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Nontrivial Topology Dynamical Corrections and the Magnetic Monopole-like Effect in Minkowski Spacetime

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Abstract: We investigate two physical systems within a spacetime region affected by the nontrivial topology. The set-up for our analysis is a Minkowski metric perturbed by elements reflecting the topological nontriviality. These elements arise when exploring Cartan's spinorial approach along with the exotic spinors counterpart. This evinced nontrivial topology corrections in the free particle dynamics and charged particles coupled to an external electromagnetic field. As a complement, we show the appearance of a magnetic monopole-like effect.

Keywords: nontrivial topology; exotic spinors; Poincarè symmetry violation; magnetic monopole-like; correcting dynamics in Minkowski space

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1. Introduction

The idea that physically reasonable spacetimes must exhibit the capability of admitting spinor fields that are globally defined was argued a long time ago [1–3]. Such spacetimes are said to have a spinor structure [2,4]. However, an important but less-mentioned fact about spinor structures is that they are not generally unique, in a sense that we shall formally define in the next section. The topology of the underlying spacetime manifold dictates this property. When a spacetime is not simply connected, meaning that it is not path-connected or its fundamental group is not trivial [5], there is no unique way to define the spinors. Instead, there are inequivalent possibilities, each of which is in a one-to-one correspondence with a certain cohomology group of spacetime. If the cohomology group associated with a spacetime is nontrivial, the spinors associated with that spacetime are said to be exotic. The term "exotic" here does not mean strange or unusual but instead refers to the fact that these spinors are not equivalent to the standard one defined in simply connected spacetimes. In this paper, we are interested in some physical implications of the nontriviality of the spacetime relating to the existence of an exotic structure.

Another way to obtain mathematical and physical intuition about spinors is through a protocol first introduced by Cartan regarding how spinors are understood as the square root of the geometry. Cartan's approach says nothing about nontrivial topologies. Therefore, merging its view with exotic spinors demands input corrections from nontrivial topology formalism somewhere. Thus, the background question we may ask, motivating the developments to be reported here, is the following: What kind of geometry would result from using square exotic, instead of regular, spinors? By considering the formalism of spinors in the presence of a nontrivial spacetime topology [6–9], the exotic spinors, along with Cartan's viewpoint of spinors [10,11], we were able to evince geometrically, so to speak, nontrivial topology effects, arriving at a quadratic form taking into account such effects [12]. The basic idea is to express the differentials in terms of a basis considering a derivative correction entering into the appreciation of exotic spinors. Then, by understanding the quadratic form as an element of $(TM)^* \otimes (TM)^*$, we can study the implications of the additional terms. In [12], after setting the motivations coming from the formalism, a physical effect was studied in which the nontrivial topology influenced the field modes as an external source, leading to a quasinormal-like behavior. Recently, with the aid of a deformed left contraction, it was also possible to glance at spinors' classification in a scenario involving a nontrivial topology [13], such as the one presented in [12].

There are potentially numerous physical effects that warrant further exploration in the scenario of a nontrivial topology. The purpose of this paper is to investigate two such effects, which have been extensively studied in the standard case of a trivial topology as documented in various field theory textbooks. The first effect pertains to the dynamics of a massive point-charged particle, including its coupling with an external electromagnetic field. While we are interested in setting up the effects on general grounds, several particular systems may be derived from this analysis. The second effect concerns modifications in the homogeneous Maxwell equations. The alluded modifications appear as a natural consequence of implementing nontrivial topology effects into (exterior) derivatives. As a result, we found additional terms in the standard electromagnetic field strength, arising then as side effects of introducing a nontrivial topology. Furthermore, we observe the emergence of a magnetic monopole-like effect, which could provide another perspective on the interplay between a nontrivial topology and electromagnetic interaction. By investigating these effects, we hope to contribute to understanding the fundamental implications and importance of considering the topological properties of spacetime.

We organized this paper as follows. The next section is devoted to setting the essential mathematical background, from exotic spinors to the corrections in the quadratic form, as introduced in [12]. This was written to give a path to be followed within the algebraic or topological mathematical formalism. Since it stands for a more formal section, we start exploring a more physical system presenting an analog idea. Section 3 covers an investigation into the aforementioned physical systems. The correction in the geodesic motion of a massive particle with and without an external electromagnetic field is presented. Even when dealing with more or less familiar cases, we take care of them openly and discuss all the steps, calling attention to the modifications imputed to the corrected quadratic form. This section also presents a magnetic monopole-like effect due exclusively to the nontrivial topology terms. In the last section, we conclude the paper.

