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Derivation of Emergent Spacetime Metric, Gravitational Potential and Speed of Light in Superfluid Vacuum Theory

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Abstract: Within the frameworks of the logarithmic superfluid model of physical vacuum, we demonstrate the emergence of four-dimensional curved spacetime from the dynamics of quantum Bose liquid in three-dimensional Euclidean space. We derive the metric tensor of this spacetime and study its special cases and limits, such as the linear-phase flow and linearized gravity limit. We show that the value of speed of light, which is a fundamental parameter in a theory of relativity, is a derived notion in superfluid vacuum theory: its value is a combination of the Planck constant and original parameters of the background superfluid. As for the gravitational potential, then it can be defined in terms of the quantum information entropy of the background superfluid. Thus, relativistic gravity and curved spacetime are shown to result from the dynamics of quantum excitations of the background superfluid being projected onto the measurement apparatus of a relativistic observer.

Keywords: emergent spacetime; modified gravity; quantum gravity; superfluid vacuum; logarithmic fluid

1. Introduction

In superfluid vacuum theory (SVT), Lorentz symmetry and four-dimensional spacetime are approximate notions, which emerge in the low-momentum small-fluctuations limit of the background superfluid's dynamics; a pedagogical introduction can be found in the monographs [1,2]. This background superfluid is a quantum object whose dynamics is defined in three-dimensional Euclidean space, the latter, however, being unobservable, or "fictitious", for someone detecting only small fluctuations of the superfluid wavefunction. Instead, such an observer perceives four-dimensional spacetime with a pseudo-Riemannian metric tensor of a certain form, for which reason they are called a relativistic or R-observer in what follows, to differentiate from the F(ull)-observers. This mapping is the main underlying feature of superfluid vacuum theory, making this post-relativistic approach fundamentally different from habitual modified gravity theories [3,4].

This naturally simplifies the problem of gravity's quantization: instead of quantizing the spacetime (which is essentially a set of length- and time-measurement rules and mappings, or "rods and clocks"), one quantizes the matter, being the background quantum liquid in this case, in the conventional quantum-mechanical way; and then projects the results onto what is "seen" by R-observer's measuring apparatus. In other words, a theory of gravitational interaction, at both the classical and quantum level, can be formulated as a subset of the theory of physical vacuum, with the latter being treated as a state of matter, although somewhat different from those we usually deal with [5]. The general theory of condensed matter itself has clear physical foundations and methods which have rich history and solid experimental evidence. This should help to narrow down a set of mathematical variations of an SVT-based quantum gravity and cosmology, as well as to explore the applicability borders of a relativistic theory of gravity—and to go beyond, towards a generalization thereof.

However, the difference exists here from canonical condensed matter theory: in superfluid vacuum theory, one still has to establish the "dictionary" of the above-mentioned

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R-observer projection, which would translate between the R- and F-observer's pictures. This is a nontrivial mathematical task on its own, because of the complexity of fluid dynamics on one side and Riemannian geometry on another. Nevertheless, it is known to be largely possible in the low-momentum small-fluctuations limit of SVT, via the formal mapping known as the fluid/gravity correspondence [6]; the latter being usually used for mimicking general relativistic effects within suitable laboratory materials.

In order to fully achieve the goal of describing the physical vacuum, the background superfluid's microscopical structure must be further specified on a theoretical level, for which purpose one needs specific models. One of the popular models in a theory of diluted Bose–Einstein condensates, described by Gross–Pitaevskii (or cubic Schrödinger) equation, is unlikely suitable for this purpose because it is based on the assumption of the two-body delta-singular interparticle interaction potential, which is a way too restricting to be applicable to the case of physical vacuum.

