the processes $e^+e^- \to \phi(2170) \to \phi\pi\pi(K\bar{K})$. The strong pion–pion final-state interactions, especially the $K\bar{K}$ coupled channel in the S wave, are taken into account in a model-independent way using the Omnès function solution. Through fitting the experimental data of the $\pi\pi$ and $\phi\pi$ invariant mass distributions of the $e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-$ process, the low-energy constants in the chiral Lagrangian are determined. The theoretical prediction for the cross sections' ratio $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$ is given, which could be useful for selecting the physical solution, when the fit to the $e^+e^- \to \phi K^+K^-$ cross-section distribution is available in the future. Our results suggest that above the kinematical threshold of $\phi K\bar{K}$, the mechanism $e^+e^- \to \phi K^+K^-$, with the kaons rescattering to a pion pair, plays an important role in the $e^+e^- \to \phi\pi^+\pi^-$ transition.

Keywords: dispersion theory; final-state interaction; strangeoniumlike; exotic state

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1. Introduction

The vector strangeoniumlike state $\phi(2170)$ was first discovered in 2006 by the BaBar Collaboration in the initial-state radiation process $e^+e^- \to \gamma_{ISR}\phi f_0(980)$ [1–4] and was later confirmed by BESII [5], Belle [6], and BESIII [7,8] collaborations. Its mass and width were measured to be $M = 2188 \pm 10$ MeV and $\Gamma = 83 \pm 12$ MeV, respectively, and its spin-parity quantum number is $J^{PC} = 1^{--}$ [9]. The nature of $\phi(2170)$ has remained controversial, and models have been proposed to interpret the $\phi(2170)$ as a hybrid state [10], an excited strangeonium [11], a hidden-strangeness baryon–antibaryon state ($qqs\bar{q}\bar{q}\bar{s}$) [12], a bound state of $\Lambda\bar{\Lambda}(^3S_1)$ [13], a tetraquark state [14–19], and a dynamically generated state in the $\phi f_0(980)$ system [20,21] or the $\phi K\bar{K}$ system [22–25].

Since both $\phi(2170)$ and $Y(4230)$ are observed in e^+e^- annihilation through initial state radiation, $\phi(2170)$ is often taken as the strange analogue of $Y(4230)$. Similar to the observation of $Z_c(3900)$ in the $J/\psi\pi$ invariant mass spectrum in $Y(4230) \to J/\psi\pi\pi$ process, recently, the BESIII Collaboration searched for a strangeoniumlike structure Z_s decaying into $\phi\pi$ in the $\phi(2170) \to \phi\pi\pi$ process [26]. No Z_s signal was observed in the $\phi\pi$ invariant mass spectrum. On the other hand, the Born cross sections for the channel $e^+e^- \to \phi K^+K^-$ were measured for the first time at center-of-mass energies between 2.100 and 3.080 GeV [27]. In this work, we study the $\pi\pi$ and $\phi\pi$ invariant mass spectrum of $e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-$ process and the ratio of cross sections $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$. As shown in Figure 1, the quark lines of the ϕ and $\pi\pi$ final states are disconnected, and thus at tree level, the leading electromagnetic contributions to the $e^+e^- \to \phi\pi^+\pi^-$ process from the exchange of a virtual photon is suppressed [25]. The $\pi\pi$ invariant mass increases to more than 1.1 GeV, and in this energy region, there are strong coupled-channel final-state interactions (FSIs), especially in the S wave. In this work, we take into account the strong FSIs model-independently using dispersion theory and study the contribution of the mechanism $e^+e^- \to \phi K^+K^-$ with the kaons rescattering to a pion pair to $e^+e^- \to \phi\pi^+\pi^-$ transition. At low energies, the amplitude should agree with the leading chiral results; therefore, the subtraction terms in the dispersion relations can be determined by matching them to the chiral contact terms. For the leading contact couplings for $\phi(2170)\phi PP$, where P denotes the pseudoscalar meson π or K, we construct the chiral Lagrangians in the spirit of the chiral effective field theory (χEFT) [28]. The parameters are then fixed by fitting to the BESIII data. The relevant Feynman diagrams considered are given in Figure 2.