2. Mathematical Preliminaries

Before revising the basic steps of formalism, and since we shall go through some pieces of algebraic topology in a broad brush, let us observe some insightful aspects of metric correction terms due to a nontrivial topology through the nonlinear sigma model analogy. The base manifold is the usual Minkowski space $\mathbb{R}^{1+3}$, upon which we consider a set of $N+1$ scalar fields $\varphi^M(x)$, $M = m, N+1$ and $m = 1, \cdots, N$. The fields are subject to the constraint $\sum_M \varphi^M(x)\varphi^M(x) = 1$, which can trivially be solved by $\varphi^{N+1} = \pm\Big[1 - \sum_m \varphi^m(x)\varphi^m(x)\Big]^{1/2}$. This constraint is sufficient to set the so-called target space mapping:

$$\begin{aligned} \mathbb{R}^{1+3} \quad &\to \quad S^N \cong SO(N+1)/SO(N) \\ x^\mu \quad &\mapsto \quad \varphi^m(x). \end{aligned}$$

The target space has a trivial topology, but the geometrical constraint shall induce corrections upon a metric given in terms of the fields. Let us start from the free Lagrangian density (with a sum over all indexes assumed) $\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu\varphi^M(x)\partial^\mu\varphi^M(x)$, which is simply decomposed as

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu\varphi^m(x)\partial^\mu\varphi^m(x) + \frac{1}{2}\partial_\mu\Big(\pm\sqrt{1-\varphi^m(x)\varphi^m(x)}\Big)\partial^\mu\Big(\pm\sqrt{1-\varphi^n(x)\varphi^n(x)}\Big). \quad (1)$$

By rewriting the first term and performing the derivatives, one arrives at

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\left(\delta^{mn} + \frac{\varphi^m(x)\varphi^n(x)}{1-\varphi^k(x)\varphi^k(x)}\right)\partial_\mu\varphi^m(x)\partial^\mu\varphi^n(x) \equiv \frac{1}{2}g_{mn}(\varphi)\partial_\mu\varphi^m(x)\partial^\mu\varphi^n(x) \tag{2}$$

In addition, a metric correcting the δ^{mn} term is reached due to a geometrical constraint. The parallel we will see is that a region with a nontrivial spacetime topology shall induce a metric correction very much as in the spirit just shown via this nonlinear sigma model.

More than one spinorial structure arises in a spacetime that is not simply connected [7]. Spinors belonging to a different spinorial structure from the usual one are called exotic spinors [6,7]. The appropriate steps for connecting exotic spinors to a corrected spacetime metric were developed and motivated in [12]. Here, we shall pinpoint some steps to a certain context. Let $\cup_{i\in\mathbb{N}^*}U_i$ be a simple covering of the spacetime endowed with a nontrivial topology $\mathcal{M}$. Since we are interested in spinorial representations, consider h_{ij} to be functions relating the intersections between the coverings to the $Spin(1,3)$ group (i.e., $h_{ij} : U_i \cap U_j \subset \mathcal{M} \to Spin(1,3)$). The different patching in the spacetime (due to the nontrivial topology) is reflected in the existence of $\tilde{h}_{ij}$, also mapping $U_i \cap U_j \to Spin(1,3)$ such that $\tilde{h}_{ij}(x) = h_{ij}(x)C_{ij}(x)$ for $x \in U_i \cap U_j$, where $C_{ij} \in \mathbb{Z}_2 \hookrightarrow Spin(1,3)$ are the transition functions. As the spacetime is not simply connected by the relation between the de Rham and Čech cohomologies for base differential manifolds [14], the transition functions represent the cocycle $C_{ij} : U_i \cap U_j \to \mathbb{Z}_2$. A spinor field on $\mathcal{M}$, ψ, is a section of the frame bundle $P_{Spin(1,3)} \times_\sigma \mathbb{C}^4$, where $\sigma = \{(1/2,0),(0,1/2),(1/2,0)\oplus(0,1/2)\}$. Conceivably, $\tilde{P}_{Spin(1,3)} \times_\sigma \mathbb{C}^4 \ni \tilde{\psi}$ for the exotic spinors. By denoting with $Cl(1,3)$ the Clifford algebra and $M(4,\mathbb{C})$ the set of 4 complex matrices, one can define the linear mapping