In previous works on the theme, to mention the landmark works [7,8], we advocated a superfluid vacuum theory based on the assumption that the background superfluid belongs to a class of logarithmic fluid models. The latter have already found applications in modeling quantum Bose liquids produced in a laboratory, such as helium superfluids and Bose–Einstein condensates (BEC) of alkali atoms [9–12]. Moreover, those studies resulted in a strong intuitive feeling that the logarithmic models are a crucial ingredient and theoretical tool necessary for a consistent explanation/description of vacuum effects in laboratory quantum Bose liquids. It was thus natural to apply logarithmic fluid models to the physical vacuum itself.

The outline of this paper follows. In Section 2, we give an idea of how relativistic gravity emerges from the non-relativistic background superfluid dynamics: we show that small fluctuations of the superfluid density are governed by Lorentz covariant equations in curved spacetime; in Section 2.2 we derive the metric tensor of such spacetime. In Section 3, we consider various approximations and spacial cases of the emergent spacetime. Section 4 is devoted to the linearized limit of the emergent spacetime, where we derive both the effective gravitational potential and the speed of light, in the conformal Newtonian gauge. Conclusions are drawn in Section 5.

2. Spacetime as an Emergent Phenomenon

In order to see the emergence of spacetime from superfluid vacuum, one needs to perform two consecutive mappings. First, one rewrites the original three-dimensional wave equation describing the motion of the logarithmic Bose liquid in the fluid-mechanical form, which is broadly called the fluid-Schrödinger analogy, see Section 2.1. Second, one demonstrates that small fluctuations of superfluid's wavefunction obey the equations of motion which have the local Lorentz symmetry and derives the corresponding spacetime interval, see Section 2.2.

2.1. Fluid-Schrödinger Analogy

In this section, a formal mapping is derived between wave equations of type (2) and motion equations for inviscid flow of fluid substances.

Let us also assume that the state of such a substance can be described by a single complex-valued function defined in the three-dimensional Euclidean space, $\Psi = \Psi(\mathbf{x}, t)$, which obeys the normalization condition

$$\int_{\mathcal{V}} |\Psi|^2 d\mathcal{V} = \int_{\mathcal{V}} \rho \, d\mathcal{V} = M > 0, \tag{1}$$

where $\rho = \rho(\mathbf{x}, t) = |\Psi(\mathbf{x}, t)|^2$ is a fluid density M and $\mathcal{V}$ are the total mass and volume of the material.

Let us consider a nonlinear $U(1)$-symmetric Schrödinger equation of the general form:

$$i\partial_t \Psi = \left[-\frac{\mathcal{D}}{2} \boldsymbol{\nabla}^2 + \frac{1}{\hbar} V(\mathbf{x}, t) - \frac{1}{\hbar} b(\mathbf{x}, t) \ln\Big(|\Psi|^2 / \rho_0 \Big) \right] \Psi, \tag{2}$$

where $b(\mathbf{x}, t)$ is a real-valued coupling function, $\mathcal{D}$ and ρ_0 are positive constants, and $V(\mathbf{x}, t)$ is external potential, often called trap potential in the conventional BEC theory. While the parameter ρ_0 is just a scaling one, the coupling function b is a dynamical value related to quantum temperature, a more detailed discussion can be found in [13] (there were also extensive mathematical studies of the logarithmic nonlinear models and around, to mention just very recent literature [14–24]).

The constant parameter $\mathcal{D}$ can be written as

$$\mathcal{D} = \hbar/m, \tag{3}$$

where m is a mass of a background superfluid's particle. The latter should not be confused with the relativistic particles which are defined based on irreducible representations of the Poincaré group and thus emerge together with superfluid-induced spacetime. Instead, the superfluid particle is a Euclidean quantum-mechanical object which is thus unobservable by an R-observer directly. Moreover, this particle cannot be observed by the full observer either, if the background superfluid is in the state close to the Bose–Einstein condensate. In that case, according to the BEC theory, the full observer would "see" a condensate's wave function which is a collective state, not a plain cloud of original particles. The superfluid particle can be detected on its own only if the condensate is broken, which can be regarded as the "vacuum breakdown", from the viewpoint of an R-observer.