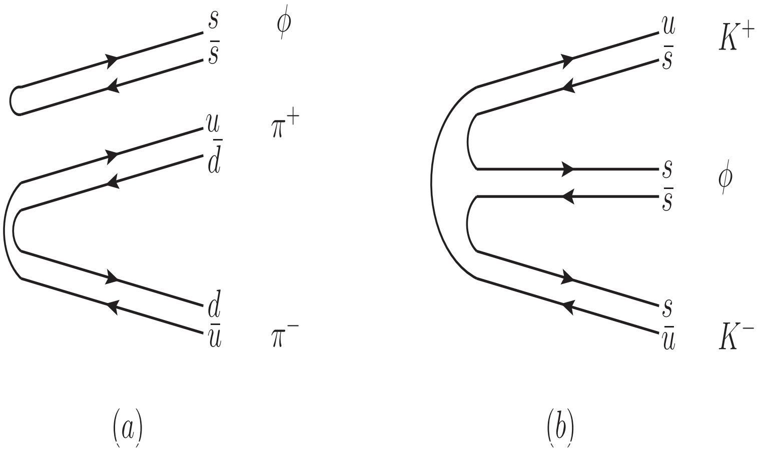

Figure 1. Quark diagrams of the final states of $\phi\pi\pi$ and $\phi K\bar{K}$. (**a**) the quark lines of the ϕ ; (**b**) the quark lines of the π.

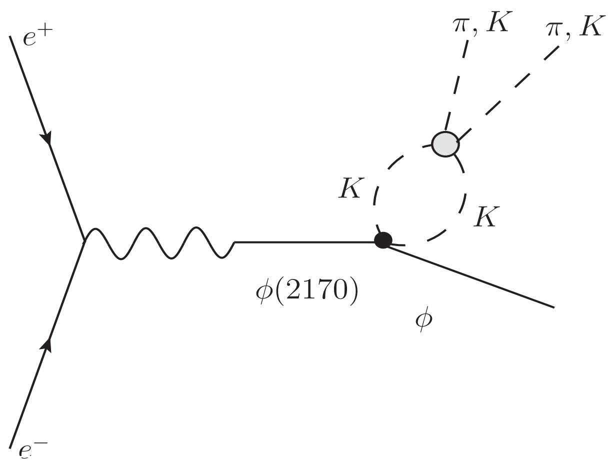

Figure 2. Feynman diagrams considered for $e^+e^- \to \phi(2170) \to \phi\pi\pi(\phi K\bar{K})$. The gray blob denotes the effects of FSI.

This paper is organized as follows. In Section 2, we present the theoretical framework and the calculation of the amplitudes as well as the dispersive treatment of the FSI. In Section 3, we fit the experimental data of the $\pi\pi$ and $\phi\pi$ invariant mass distribution to determine the coupling constants and discuss the contribution of $e^+e^- \to \phi K^+K^-$ with the kaons rescattering to a pion pair to the $e^+e^- \to \phi\pi^+\pi^-$ transition. A summary is given in Section 4.

2. Theoretical Framework

2.1. Lagrangians

The ϕ meson can be decomposed into SU(3) singlet and octet components of light quarks,

$$|\phi\rangle = s\bar{s} = \frac{\sqrt{3}}{3}|V_1\rangle - \frac{\sqrt{6}}{3}|V_8\rangle\,, \tag{1}$$

where $|V_1\rangle \equiv \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$, and $|V_8\rangle \equiv \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$. In the $\phi(2170) \to \phi PP$ transition, the two pseudoscalars in the final state must come from light-flavor sources. There are two types of sources that the two pseudoscalars may come from, one is the possible light-quark components contained in the $\phi(2170)$ (e.g., in the $\phi f_0(980)$ molecule or the tetraquark scenarios), and the other possibility is that the two pseudoscalars are excited by the $\phi(2170)$ from vacuum (e.g., in the pure $s\bar{s}$ or the hybrid state scenarios). In our study, we do not distinguish these two types of pseudoscalars sources but take them into account in a unified scheme, and both are "provided" by the $\phi(2170)$. If the $\phi(2170)$ contains no u, d quarks (as in the pure $s\bar{s}$ or the hybrid state scenarios), the light-flavor sources excited by the $\phi(2170)$ from a vacuum have to be in the form of an SU(3) singlet state. Since the structure of the $\phi(2170)$ has remained controversial, the relative strengths between the light-flavor SU(3) singlet part and SU(3) octet part acting in this transition are undetermined. Therefore, considering the two pseudoscalars sources provided by the $\phi(2170)$ in the $\phi(2170) \to \phi PP$ transition, the $\phi(2170)$ can be decomposed into SU(3) singlet and octet components of light quarks,

$$|\phi(2170)\rangle = a|Y_1\rangle + b|Y_8\rangle\,. \tag{2}$$

The values of the component strengths a and b can not be determined in this study, since they always appear in the combination of $(a - \sqrt{2}b)$ in the chiral contact amplitude for $\phi(2170) \to \phi PP$ transition that is given in Section 2.2. Expressed in terms of a 3×3 matrix in the SU(3) flavor space, it is written as

$$\frac{a}{\sqrt{3}} Y_1 \cdot \mathbb{1} + \frac{b}{\sqrt{6}} Y_8 \cdot \mathrm{diag}(1, 1, -2). \tag{3}$$