$$\rho : Spin(1,3) \subset Cl(1,3) \quad \to \quad M(4,\mathbb{C})$$

and functions (assumed to exist for the general properties of $\check{H}(M,\mathbb{Z}_2)$)

$$\xi_i \in U(1) : U_i \to \mathbb{C}. \tag{3}$$

If $x \in U_i \cap U_j$, then $\xi_i(x) = \rho(C_{ij})\xi_j(x)$. Since ρ is faithful such that $\rho(C_{ij}) = \pm 1$, the local spinorial sections are connected by

$$\psi_i = \rho(h_{ij})\psi_j, \tag{4}$$

This is similar for local exotic spinors; that is, $\tilde{\psi}_i = \rho(\tilde{h}_{ij})\tilde{\psi}_j$. Now, from $\tilde{h}_{ij}(x) = h_{ij}(x)C_{ij}(x)$, we have $\tilde{\psi}_i = \rho(h_{ij})\rho(C_{ij})\tilde{\psi}_j$. Notice that $\xi_j(x) = \rho(C_{ji})\xi_i(x) = \rho(C_{ij})\xi_i(x)$, and by inserting $\rho(\xi_i)$ in the left and right sides in the above expression of $\tilde{\psi}$, we arrive at

$$\rho(\xi_i)\tilde{\psi}_i = \rho(h_{ij})\rho(\xi_j)\tilde{\psi}_j. \tag{5}$$

When comparing Equations (4) and (5), we readily obtain $\psi_k = \rho(\xi_k)\tilde{\psi}_k$ in the same local chart U_k of a simple covering of $\mathcal{M}$. As a matter of fact, $\xi_i^2 = \xi_j^2$ in $U_i \cap U_j$, and therefore, there is a global uniquely defined function, say $\xi : \mathcal{M} \to \mathbb{C}$, such that $\xi(x) = \xi_i^2(x)$. This is the final piece to write for a globally defined relation between the usual and exotic spinors, given in a straightforward fashion by

$$\psi = \rho(\xi)\tilde{\psi}. \tag{6}$$

From this last expression, it is possible to compute the dynamics of an exotic spinor very similar to a covariant derivative in standard gauge theories, requiring that $\partial\psi = \rho(\xi)\tilde{\nabla}\tilde{\psi}$ (see, for instance, [13]). This reasoning immediately leads to

$$\tilde{\nabla} = \partial + \rho^{-1}(\xi)\nabla\rho(\xi). \tag{7}$$

There are two points to be stressed. First, notice that had we worked a representation mapping $\mathbb{Z}_2$ into the identity, then Equation (6) would present no difference between the exotic and usual fields. That is why fermionic fields are the only ones with inequivalent exotic counterparts. Secondly, and more important to our purposes, when acting upon a spinor, the derivative in Equation (7) shall ultimately operate (with the correction term) in spinor entries. This last point is particularly relevant when contrasting exotic spinors with Cartan's construction.

It is a well-known fact that, via Cartan's formalism, a given spacetime point $P = (t, x, y, z)$ may be expressed as $\sqrt{2}P = (\zeta\zeta^* + \xi\zeta^*, \zeta\zeta^* + \xi\zeta^*, i(\xi\zeta^* - \zeta\xi^*), \zeta\zeta^* - \xi\xi^*)$, where ζ and ξ are the spinor entries [3]. This fact is quite remarkable, and we shall explore this fact along with the formalism of exotic spinors. All the details are discussed in [12], but some steps may be pinpointed here.