Furthermore, our wavefunction can be always written in the Madelung form [25,26]

$$\Psi = \sqrt{\rho} \exp(i\mathcal{S}), \tag{4}$$

where $\mathcal{S} = \mathcal{S}(\mathbf{x}, t)$ is a phase of wavefunction, which will be related to the fluid velocity in what follows. If one applies the Madelung ansatz to the wave Equation (2), and separates real and imaginary parts, then one obtains a set of two equations for the wavefunction's amplitude squared and phase

$$\begin{aligned}
&\partial_t \rho + \mathcal{D}\boldsymbol{\nabla} \cdot (\rho \boldsymbol{\nabla} \mathcal{S}) = 0, \\
&\partial_t \mathcal{S} - \frac{\mathcal{D}}{2}\left[\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla}\sqrt{\rho}}{\sqrt{\rho}}\right) + \frac{(\boldsymbol{\nabla}\sqrt{\rho})^2}{\rho} - (\boldsymbol{\nabla}\mathcal{S})^2\right] = \frac{1}{\hbar} b(\mathbf{x}, t) \ln(\rho/\rho_0) - \frac{1}{\hbar} V(\mathbf{x}, t),
\end{aligned} \tag{5}$$

where a dot denotes an inner scalar product.

Furthermore, we take a gradient of the last equation and introduce

$$\mathbf{u} = \mathcal{D}\boldsymbol{\nabla}\mathcal{S}, \tag{6}$$

the fluid velocity. We then obtain fluid-mechanical equations for mass and momentum conservation

$$\begin{aligned}
&\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \\
&\rho D_t \mathbf{u} = \rho[\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u}] = \boldsymbol{\nabla} \cdot \mathbb{T} + \mathbf{f}_V,
\end{aligned} \tag{7}$$

where we denoted

$$\mathbf{f}_V = -\frac{\mathcal{D}}{\hbar} \rho \boldsymbol{\nabla} V, \;\; \mathbb{T} = -\frac{\mathcal{D}^2}{4\rho} \boldsymbol{\nabla}\rho \otimes \boldsymbol{\nabla}\rho - \tilde{p}\,\mathbb{I}, \;\; \tilde{p} = p - \frac{1}{4}\mathcal{D}^2 \boldsymbol{\nabla}^2 \rho, \tag{8}$$

where $D_t = D/Dt$ is a material derivative, $\mathbf{f}_V$ is the body force per unit volume, and $\mathbb{T}$ is the stress tensor being of the Korteweg form with capillary effects [27,28], $\mathbb{I}$ is the identity matrix, $\tilde{p}$ is the capillary pressure, and $p = p(\mathbf{x}, \rho)$ is the pressure given by the equation of state in a differential form

$$\boldsymbol{\nabla} p = -\frac{\mathcal{D}}{\hbar} \rho \boldsymbol{\nabla}[b(\mathbf{x}, t) \ln(\rho/\rho_0)]. \tag{9}$$

In the absence of heat transfer, Equations (6)–(9) form a closed system, which describes the isothermal inviscid flow of the fluid with internal capillarity. Because this system is equivalent to the original system (5), in what follows we shall use both sets of equations.

Finally, to gain more clear understanding of the fluid described by the equation of state (9), let us rewrite the latter in the form

$$\boldsymbol{\nabla}\left(p + \frac{\mathcal{D}}{\hbar} b\rho\right) = \frac{\mathcal{D}}{\hbar}\rho[1 - \ln(\rho/\rho_0)]\boldsymbol{\nabla} b, \tag{10}$$

which can be easily integrated for a case of the constant or slowly varying coupling function:

$$p = c_s^2 \rho + \mathcal{O}(\boldsymbol{\nabla} b), \tag{11}$$

where we introduced the averaged speed of acoustic oscillations

$$c_s^2 = -\frac{\mathcal{D}}{\hbar} b(\mathbf{x}, t), \tag{12}$$

and $\mathcal{O}(f)$ represents terms of order f. Same result can be obtained for a case of anisotropic logarithmic fluid by choosing a frame of reference aligned along the vector $\boldsymbol{\nabla} b$, such that $\boldsymbol{\nabla} b = |\boldsymbol{\nabla} b|\, \mathbf{n}$; then Equation (10) can be exactly integrated in the transverse directions to the normal $\mathbf{n}$, hence we obtain $p_\perp = c_s^2 \rho$.