The effective Lagrangian for the $\phi(2170)\phi\pi\pi$ and $\phi(2170)\phi K\bar{K}$ contact couplings, at leading order in the chiral expansion, reads [28,29]

$$\begin{aligned}\mathscr{L}_{\phi(2170)\phi PP} =& g_1\langle Y_1^\mu V_{1\mu}\rangle\langle u_\nu u^\nu\rangle - \sqrt{2}g_1\langle Y_1^\mu\rangle\langle V_{8\mu}u_\nu u^\nu\rangle + g_8\langle V_{1\mu}\rangle\langle Y_8^\mu u_\nu u^\nu\rangle - \sqrt{2}g_8\langle Y_8^\mu V_{8\mu}u_\nu u^\nu\rangle \\ &+ h_1\langle Y_1^\mu V_{1\nu}\rangle\langle u_\mu u^\nu\rangle - \sqrt{2}h_1\langle Y_1^\mu\rangle\langle V_{8\nu}u_\mu u^\nu\rangle + h_8\langle V_{1\nu}\rangle\langle Y_8^\mu u_\mu u^\nu\rangle - \sqrt{2}h_8\langle Y_8^\mu V_{8\nu}u_\mu u^\nu\rangle + \mathrm{H.c.}\,,\end{aligned} \tag{4}$$

where $\langle \ldots \rangle$ denotes the trace in the SU(3) flavor space. In Equation (4), the Lagrangian is constructed by placing the SU(3) singlet parts and the SU(3) octet parts into different SU(3) flavor traces. The SU(3) octet of the pseudo-Goldstone bosons from the spontaneous breaking of chiral symmetry can be filled nonlinearly into

$$u_\mu = i\left(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\right), \qquad u = \exp\left(\frac{i\Phi}{\sqrt{2}F}\right), \tag{5}$$

with the Goldstone fields

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}. \tag{6}$$

Here, F is the pion decay constant in the chiral limit, and we take the physical value 92.1 MeV for it.

The gauge-invariant $\gamma^*(\mu)$ and $\phi(2170)(\nu)$ coupling is given by

$$iV_{\gamma^{*\mu}Y^\nu} = 2i(g^{\mu\nu}p^2 - p^\mu p^\nu)c_\gamma\,, \tag{7}$$

where p is the momentum of the virtual photon γ^*.

2.2. Amplitudes of $\phi(2170) \to \phi PP$ Processes

The decay amplitude of $\phi(2170)(p_a) \to \phi(p_b)P(p_c)P(p_d)$ can described in terms of the Mandelstam variables

$$s = (p_c + p_d)^2\,, \qquad t_P = (p_a - p_c)^2\,, \qquad u_P = (p_a - p_d)^2\,,$$
$$3s_{0P} \equiv s + t_P + u_P = M^2_{\phi(2170)} + M^2_\phi + 2m^2_P\,. \tag{8}$$

The variables t_P and u_P can be expressed in terms of s and the scattering angle θ according to

$$t_P = \frac{1}{2}[3s_{0P} - s + \kappa_P(s)\cos\theta]\,, \qquad u_P = \frac{1}{2}[3s_{0P} - s - \kappa_P(s)\cos\theta]\,,$$
$$\kappa_P(s) \equiv \sigma_P \lambda^{1/2}(M^2_{\phi(2170)}, M^2_\phi, s)\,, \qquad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}\,, \tag{9}$$

where θ is defined as the angle between the positive pseudoscalar meson and the $\phi(2170)$ in the rest frame of the PP system, and $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$ is the Källén triangle function. We define $\mathbf{q}$ as the 3-momentum of the final ϕ in the rest frame of the $\phi(2170)$ with

$$|\mathbf{q}| = \frac{1}{2M_{\phi(2170)}}\lambda^{1/2}(M^2_{\phi(2170)}, M^2_\phi, s)\,. \tag{10}$$

Using the Lagrangians in Equation (4), we can calculate the chiral contact terms for the $\phi(2170) \to \phi\pi^+\pi^-$ and $\phi(2170) \to \phi K^+K^-$ processes

$$M^{\chi,\pi}(s,\cos\theta) = 0\,,$$
$$M^{\chi,K}(s,\cos\theta) = -\frac{3}{F^2}\Big[2\big(g_1 - \sqrt{2}g_8\big)p_c\cdot p_d\epsilon_Y\cdot\epsilon_\phi + \big(h_1 - \sqrt{2}h_8\big)\big(p_c\cdot\epsilon_Y p_d\cdot\epsilon_\phi + p_c\cdot\epsilon_\phi p_d\cdot\epsilon_Y\big)\Big]\,. \tag{11}$$

Notice that the quark lines of the ϕ and $\pi\pi$ final states are disconnected; therefore, at tree level, the leading electromagnetic contributions to the $e^+e^- \to \phi\pi^+\pi^-$ process from the exchange of a virtual photon is suppressed [25]. The amplitude $M^{\chi,\pi}(s,\cos\theta) = 0$ in Equation (11) agrees with this observation. Thus, the mechanism $e^+e^- \to \phi K\bar{K}$ with the kaons rescattering to a pion pair may be an important contribution to $e^+e^- \to \phi\pi^+\pi^-$.