Recall the standard procedure for finding differentials. Denoting with $\{e_i\}$ the basis of $\mathbb{R}^n$ and $\{dx^j\}$ the basis of $(\mathbb{R}^n)^*$, consider a vector field, say $h \in \mathbb{R}^n$, upon which the differential of a smooth function f acts in $x_0 \in \mathbb{R}^n$ (i.e., $df(x_0)(h) = \sum_i [\partial f/\partial x^i]_{x_0} h^i$). Through the usual linear orthogonal projections $\pi^i : \mathbb{R}^n \to \mathbb{R}$, such as $d\pi(h) = dx^i(h) = h^i$, the differential expression may be written abstractly independent of acting upon h. This is the crevice we shall explore to incorporate exotic effects. According to Cartan, $x^\mu \propto (\xi\xi^*)$, leading to $d\pi^\mu = dx^\mu \propto d(\xi\xi*)^\mu = \partial_\mu(\xi\xi^*)dx^\mu = (\partial_\mu(\xi)\xi^* + \xi\partial_\mu(\xi^*))dx^\mu$, and without exotic spinor considerations, no effect is felt, since we may rewrite the expression backward. The situation changes if we consider exotic spinor entries, for which derivatives are shifted due to the nontrivial topology. Motivated by Equation (7), we shall perform $\partial_\mu \mapsto \partial_\mu + \partial_\mu\theta$ for $\theta \in \mathbb{R}$, by means of which (with a simple rescaling in θ) it can be readily verified that $d\pi^\mu = dx^\mu + x^\mu d\theta$ (with $d\theta = \partial_\mu\theta dx^\mu$), culminating in

$$df(x_0) = \partial_\mu f|_{x_0} dx^\mu + \partial_\mu f|_{x_0} x^\mu d\theta. \tag{8}$$

In every step of the above reasoning (and from now on), as long as θ is (or may be regarded as) constant, the effects of the nontrivial topology are disregarded. Whenever this is the case, the standard expressions are recovered. The shift $dx^\mu \mapsto dx^\mu + x^\mu d\theta$ is at the heart of the effects we shall explore in this paper. Upon careful examination, it is now apparent that the customary spacetime transformations cannot be regarded as universal symmetries within the current context. This is mainly due to the inclusion of θ terms in the metric, which fundamentally undermine the notion of inertial frames. We started noticing that this brings about important consequences when investigating the quadratic form[1]

$$\tilde{\eta} = \eta_{\mu\nu}(dx^\mu + x^\mu d\theta) \otimes (dx^\nu + x^\nu d\theta) \tag{9}$$

whose action upon the base vectors $\tilde{\eta}(e_\mu, e_\nu)$ yields

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} + x_\mu\partial_\nu\theta + x_\nu\partial_\mu\theta + x^2\partial_\mu\theta\partial_\nu\theta. \tag{10}$$

The introduction of the derivative of θ within $\tilde{\eta}_{\mu\nu}$ might point toward a curved spacetime rather than a flat one. Although exploring this avenue of investigation appears promising, we shall defer it to future research, as explicitly indicated in Section 4. We have argued (see [12,13]) that the θ corrections may be interpreted as follows: a correction necessary in a certain region, say $\tilde{\mathbb{R}}^{1+3}$, with a trivial topology but near enough to the nontrivial topology region. For this idea, we may think of the nontrivial topology localized in a finite region $\mathcal{R} \supseteq \mathbb{R} \times (\mathbb{R}^2 \times S^1)$. When we say near enough, we mean that the nontrivial topology affects physical systems in a given neighborhood $\mathcal{V}$ of $\mathcal{R}$ ($\mathcal{V} \supset \mathcal{R}$) such that the net effect perturbs the usual case in $\mathcal{V}\backslash\mathcal{R} \simeq \tilde{\mathbb{R}}^{1+3}$ but may be neglected in other domains of the spacetime. Yet, in $\tilde{\mathbb{R}}^{1+3}$, the first $\theta(x)$ derivative shall be taken as a small value so that terms such as $\partial^2\theta$ and $(\partial\theta)^2$ are negligible. Incidentally, we note that in natural units (θ is dimensionless), the correction terms $\partial\theta$ scale as $(length)^{-1}$, and the θ correction terms can then be thought of as a high-energy effect.

3. Physical Effects

In order to establish the set-up for our further analysis, notice that $\tilde{\eta}(\Delta x^\mu, \Delta x^\mu)$, where Δx^μ stands for spacetime displacements. This leads, in the infinitesimal limit, to

$$d\tilde{s}^2 = ds^2 + 2x_\mu \partial_\alpha \theta dx^\mu dx^\alpha, \tag{11}$$

in $\tilde{\mathbb{R}}^{1+3}$, where $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ as usual. The explicit appearance of x^μ terms in the quadratic form requires due care. This term breaks the Poincarè symmetries, and its consequences need to be investigated case by case (including special attention to experimental bounds [15,16]). In [12], it is shown that several considerations may taken to appreciate the physical consequences without possible ambiguity coming from these factors in some cases. Regarding the general analysis we are about to conduct, it is relevant to note that for an observer for which $d\vec{x}$ is null, we are left with $d\tilde{s}^2 = (1 + 2t\dot{\theta})dt^2$, where a dot stands for a derivative with respect to time. Hence, as can be seen, the very concept of proper time is cumbersome here. Nevertheless, for reinforcing that the dimension of $d\tilde{s}$ is *lenght*, we shall adopt a somewhat pragmatic posture, bearing in mind that when $\theta \to cte$, all the usual concepts and interpretations apply. This allows us to deal with the consequences of the θ terms in the line element.