Equations (11) and (12) indicate that our quantum Bose liquid behaves as an ideal fluid in the leading-order approximation, because c_s^2 does not dependent on density. This feature originates from the logarithmic term in Equation (2), which makes logarithmic fluid models outstanding in their category. It also connects this theory to the origin of Einstein's second postulate, as discussed in more details in [7] and below.

To date, we were describing the picture seen by an F-observer, operating in Euclidean three-dimensional space. What does the R-observer see?

2.2. Superfluid-Spacetime Correspondence

In this section, we discuss the projection of the background superfluid effects onto the R-observer's measuring apparatus. To begin with, let us consider small perturbations of logarithmic superfluid. We therefore introduce some shorthand notations

$$\phi \equiv -\mathcal{D}\,\mathcal{S}, \; \mathbf{u} = -\boldsymbol{\nabla}\phi, \tag{13}$$

and perform the following expansions

$$\rho = \rho_{(0)} + \rho_{(1)}, \; \phi = \phi_{(0)} + \phi_{(1)}, \; \mathbf{u} = \mathbf{u}_{(0)} + \mathbf{u}_{(1)}, \tag{14}$$

where bracketed subscripts denote the orders of approximation, such that $|\rho_{(1)}| \ll |\rho_{(0)}|$ and so on.

Then we linearize Equations (5)–(9) by keeping only first powers of the first-order values. We obtain the following set of equations

$$\partial_t \rho_{(0)} - \boldsymbol{\nabla} \cdot \left(\rho_{(0)} \boldsymbol{\nabla} \phi_{(0)} \right) = 0, \tag{15}$$

$$\partial_t \rho_{(1)} - \boldsymbol{\nabla} \cdot \left(\rho_{(0)} \boldsymbol{\nabla} \phi_{(1)} + \rho_{(1)} \boldsymbol{\nabla} \phi_{(0)} \right) = 0, \tag{16}$$

$$\partial_t \phi_{(0)} - \frac{1}{2} \left(\boldsymbol{\nabla} \phi_{(0)} \right)^2 = -\frac{\mathcal{D}}{\hbar} b \ln\left(\frac{\rho_{(0)}}{\rho_0} \right) + \frac{\mathcal{D}}{\hbar} V(\mathbf{x}, t) - \frac{\mathcal{D}^2}{4} \left[\frac{\Delta \rho_{(0)}}{\rho_{(0)}} - \frac{1}{2} \left(\frac{\boldsymbol{\nabla} \rho_{(0)}}{\rho_{(0)}} \right)^2 \right], \tag{17}$$

$$\partial_t \phi_{(1)} - \boldsymbol{\nabla} \phi_{(0)} \cdot \boldsymbol{\nabla} \phi_{(1)} = -\frac{\mathcal{D}}{\hbar} b \frac{\rho_{(1)}}{\rho_{(0)}} + \frac{\mathcal{D}^2}{4} \left[\frac{\rho_{(1)} \Delta \rho_{(0)}}{\rho_{(0)}^2} - \frac{\Delta \rho_{(1)}}{\rho_{(0)}} + \frac{\boldsymbol{\nabla} \rho_{(0)} \cdot \left(\rho_{(0)} \boldsymbol{\nabla} \rho_{(1)} - \rho_{(1)} \boldsymbol{\nabla} \rho_{(0)} \right)}{\rho_{(0)}^3} \right], \tag{18}$$

whereas the perturbative expansion of the pressure can be done using equations of state (10) or (11); although this three-dimensional pressure p is not really needed for further computations related to spacetime emergence, see also remarks at the end of this section.