The appropriate helicity amplitudes $M^{\chi,\pi(K)}_{\lambda_1\lambda_2}(s,\cos\theta)$, with $\lambda_1(\lambda_2)$ denoting the $\phi(2170)(\phi)$ helicities, respectively, are obtained by inserting the explicit expressions for the polarization vectors $\epsilon^\mu(p_i,\lambda)$ occurring in the amplitudes Equation (11), taken from ref. [30],

$$\epsilon^\mu(p_{\phi(2170)}, \pm 1) = \begin{pmatrix} 0 \\ \frac{\mp\cos\theta}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \\ \frac{\pm\sin\theta}{\sqrt{2}} \end{pmatrix}, \ \epsilon^\mu(p_{\phi(2170)}, 0) = \begin{pmatrix} \frac{\mathbf{q}}{M_{\phi(2170)}} \\ \frac{\sqrt{M^2_{\phi(2170)}+\mathbf{q}^2}}{M_{\phi(2170)}}\sin\theta \\ 0 \\ \frac{\sqrt{M^2_{\phi(2170)}+\mathbf{q}^2}}{M_{\phi(2170)}}\cos\theta \end{pmatrix},$$
$$\epsilon^\mu(p_\phi, \pm 1) = \begin{pmatrix} 0 \\ \frac{\pm\cos\theta}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \\ \frac{\mp\sin\theta}{\sqrt{2}} \end{pmatrix}, \ \epsilon^\mu(p_\phi, 0) = \begin{pmatrix} \frac{\mathbf{q}}{M_\phi} \\ -\frac{\sqrt{M^2_\phi+\mathbf{q}^2}}{M_\phi}\sin\theta \\ 0 \\ -\frac{\sqrt{M^2_\phi+\mathbf{q}^2}}{M_\phi}\cos\theta \end{pmatrix}. \tag{12}$$

Note, in this study, we need to perform the partial-wave projections of the PP system to take into account the final-state interactions. We can analytically continue the $\phi(2170) \to \phi PP$ decay amplitude to the $\phi(2170)\phi \to PP$ scattering amplitude, since the partial-wave decomposition to the latter is easier. Therefore, the expressions for the polarization vectors given in Equation (12) are defined in the PP rest frame.

The partial-wave projection of the $\phi(2170) \to \phi PP$ helicity amplitudes is given as

$$M^{\chi,\pi(K),l}_{\lambda_1\lambda_2}(s) = \frac{2l+1}{2}\int \mathrm{d}\cos\theta\, d^l_{\lambda_1-\lambda_2,0}(\theta) M^{\chi,\pi(K)}_{\lambda_1\lambda_2}(s,\cos\theta)\,, \tag{13}$$

where $d^l_{\lambda_1-\lambda_2,0}(\theta)$ are the small Wigner-d functions.

2.3. Final-State Interactions with a Dispersive Approach, Omnès Solution

The strong FSIs between two pseudoscalar mesons can be taken account of model-independently using dispersion theory. Since the invariant mass of the pion pair reaches above the $K\bar{K}$ threshold, we take into account the coupled-channel ($\pi\pi$ and $K\bar{K}$) FSI for the dominant S-wave component, while for the D wave, only the single-channel FSI is considered. Similar methods to consider the FSI have been applied previously, e.g., in refs. [29,31–41].

For $\phi(2170) \to \phi PP$, the partial-wave decomposition of the helicity amplitude including FSIs reads

$$M^{P,\mathrm{decay}}(s,\cos\theta) = \sum_{\lambda_1\lambda_2}\sum_{l=0}^{\infty} M^{P,l}_{\lambda_1\lambda_2}(s) d^l_{\lambda_1-\lambda_2,0}(\theta)\,. \tag{14}$$

For the S wave, we take into account the two-channel rescattering effects. The two-channel unitarity condition reads

$$\mathrm{disc}\,\mathbf{M}^0_{\lambda_1\lambda_2}(s) = 2iT^{0*}_0(s)\Sigma(s)\mathbf{M}^0_{\lambda_1\lambda_2}(s), \tag{15}$$

where the two-dimensional vectors $\mathbf{M}^0_{\lambda_1\lambda_2}(s)$ contain both the $\pi\pi$ and the $K\bar{K}$ final states,

$$\mathbf{M}^0_{\lambda_1\lambda_2}(s) = \begin{pmatrix} M^{\pi,0}_{\lambda_1\lambda_2}(s) \\ \frac{2}{\sqrt{3}} M^{K,0}_{\lambda_1\lambda_2}(s) \end{pmatrix}. \tag{16}$$