It may be helpful to keep in mind that, since we are in $\tilde{\mathbb{R}}^{1+3}$, the nontrivial topology effects act as a kind of external source, disturbing the physical system at hand. In what follows, we shall evince two interesting situations in which this happens.

3.1. Relativistic Massive Particle Dynamics in $\tilde{\mathbb{R}}^{1+3}$

Let us begin proposing an action borrowing its functional form from the usual massive relativistic particle dynamics but with $d\tilde{s}$ as an element to be integrated; in other words, we have

$$\tilde{S} = \kappa \int_{(\alpha)} d\tilde{s}, \tag{12}$$

where the integration is performed along a lineworld in $\tilde{R}^{1+3}$ and κ is a constant to be determined. We shall extremize Equation (12) to reach the relativistic particle equation of motion. Notice that, of course, $\delta d\tilde{s}^2 = 2d\tilde{s}\delta d\tilde{s}$, while on the other hand, we have

$$\delta d\tilde{s}^2 = \delta\{(\eta_{\mu\nu} + 2x_\mu \partial_\nu \theta) dx^\mu dx^\nu\} \equiv \delta(\tilde{\eta}_{\alpha\beta} dx^\alpha dx^\beta) \tag{13}$$

and therefore

$$\delta d\tilde{s} = \tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \delta dx^\beta + \frac{1}{2} \delta \tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}} d\tilde{s}. \tag{14}$$

Within the discussed approximations $\delta\tilde{\eta}_{\alpha\beta} \approx 2\delta x_\alpha \partial_\beta \theta$ and the variation in Equation (12), when taking into account Equation (14), the result reads as follows:

$$\delta\tilde{S} = \kappa \int_{(\alpha)} \Big\{ \tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \delta dx^\beta + \delta x_\alpha \partial_\beta \theta \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}} d\tilde{s} \Big\}. \tag{15}$$

The first term under integration may be rewritten as

$$\tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \delta dx^\beta = d\Big(\tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \delta x^\beta \Big) - d\tilde{\eta}_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \delta x^\beta - \tilde{\eta}_{\alpha\beta} d\Big(\frac{dx^\alpha}{d\tilde{s}} \Big) \delta x^\beta. \tag{16}$$

Upon integration, since the endpoints are fixed, the first term of Equation (16) vanishes. Aside from that, $d\tilde{\eta}_{\alpha\beta} \approx 2dx_\alpha \partial_\beta \theta$, and a bit of usual algebra leads to

$$\delta\tilde{S} = \int_{(\alpha)} \Big\{ -\kappa \tilde{\eta}_{\alpha\beta} \frac{d^2 x^\alpha}{d\tilde{s}^2} - 2\kappa \partial_\beta \theta \frac{dx^\alpha}{d\tilde{s}} \frac{dx_\alpha}{d\tilde{s}} + \kappa \partial_\alpha \theta \frac{dx^\alpha}{d\tilde{s}} \frac{dx_\beta}{d\tilde{s}} \Big\} \delta x^\beta d\tilde{s}. \tag{17}$$

The imposition $\delta\tilde{S} = 0$, $\forall\, \delta x^\beta$, along with the expression for $\tilde{\eta}_{\alpha\beta}$, leads to

$$-\kappa\left[\frac{d^2x_\beta}{d\tilde{s}^2}+2\partial_\beta\theta\left(x_\alpha\frac{d^2x^\alpha}{d\tilde{s}^2}+\frac{dx_\alpha}{d\tilde{s}}\frac{dx^\alpha}{d\tilde{s}}\right)\right]+\kappa\partial_\alpha\theta\frac{dx^\alpha}{d\tilde{s}}\frac{dx_\beta}{d\tilde{s}}=0. \quad (18)$$