Furthermore, from Equation (18) we derive

$$\rho_{(1)} = \frac{\rho_{(0)}}{A_b} \left(\partial_t \phi_{(1)} - \boldsymbol{\nabla} \phi_{(0)} \cdot \boldsymbol{\nabla} \phi_{(1)} + B \right), \tag{19}$$

where

$$A_b = A + c_s^2 = A - \frac{\mathcal{D}}{\hbar} b(\mathbf{x}, t), \tag{20}$$

and

$$A = \frac{\mathcal{D}^2}{4} \left[\frac{\Delta \rho_{(0)}}{\rho_{(0)}} - \left(\frac{\boldsymbol{\nabla} \rho_{(0)}}{\rho_{(0)}} \right)^2 \right] = \frac{\mathcal{D}^2}{4} \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla} \rho_{(0)}}{\rho_{(0)}} \right), \tag{21}$$

$$B = \frac{\mathcal{D}^2}{4} \left(\frac{\Delta \rho_{(1)}}{\rho_{(0)}} - \frac{\boldsymbol{\nabla} \rho_{(0)} \cdot \boldsymbol{\nabla} \rho_{(1)}}{\rho_{(0)}^2} \right) = \frac{\mathcal{D}^2}{4} \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla} \rho_{(1)}}{\rho_{(0)}} \right). \tag{22}$$

Substituting Equation (19) into Equation (16), we obtain

$$-\partial_t \left[\frac{\rho_{(0)}}{A_b} \left(\partial_t \phi_{(1)} - \boldsymbol{\nabla} \phi_{(0)} \cdot \boldsymbol{\nabla} \phi_{(1)} + B \right) \right] + \boldsymbol{\nabla} \cdot \left[\rho_{(0)} \boldsymbol{\nabla} \phi_{(1)} + \frac{\rho_{(0)}}{A_b} \boldsymbol{\nabla} \phi_{(0)} \left(\partial_t \phi_{(1)} - \boldsymbol{\nabla} \phi_{(0)} \cdot \boldsymbol{\nabla} \phi_{(1)} + B \right) \right] = 0, \tag{23}$$

which can be rewritten in the explicitly four-dimensional Lorentz-covariant form

$$\partial_\mu (f^{\mu\nu} \partial_\nu \phi_{(1)}) + \partial_\mu (f^{0\mu} B) = 0, \tag{24}$$

where we assume the Einstein summation convention for repeating Greek indices labeling the coordinates $x^\mu = \{t, \mathbf{x}\}$ and introduce the rank-4 matrix

$$f^{\mu\nu} \propto \frac{\rho_{(0)}}{A_b} \left(\begin{array}{c|c} -1 & \partial_j \phi_{(0)} \\ \cdots & \cdots \\ \partial_i \phi_{(0)} & A_b \delta_{ij} - \partial_i \phi_{(0)} \partial_j \phi_{(0)} \end{array} \right), \tag{25}$$

where Latin indices label spatial coordinates and the proportionality symbol indicates that this matrix is defined up to a multiplicative constant. The matrix determinant (without a multiplicative constant) is given by $f = -\rho_{(0)}^4 / A_b$.

Furthermore, we restore the wavefunction notation

$$\phi_{(0)} = -\mathcal{D}\,\mathcal{S},\; \phi_{(1)} = -\mathcal{D}\,\delta\mathcal{S},\; \rho_{(0)} = |\Psi|^2, \tag{26}$$

and rewrite Equation (24) in the form

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\mathcal{S}\right) - \mathcal{D}^{-1}\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{0\mu}B\right) = 0, \tag{27}$$