The two-dimensional matrices $T^0_0(s)$ and $\Sigma(s)$ are represented as

$$T^0_0(s) = \begin{pmatrix} \frac{\eta^0_0(s)e^{2i\delta^0_0(s)}-1}{2i\sigma_\pi(s)} & |g^0_0(s)|e^{i\psi^0_0(s)} \\ |g^0_0(s)|e^{i\psi^0_0(s)} & \frac{\eta^0_0(s)e^{2i\left(\psi^0_0(s)-\delta^0_0(s)\right)}-1}{2i\sigma_K(s)} \end{pmatrix}, \tag{17}$$

and $\Sigma(s) \equiv \mathrm{diag}\big(\sigma_\pi(s)\theta(s-4m_\pi^2), \sigma_K(s)\theta(s-4m_K^2)\big)$. There are three input functions in the $T^0_0(s)$ matrix: the $\pi\pi$ S-wave isoscalar phase shift $\delta^0_0(s)$ and the modulus and phase of the $\pi\pi \to K\bar{K}$ S-wave amplitude $g^0_0(s) = |g^0_0(s)|e^{i\psi^0_0(s)}$. We use the parametrization of the $T^0_0(s)$ matrices given in refs. [42,43]. Note that the relation between the inelasticity parameter $\eta^0_0(s)$ in Equation (17) and the modulus $|g^0_0(s)|$

$$\eta^0_0(s) = \sqrt{1-4\sigma_\pi(s)\sigma_K(s)|g^0_0(s)|^2\theta(s-4m_K^2)}\,. \tag{18}$$

These inputs are used up to $\sqrt{s_0} = 1.3\,\mathrm{GeV}$, and above s_0, the $f_0(1370)$ and $f_0(1500)$ resonances coupling strongly to 4π contribute further inelasticities [44,45]. Above s_0, we smoothly guide the phases $\delta^0_0(s)$ and ψ^0_0 to 2π by means of [46]

$$\delta(s) = 2\pi + (\delta(s_0) - 2\pi)\frac{2}{1+(s/s_0)^{3/2}}\,. \tag{19}$$

The solution of the coupled-channel unitarity condition in Equation (15) is given by

$$\mathbf{M}^0_{\lambda_1\lambda_2}(s) = \Omega(s)\mathbf{P}^{n-1}(s)\,, \tag{20}$$

where $\Omega(s)$ satisfies the homogeneous coupled-channel unitarity relation

$$\mathrm{Im}\,\Omega(s) = T^{0*}_0(s)\Sigma(s)\Omega(s), \qquad \Omega(0) = \mathbb{1}\,, \tag{21}$$

and its numerical results have been computed, e.g., in refs. [46–49].

For the D wave, the single-channel FSI is considered. In the elastic PP rescattering region, the partial-wave unitarity condition is

$$\mathrm{Im}\, M^{P,2}_{\lambda_1\lambda_2}(s) = M^{P,2}_{\lambda_1\lambda_2}(s)\sin\delta^0_2(s)e^{-i\delta^0_2(s)}\,, \tag{22}$$

where the phase of the D-wave isoscalar amplitude δ^0_2 coincides with the PP elastic phase shift, as required by Watson's theorem [50,51]. The Omnès solution to Equation (22) reads

$$M^{P,2}_{\lambda_1\lambda_2}(s) = \Omega^0_2(s)P^{n-1}_2(s)\,, \tag{23}$$

where the polynomial $P^{n-1}_2(s)$ is a subtraction function, and the Omnès function is defined as [52]

$$\Omega^0_2(s) = \exp\left\{\frac{s}{\pi}\int_{4m^2_\pi}^{\infty}\frac{\mathrm{d}x}{x}\frac{\delta^0_2(x)}{x-s}\right\}. \tag{24}$$

We use the result of $\delta^0_2(s)$ given in ref. [53], which is smoothly continued to π for $s\to\infty$.