It is worth noticing that if $\partial\theta \to 0$, then $d\tilde{s} \to ds$, and we recover the standard case for $\kappa = -m$. This straightforward procedure sets up the κ constant. Moreover, the term in parenthesis may be recast in a simple form so that we are left with

$$m\frac{d^2x_\beta}{d\tilde{s}^2}=m\partial_\alpha\theta\frac{dx^\alpha}{d\tilde{s}}\frac{dx_\beta}{d\tilde{s}}-m\partial_\beta\theta\frac{d^2(x_\alpha x^\alpha)}{d\tilde{s}^2}, \quad (19)$$

as the dynamical equation for the massive particle. As evident from the equation of motion, the effect of the nontrivial topology engenders an external force term (even for the free particle case). This effect is the pure particle dynamical counterpart of the quasinormal-like behavior reported in [12] in the context of fields.

It is common to couple the particle to an external electromagnetic field to recover the Lorentz force term, driving the dynamics in the presence of an external field. Let us investigate this case in detail. Along with the discussion in the preceding section, we shall also expect additional terms from the nontrivial topology.

We start with an electromagnetic action coupling a particle with a charge e to the electromagnetic potential. To accomplish this, we start with $\tilde{A} = A_\mu d\tilde{x^\mu}$ and implement the previously obtained shift, arriving at $\tilde{A} = A_\mu(dx^\mu + x^\mu d\theta)$. Since $d\theta = \partial_\alpha\theta dx^\alpha$, after adequately relabeling the indices of the second term, we have $\tilde{A} = (A_\mu + A_\alpha x^\alpha\partial_\mu\theta)dx^\mu$. Therefore, the action is given by

$$\tilde{S}_{em} = -e\int_{(\alpha)}\tilde{A}, \quad (20)$$

where the $-e$ constant factor was introduced by following the same reasoning establishing $\kappa = -m$ in our previous analysis. The variation in Equation (20) may be written as

$$\delta\tilde{S}_{em} = \delta S_{em} - e\,\delta\int_{(\alpha)}A_\alpha x^\alpha\partial_\mu\theta dx^\mu, \quad (21)$$

where δS_{em} stands for the usual textbook case

$$\delta S_{em} = -e\int_{(\alpha)}F_{\beta\delta}\frac{dx^\delta}{d\tilde{s}}\delta x^\beta d\tilde{s}, \quad (22)$$

with $F_{\beta\delta}$ denoting the standard electromagnetic field strength. The last term of Equation (21) will be analyzed here. It can be expressed as

$$\delta\int_{(\alpha)}A_\alpha x^\alpha\partial_\mu\theta dx^\mu=\int_{(\alpha)}\left[\frac{\partial A_\alpha}{\partial x^\beta}x^\alpha\partial_\mu\theta\frac{dx^\mu}{d\tilde{s}}+A_\beta\partial_\mu\theta\frac{dx^\mu}{d\tilde{s}}\right]\delta x^\beta d\tilde{s}+\int_{(\alpha)}A_\alpha x^\alpha\partial_\mu d\delta x^\mu. \quad (23)$$

In turn, the last integral may be recast, noticing that

$$d(A_\alpha x^\alpha\partial_\mu\theta) = \partial_\beta A_\alpha dx^\beta x^\alpha\partial_\mu\theta + A_\alpha dx^\alpha\partial_\mu\theta \quad (24)$$

Therefore, as the left-hand side vanishes under integration, we have

$$\int_{(\alpha)}A_\alpha x^\alpha\partial_\mu\theta d\delta x^\mu = -\int_{(\alpha)}\partial_\mu A_\alpha x^\alpha\partial_\beta\theta\frac{dx^\mu}{d\tilde{s}}d\tilde{s}\delta x^\beta-\int_{(\alpha)}A_\alpha\partial_\beta\theta\frac{dx^\alpha}{d\tilde{s}}d\tilde{s}\delta x^\beta. \quad (25)$$

Inserting Equation (25) back into Equation (23) leads to

$$\delta\int_{(\alpha)}A_\alpha x^\alpha\partial_\mu\theta dx^\mu=\int_{(\alpha)}\left\{(\partial_\beta A_\alpha\partial_\mu\theta-\partial_\mu A_\alpha\partial_\beta\theta)x^\alpha\frac{dx^\mu}{d\tilde{s}}+(A_\beta\partial_\alpha\theta-A_\alpha\partial_\beta\theta)\frac{dx^\alpha}{d\tilde{s}}\right\}\delta x^\beta d\tilde{s} \quad (26)$$

The electromagnetic coupling action variation in Equation (21) is given by Equations (22) and (26). As can be seen from the equation above, if no correction comes from the nontrivial topology terms, then the only contribution is the standard one. Therefore, apart from the pure dynamical "external force" corrections in Equation (19), in the presence of an electromagnetic field, we also have nontrivial topology effects cascading down to $\delta\tilde{S}_{em}$.