where we introduced the metric tensor

$$g^{\mu\nu} \propto \frac{1}{\sqrt{-f}}f^{\mu\nu} \propto \frac{1}{\rho\sqrt{A_b}}\begin{pmatrix} -1 & \vdots & -\mathcal{D}\partial_j\mathcal{S} \\ \cdots & \cdot & \cdots \\ -\mathcal{D}\partial_i\mathcal{S} & \vdots & A_b\delta_{ij} - \mathcal{D}^2\partial_i\mathcal{S}\,\partial_j\mathcal{S} \end{pmatrix}, \tag{28}$$

$$g_{\mu\nu} = \left(g^{\mu\nu}\right)^{-1} \propto \frac{\rho}{\sqrt{A_b}}\begin{pmatrix} -N^2 & \vdots & -\mathcal{D}\partial_j\mathcal{S} \\ \cdots & \cdot & \cdots \\ -\mathcal{D}\partial_i\mathcal{S} & \vdots & \delta_{ij} \end{pmatrix}, \tag{29}$$

where

$$N^2 \equiv A_b - \mathcal{D}^2(\nabla\mathcal{S})^2 = \frac{\mathcal{D}}{2}\left(\partial_t\mathcal{S} + \frac{1}{\hbar}V\right) - \frac{\mathcal{D}}{\hbar}b\left[1 + \frac{1}{2}\ln(\rho/\rho_0)\right] - \frac{\mathcal{D}^2}{4}\frac{\Delta\rho}{\rho}, \tag{30}$$

$$A_b = A - \frac{\mathcal{D}}{\hbar}b,\; A = \frac{\mathcal{D}^2}{4}\nabla\cdot\left(\frac{\nabla\rho}{\rho}\right),\; B = \frac{\mathcal{D}^2}{4}\nabla\cdot\left(\frac{\nabla\delta\rho}{\rho}\right), \tag{31}$$

where we also used the second of Equation (5) to derive Equation (30). The conformal factor of the metric plays a role of the gauging function, determining the choice of units for clocks and measuring rods. Due to this gauge freedom, the physical frame is a priori unknown, but its choice can be made based on the physical properties of spacetime.

Equation (27) indicate that fluctuations of the three-dimensional background superfluid's wavefunction can indeed be described as relativistic fields in four-dimensional spacetime, thus confirming the aforesaid. In the SVT framework, they obviously have a finite range of applicability, namely, outside special points or domains, such as spacetime singularities or null surfaces, where the small-fluctuation approximation of SVT is unlikely to be valid. In those cases, one has to resort to the F-observer formulation of SVT in terms of the background superfluid.

In this picture, massless excitations, such as photons, are somewhat analogous to acoustic waves, or phonons, which also obey a linear dispersion law and propagate along null geodesics of the induced metric. One can easily show that in the (conformally) flat limit, source-free Maxwell equations are insensitive to the conformal factor [29,30]. Other relativistic particles can be viewed as projections of small excitations of the background superfluid onto the R-observer's measuring apparatus: because of the emergent Lorentz symmetry, they must transform according to the irreducible representations of the Poincaré group.

Furthermore, Einstein field equations can be now interpreted as a definition for an effective stress–energy tensor, describing the matter, which would be observed by the R-observers:

$$\widetilde{T}_{\mu\nu} \equiv \kappa^{-1}\left[R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\right], \tag{32}$$

where $\kappa = 8\pi G/c_{(0)}^2$, G is the Newtonian constant of gravitation, $c_{(0)} \approx c$, and $c \approx 2.998 \times 10^{10}$ cm s^{-1} is historically called the *speed of light in vacuum*. A theoretical value of $c_{(0)}$ can be derived from the linearized limit of the metric (29), provided that both the superfluid wavefunction and physical frame are known, cf. Section 4 below.

The physical meaning of the stress–energy tensor (32) is obvious: it describes the gravitating matter which is "seen" by R-observers by means of observing geodesics of test particles, therefore, Einstein field equations are automatically valid for such observers, because the reverse procedure, the derivation of the original metric from this stress–energy tensor, is always possible. One can even reconstruct a corresponding action functional, one example being given in Section 5.3 of [7].