On the other hand, at low energies, the partial-wave amplitudes $\mathbf{M}^0_{\lambda_1\lambda_2}(s)$ and $M^2_{\lambda_1\lambda_2}(s)$ should match those from χEFT. Namely, if one switches off the FSI with $s=0$, $\Omega(0)=\mathbb{1}$, and $\Omega^0_2(0)=1$, the subtraction functions should agree well with the low-energy chiral amplitudes given in Equation (13). Thus, for the S wave, the integral equation takes the form

$$\mathbf{M}^0_{\lambda_1\lambda_2}(s) = \Omega(s)\mathbf{M}^{\chi,0}_{\lambda_1\lambda_2}(s)\,, \tag{25}$$

where $\mathbf{M}^{\chi,0}_{\lambda_1\lambda_2}(s) = \left(M^{\chi,\pi,0}_{\lambda_1\lambda_2}(s), 2/\sqrt{3}\,M^{\chi,K,0}_{\lambda_1\lambda_2}(s)\right)^T$, while for the D wave, it reads

$$M^{P,2}_{\lambda_1\lambda_2}(s) = \Omega^0_2(s)M^{\chi,P,2}_{\lambda_1\lambda_2}(s)\,. \tag{26}$$

Note, that as given in Equation (11), the chiral contact amplitudes for $\phi(2170)\to\phi\pi^+\pi^-$ process $M^{\chi,\pi,0(2)}_{\lambda_1\lambda_2}(s)$ equal 0, and only the nonzero amplitudes $M^{\chi,K,0(2)}_{\lambda_1\lambda_2}(s)$ affect the numerical calculation.

The polarization-averaged modulus square of the $e^+e^-\to\phi(2170)\to\phi\pi^+\pi^-$ amplitude can be written as

$$|\bar{M}(E^2,s,\cos\theta)|^2 = \frac{4\pi\alpha c^2_\gamma|M^{\pi,\mathrm{decay}}(s,\cos\theta)|^2}{3|E^2-M^2_{\phi(2170)}+iM_{\phi(2170)}\Gamma_{\phi(2170)}|^2M^2_\phi}\left[8M^2_\phi E^2+(s-E^2-M^2_\phi)^2\right], \tag{27}$$

where E is the center-of-mass energy of the e^+e^- system, and we set the $\gamma^*\phi(2170)$ coupling constant c_γ to 1 since it can be absorbed into the overall normalization in the fitting of the event distributions. Here, we use the energy-independent width for the $\phi(2170)$, and the values of the $\phi(2170)$ mass are taken as 2125 MeV, which is the center-of-mass energy measured by BESIII detector in ref. [26]. The width of $\phi(2170)$ is taken as 100 MeV from PDG [9].

The $\pi\pi$ invariant mass distribution of $e^+e^-\to\phi\pi^+\pi^-$ reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}m_{\pi\pi}} = \int_{-1}^{1}\frac{|\bar{M}(E^2,s,\cos\theta)|^2|\mathbf{k}^*_3||\mathbf{k}_5|}{128\pi^3|\mathbf{k}_1|E^2}\mathrm{d}\cos\theta\,, \tag{28}$$

where $\mathbf{k}_1$ and $\mathbf{k}_5$ represent the 3-momenta of $e^\pm$ and ϕ in the center-of-mass frame, respectively, and $\mathbf{k}^*_3$ denotes the 3-momenta of $\pi^\pm$ in the rest frame of the $\pi\pi$ system. They are given as

$$|\mathbf{k}_1| = \frac{E}{2}\,,\quad |\mathbf{k}^*_3| = \frac{1}{2}\sqrt{s-4m^2_\pi}\,,\quad |\mathbf{k}_5| = \frac{1}{2E}\lambda^{1/2}(E^2,s,M^2_\phi)\,. \tag{29}$$

The $\phi\pi$ invariant mass distribution of $e^+e^- \to \phi\pi^+\pi^-$ reads

$$\frac{d\sigma}{dm_{\phi\pi}} = \int_{s_-}^{s_+} \frac{|\bar{M}(E^2, s, \cos\theta)|^2 |\mathbf{k}_3^*||\mathbf{k}_5| m_{\phi\pi}}{64\pi^3 |\mathbf{k}_1| E^2 \kappa_\pi(s)\sqrt{s}} ds\,, \tag{30}$$

where

$$s_\pm = \frac{1}{4m_{\phi\pi}^2}\left\{ (E^2 - M_\phi^2)^2 - \left[\lambda^{\frac{1}{2}}(E^2, m_{\phi\pi}^2, m_\pi^2) \mp \lambda^{\frac{1}{2}}(m_{\phi\pi}^2, m_\pi^2, M_\phi^2)\right]^2 \right\}. \tag{31}$$

3. Phenomenological Discussion

Fitting to the BESIII Data

In this work, we performed fits simultaneously by taking into account the experimental data sets of the $\pi\pi$ and $\phi\pi$ invariant mass distributions of $e^+e^- \to \phi\pi\pi$ measured at a center-of-mass energy $E = 2.125$ GeV by the BESIII Collaboration [26]. The charged and neutral-pion final states data are taken account of simultaneously.