When taking all the effects into account (that is, $\delta\tilde{S}_{total} = \delta\tilde{S} + \delta\tilde{S}_{em} = 0 \;\forall\; \delta x^{\beta}$), Equations (17), (22) and (26) give the following complete equation of motion:

$$
\begin{aligned}
m\frac{d^2x_\beta}{d\tilde{s}^2} &= eF_{\beta\alpha}\frac{dx^\alpha}{d\tilde{s}} + m\partial_\alpha\theta\frac{dx^\alpha}{d\tilde{s}}\frac{dx_\beta}{d\tilde{s}} - m\partial_\beta\theta\frac{d^2(x_\alpha x^\alpha)}{d\tilde{s}^2} \\
&+ e(\partial_\beta A_\alpha\partial_\mu\theta - \partial_\mu A_\alpha\partial_\beta\theta)x^\alpha\frac{dx^\mu}{d\tilde{s}} + e(A_\beta\partial_\alpha\theta - A_\alpha\partial_\beta\theta)\frac{dx^\alpha}{d\tilde{s}}.
\end{aligned}
\tag{27}
$$

Note that in the absence of $\partial\theta$ terms, we are left with the usual Lorentz force equation, as expected. In $\tilde{\mathbb{R}}^{1+3}$, the dynamics of a free particle of mass m and charge e subject to an external electromagnetic field and under the effects of a nontrivial topology in the presented context are shown. The resulting equation is quite complicated, and extracting the particle's physical motion may be challenging even in simple situations (e.g., external electric and magnetic constant fields). In any case, deviations from the standard motion are, of course, expected and may probe, at least argumentatively, the unusual spacetime effects. While this situation may be simplified for particular linear θ function sub-cases, we would like to investigate an effect entirely imputed to the electromagnetic field dynamics in $\tilde{\mathbb{R}}^{1+3}$.

3.2. Magnetic Monopole-like Effect

From the exterior derivative in $\tilde{\mathbb{R}}^{1+3}$, several potentially interesting effects were reported in [12]. Here, we shall pursue a pure electrodynamics effect based on the simple fact that the new field strength $\tilde{F}$ presents additional terms coming from the nontrivial topology, which resembles the non-abelian obstructions to the validity of the partial (not covariant) derivative Jacobi equation in Yang–Mills theory. However, as the additional terms do not come from any sophistication in the gauge group, there is no need for any symmetry-breaking mechanism to recover electromagnetism. A similar role, so to speak, was already performed here by Lorentz symmetry breaking due to the spacetime topology. In fact, as reported in [12], some expected results of exterior calculus present differences in $\tilde{\mathbb{R}}^{1+3}$. Here, we shall report the most direct consequence of breaking $\nabla\cdot(\nabla\times\cdot) = 0$.

From $\tilde{A} = (A_\mu + A_\alpha x^\alpha\partial_\mu\theta)dx^\mu$, and taking into account the correction in the exterior derivative (see the appendix of [12] for a complete account of that), we have

$$\tilde{F} = \partial_\nu(A_\mu + A_\alpha x^\alpha\partial_\mu\theta)dx^\mu \wedge d\tilde{x}^\nu, \tag{28}$$

from which we see that

$$\tilde{F} = \frac{1}{2}\tilde{F}^{\mu\nu}dx_\mu \wedge dx_\nu, \tag{29}$$

where

$$\tilde{F}^{\mu\nu} = F^{\mu\nu} + 2(A^\mu + x^\alpha\partial^\mu A_\alpha)\partial^\nu\theta + \partial_\alpha A^\nu x^\alpha\partial^\mu\theta. \tag{30}$$