It should be stressed that this effective matter cannot be merely viewed as the relativistic perfect fluid with the values of density and pressure taken from Section 2.1. Those values refer to the non-relativistic three-dimensional superfluid observed by F-observers but not by R-observers. The latter see a rather different matter and environment instead—for example, inside the flow described in Section 5.3 of [7], R-observers see themselves embedded in the Friedmann–Lemaître–Robertson–Walker-type universe with a scalar dilaton field. In all cases, a specific form of the effective matter is determined only by the emergent metric (29) and corresponding stress-energy tensor (32).

Finally, note that SVT approach does not have to reproduce all pseudo-Riemannian manifolds, especially considering that many of those are not physically relevant anyway. Instead, even in its simplest form (inviscid, irrotational, isothermal, isentropic), our model induces spacetimes with conformal $3+1$ foliation (the generalized ADM decomposition), which is a rather large class of manifolds; it includes all physically relevant cases we know so far, both cosmological and asymptotically flat. Naturally, one can consider more general types of flow, which would result in inducing more general manifolds, if such a need occurs.

3. Approximations and Special Cases

In this section, we consider various limits and approximations of the spacetime metric (29)–(31), which allow to extract physical information without actually solving the original Equation (2).

3.1. Small-Slow Density Perturbations

Although our metric was already derived under the assumptions of small density perturbations $\delta\rho \ll \rho$, one can still impose additional smallness assumptions. For instance, one can assume that spatial variations of density perturbations are small,

$$\ell_\delta |\boldsymbol{\nabla} \delta\rho| / \rho \ll 1, \tag{33}$$

or that the rates of density variations are small

$$\ell_\rho |\boldsymbol{\nabla} \rho| / \rho \ll 1, \tag{34}$$

where ℓ's being some characteristic length scales.

If we impose these assumptions simultaneously then we can neglect both functions A and B in the leading-order approximation. Therefore, the metric tensor (29) simplifies to

$$g_{\mu\nu} \propto \frac{\rho}{\sqrt{|b|}} \begin{pmatrix} -N^2 & \vdots & -\mathcal{D}\partial_j \mathcal{S} \\ \cdots & \cdot & \cdots \\ -\mathcal{D}\partial_i \mathcal{S} & \vdots & \delta_{ij} \end{pmatrix}, \tag{35}$$

$$N^2 = \frac{\mathcal{D}}{2}\left(\partial_t \mathcal{S} + \frac{1}{\hbar} V\right) - \frac{\mathcal{D} b}{\hbar}\left[1 + \frac{1}{2}\ln(\rho/\rho_0)\right], \tag{36}$$

which is formally equivalent to neglecting terms of the order $\mathcal{D}^2 \sim \hbar^2$. Therefore, this metric could serve as a semiclassical approximation of the metric (29), which is probably sufficient for describing gravity phenomena in the regions where vacuum fluctuations are relatively small compared to the background wavefunction's value.

As usual in our formalism, spacetime described by metric (35) can have special points or domains, such as spacetime singularities, null surfaces, or any regions with fast varying superfluid density; where this small-slow-fluctuation approximation is not likely to be robust. In such cases, one has to consider a more general metric (29), or even involve the F-observer picture, i.e., the original superfluid dynamics in the "fictitious" Euclidean space.

3.2. Linear Phase

This is a special exact case which does not require solving the original Equation (2) to deduce the main properties of the corresponding spacetime which emerges. If the background superfluid is in a stationary state, with a phase linear with respect to time and spatial coordinates then we can assume that

$$\mathcal{S} = \omega t + \mathcal{D}^{-1}\mathbf{u}_0 \cdot \mathbf{x}, \;\; \mathbf{u}_0 = \text{const}, \tag{37}$$

which means that it flows with the constant velocity $\mathcal{D}\boldsymbol{\nabla}\mathcal{S} = \mathbf{u}_0 = \text{const}$, with respect to Euclidean space. As discussed before, both this flow and Euclidean space themselves are unobservable to R-observers. Instead, those observe, according to Equation (29), spacetime with the metric tensor