There are four free parameters in our fits: the combinations of the coupling constants in Equation (11) $(g_1 - \sqrt{2}g_8)$ and $(h_1 - \sqrt{2}h_8)$ and the two normalization factors N_1 and N_2 for the charged and neutral final states, respectively. By performing the χ^2 fit, we can determine the unknown combinations of the resonance couplings:

$$(g_1 - \sqrt{2}g_8) = (-0.385 \pm 0.005)\,, \qquad (h_1 - \sqrt{2}h_8) = (3.120 \pm 0.029)\,, \tag{32}$$

with $\chi^2/\text{d.o.f} = 1583.9/(332 - 4) = 4.83$.

In Figure 3, the fit results of the $\pi\pi$ and $\phi\pi$ mass spectra in $e^+e^- \to \phi\pi\pi$ are shown. For the $\pi\pi$ mass spectra, one can see that the peak around 1 GeV due to the presence of the $f_0(980)$ is described well. A small hump below 0.5 GeV cannot be reproduced in our scheme. Also note that there are differences between the shapes of the data for the modes with charged and neutral pions especially in the region close to the lower kinematical boundary and in the region close to the upper kinematical boundary. These discrepancies contribute sizeably to the value of χ^2. For the $\phi\pi$ mass spectra, no Z_s signal is observed. The data points below 1.25 GeV or above 1.8 GeV carrying small error bars contribute largely to the value of χ^2. Note that in the present scheme, we only consider the leading chiral effective Lagrangian for the $\phi(2170)\phi PP$ contact couplings, and taking account of higher order chiral coupling terms may help to reduce the value of χ^2. Nevertheless, since our theoretical predictions roughly agree with the experimental data, we discuss the fit results in more details.

Using the fitted coupling constants in Equation (32), we can calculate the cross sections' ratio $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$. At $\sqrt{s} = 2.125$ GeV, our theoretical prediction is $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$ $= 0.12 \pm 0.01$. Note that the $\phi(2170)$ is very close to the ϕKK threshold; therefore, the phase space of $e^+e^- \to \phi(2170) \to \phi K^+K^-$ is much smaller than that of $e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-$. Using the experimental cross sections measured at the same energy point $\sigma(e^+e^- \to \phi K^+K^-) = (70.6 \pm 7.2 \pm 4.9)$ pb [27] and $\sigma(e^+e^- \to \phi\pi^+\pi^-) = (436.2 \pm 6.4 \pm 30.1)$ pb [26], one obtains $\sigma(e^+e^- \to \phi K^+K^-)/\sigma(e^+e^- \to \phi\pi^+\pi^-) = 0.16 \pm 0.02$. Note that due to the constructive or destructive interferences between/among different resonances and background, the multisolution problem exists in using coherent contributions to fit the data, as have been pointed out in refs. [54–61]. As shown in ref. [61], two solutions are found in the fit to the data of $e^+e^- \to \phi\pi^+\pi^-$ with two coherent Breit–Wigner functions. The products of the branching fraction of $\phi(2170)$ to $\phi\pi^+\pi^-$ and the e^+e^- partial width in these two solutions are $68.9 \pm 7.0 \pm 3.4$ eV$/c^2$ and $6.2 \pm 1.1 \pm 0.3$ eV$/c^2$, respectively, which differ with each other by one order. Thus, it is questionable to attribute the experimental cross section of $e^+e^- \to \phi\pi^+\pi^-$ at 2.125 GeV totally to the $\phi(2170)$ intermediate state. Note that the experimental paper ref. [27] does not perform the fit to the $e^+e^- \to \phi K^+K^-$ cross-section distribution to extract the parameters of $\phi(2170)$ and other resonances. When this kind of fit is available in the future, our results could be useful for selecting the physical solution. On the other hand, if we assume that our estimation of the cross sections' ratio $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$ can be approximately extended to other energy points in the region around $\sqrt{s} = 2.125$ GeV, we may infer that the peak in the $e^+e^- \to \phi K^+K^-$ cross-section distribution must also be reflected in the $e^+e^- \to \phi\pi^+\pi^-$ cross-section distribution. Observe that there are only two obvious peaks, $\phi(2170)$ and $X(2400)$, in the experimental $e^+e^- \to \phi\pi^+\pi^-$ cross-section distribution in the region of [2.0, 2.6] GeV, and the $X(2400)$ affects the resonance parameter of $\phi(2170)$ only moderately, as its width is only about 100 MeV [61]. One may attribute the experimental cross section $\sigma(e^+e^- \to \phi K^+K^-)$ at 2.125 GeV mainly to the $\phi(2170)$ intermediate state. Therefore, using our theoretical prediction of the cross sections' ratio given above, one can determine that the lager solution of the product of the branching fraction of $\phi(2170)$ to $\phi\pi^+\pi^-$ and the e^+e^- partial width $68.9 \pm 7.0 \pm 3.4$ eV$/c^2$ is preferred, since in this solution, the peak due to the $\phi(2170)$ intermediate state is higher than the cross-section data at $\sqrt{s} = 2.125$ GeV, as shown in Figure 3 of ref. [61]. Also, one may conclude that above the kinematical threshold of $\phi K\bar{K}$ the mechanism $e^+e^- \to \ldots \to \phi K^+K^-$, with the kaons rescattering to a pion pair, may be an important contribution to $e^+e^- \to \phi\pi^+\pi^-$. In addition, using our fitted results, we can calculate the ratio of the cross sections $\sigma(e^+e^- \to \phi(2170) \to \phi\pi^0\pi^0)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-) = 0.51 \pm 0.02$, which agree with the experimental ratio given in ref. [26], 0.54 ± 0.6, within the error bar.