Now, it is possible to read the electric and magnetic field components' counterparts in $\tilde{\mathbb{R}}^{1+3}$. In particular, the magnetic field components are given by

$$
\begin{aligned}
\tilde{B}_x &= B_x + 2(A^z + x^\alpha\partial^z A_\alpha)\partial^y\theta + \partial_\alpha A^y x^\alpha\partial^z\theta, \\
\tilde{B}_y &= B_y + 2(A^x + x^\alpha\partial^x A_\alpha)\partial^z\theta + \partial_\alpha A^z x^\alpha\partial^x\theta, \\
\tilde{B}_z &= B_z + 2(A^y + x^\alpha\partial^y A_\alpha)\partial^x\theta + \partial_\alpha A^x x^\alpha\partial^y\theta.
\end{aligned}
\tag{31}
$$

The divergence of this vector will lead to $\nabla\cdot\mathbf{B}$, which is zero, but corrections will also appear due to the $\partial\theta$ terms, even in the approximate scenario explored here. After some standard algebra, the final result reads as follows:

$$\nabla \cdot \tilde{\mathbf{B}} = C_1(x^\alpha, \partial A)\partial^x\theta + C_2(x^\alpha, \partial A)\partial^y\theta + C_3(x^\alpha, \partial A)\partial^z\theta, \tag{32}$$

where the coefficients are given by

$$\begin{aligned} C_1(x^\alpha, \partial A) &= 2\partial_y A^z + x^\alpha \partial_y(\partial_\alpha A^z + \partial^z A_\alpha) + \partial_z A^y, \\ C_2(x^\alpha, \partial A) &= 2\partial_z A^x + x^\alpha \partial_z(\partial_\alpha A^x + \partial^x A_\alpha) + \partial_x A^z, \\ C_3(x^\alpha, \partial A) &= 2\partial_x A^y + x^\alpha \partial_x(\partial_\alpha A^y + \partial^y A_\alpha) + \partial_y A^x. \end{aligned} \tag{33}$$

This result suffices to evince a magnetic monopole-like behavior due to spacetime nontrivial topology effects. There are several points to be addressed. It is interesting to note that if θ is a function of time (exclusively), then no magnetic monopole-like effect is expected. When this is not the case, on the other hand, the associated "magnetic charge" is variable and actually quite involved. This may be an obstacle to handling the physical effects probing this behavior. Finally, as is well known, magnetic monopoles are not expected in the breaking $SU(2) \times U(1) \to U_{em}(1)$ [17], but they may have a place in grand unification schemes. When this is the case, the typical energy scale setting for the monopole mass is about 10^{16} GeV [18]. According to the discussion around Equation (11), it is unlikely that the approach presented here could affect a lower energy scale. In any case, it is also unclear if this formalism can be applied as an effective theory to low-energy realizations of spin ice (Dirac) string in materials [19].

4. Discussion and Outlook

The results presented here, along with the analysis in [12,13], cover some exciting and unexplored physical outputs of nontrivial topologies. In particular, this paper highlights the effects of a nontrivial topology on the dynamics of some simple but fundamental physical systems, namely the dynamics of free particles and charged particles coupled to an electromagnetic field. New phenomena have been shown by investigating the underlying corrections emerging in such systems, including the appearance of a magnetic monopole-like effect. Aside from incorporating θ terms into the quadratic form, the corrections explicitly bring about spacetime coordinate points. This leads to a scenario of explicitly broken Lorentz symmetry. Thus, when revisiting the Euler–Lagrange equations, Hamilton's principle and other field theory fundamentals would be relevant for the explicitly dependent x^μ functional. It may be a subtle task to deal with boundaries in such an endeavor, but it could bypass the problem of implementing nontrivial effects via dynamical equations. We shall delve into this point in the near future.

The results reported here not only provide a new perspective on the interplay between a nontrivial topology and electromagnetic interaction but also contribute to our understanding of the fundamental implications and importance of considering the local effects of a globally nontrivial spacetime.

We want to end this paper by pointing out some branches of research we hope to explore in the near future. We present a somewhat natural unfolding of our analysis points to curved spacetime physics. Even without curvature sources in spacetime, nontrivial topological effects incorporated into the metric would lead to a relevant scenario. Of course, since Poincarè symmetries are lost in the flat case, we do not expect a curved spacetime theory endowing full diffeomorphism invariance. However, the breaking terms could be manageable. In a different context, the underlying mathematics supporting a nontrivial localized topology is exciting and deserves a careful look. Preliminary analysis shows that a suitable deformation in the map relating the standard and exotic spin bundle, along with a given set of requirements, may suffice to constrain the nontrivial cocycle to the desired region from the mathematical point of view.

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Note

[1] We used a mostly negative signature.

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