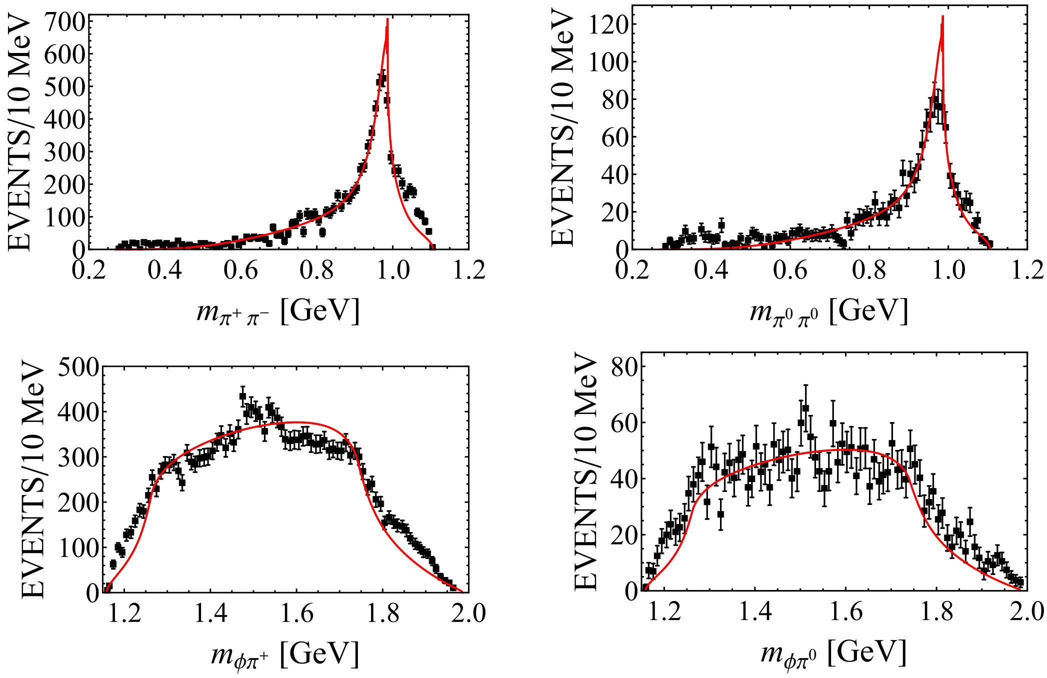

Figure 3. Fit results of the $\pi\pi$ (**top**) and $\phi\pi$ (**bottom**) invariant mass spectra in $e^+e^- \to \phi\pi^+\pi^-$ process. The charged- (**left**) and neutral-pion final states (**right**) data are taken into account simultaneously. The experimental data are taken from ref. [26].

4. Conclusions

We used dispersion theory to study the processes $e^+e^- \to \phi(2170) \to \phi\pi\pi(K\bar{K})$. The strong FSI between two pseudoscalar mesons was considered in a model-independent way, and the leading chiral amplitude acts as the subtraction function in the Omnès solution. Through fitting to the data of the $\pi\pi$ and $\phi\pi$ invariant mass spectra of $e^+e^- \to \phi(2170) \to \phi\pi\pi$, the couplings of the $\phi(2170)\phi PP$ vertex were determined. We gave the prediction of the cross sections' ratio $\sigma(e^+e^- \to \phi(2170) \to \phi K^+K^-)/\sigma(e^+e^- \to \phi(2170) \to \phi\pi^+\pi^-)$, and the result could be useful for selecting the physical solution when the fit to the $e^+e^- \to \phi K^+K^-$ cross-section distribution is available in the future. Our findings suggest that above the kinematical threshold of $\phi K\bar{K}$, the mechanism $e^+e^- \to \phi K^+K^-$ with the kaons rescattering to a pion pair plays an important role in the $e^+e^- \to \phi\pi^+\pi^-$ transition.

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