$$g_{\mu\nu} \propto \frac{\rho}{\left|\frac{\mathcal{D}}{\hbar}b - \frac{\mathcal{D}^2}{2}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)\right|^{1/2}} \begin{pmatrix} \frac{\mathcal{D}}{\hbar}b - \frac{\mathcal{D}^2}{2}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right) + \mathbf{u}_0^2 & \vdots & -(\mathbf{u}_0)_j \\ \cdots & \cdot & \cdots \\ -(\mathbf{u}_0)_i & \vdots & \delta_{ij} \end{pmatrix}, \tag{38}$$

supplemented with consistency conditions in the form of differential equations for the wavefunction amplitude squared

$$\frac{\mathcal{D}^2}{2}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right) + \frac{\mathcal{D}^2}{4}\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)^2 + \frac{\mathcal{D}}{2\hbar}[b\ln(\rho/\rho_0) - V] = \frac{\mathcal{D}}{2}\omega + \mathbf{u}_0^2, \tag{39}$$

$$\partial_t\rho + \mathbf{u}_0\cdot\boldsymbol{\nabla}\rho = 0, \tag{40}$$

as long as the initial and boundary conditions are specified. The latter of these equations can be easily solved by the method of characteristics; its solution is that the wavefunction amplitude squared must be a function of the three-dimensional scalar,

$$s = |\mathbf{x} - \mathbf{u}_0 t| = \sqrt{(\mathbf{x} - \mathbf{u}_0 t)\cdot(\mathbf{x} - \mathbf{u}_0 t)}, \tag{41}$$

while the shape of the function $\rho(s)$ itself is determined by the initial and boundary conditions.

As for Equation (39), then it can be viewed as an eigenvalue problem for the energy parameter $\hbar\omega$ and differential equation for the unknown function $\rho = \rho(s)$. Because ω is an eigenvalue parameter here, it generally contains parameters from the original wave equation and the normalization condition, thus $\omega = \omega(\mathcal{D}, b, \rho_0)$ in general; examples of some eigensolutions can be found in [13].

4. Linearized Gravity

The linearized gravity limit is instrumental for establishing the physical interpretation of metric tensor components. More specifically to our case, it yields the interpretation in terms of gravitational field potential and effective speed of light, as measured by R-observers.

In Section 5.3 of [7], it was argued that in absence of any local density inhomogeneities, the emergent spacetime must be conformally flat, at least in the leading-order approximation. The occurrence of local ("induced") matter induces distortions of that flatness. If, however, the density values of that matter are small compared to the background density, then one can assume that the distorted spacetime is a small perturbation of the (conformally) flat spacetime. Therefore, the conditions of Section 3.1 are applicable to this case.

The background superfluid can be assumed in a stationary state, therefore

$$\Psi(\mathbf{x}, t) = \exp(i\omega t)\widetilde{\Psi}(\mathbf{x}), \tag{42}$$

hence

$$|\widetilde{\Psi}|^2 = \tilde{\rho} = \rho, \;\; \partial_t \mathcal{S} = \omega, \tag{43}$$

and we also neglect trap potential for simplicity. Then metric tensor (35) takes the form

$$g_{\mu\nu} \propto \frac{\rho}{\sqrt{|b|}} \begin{pmatrix} -N^2 & \vdots & -\mathcal{D}\partial_j \mathcal{S} \\ \cdots & \cdot & \cdots \\ -\mathcal{D}\partial_i \mathcal{S} & \vdots & \delta_{ij} \end{pmatrix}, \tag{44}$$

$$N^2 \approx \frac{\mathcal{D}\,\omega}{2} - \frac{b}{2m}[2 + \ln(\rho/\rho_0)], \tag{45}$$

where we assume metric tensor's off-diagonal terms being small compared to the diagonal ones, and remember the notation (3). We remind that $\omega = \omega(\mathcal{D}, b, \rho_0)$ is an eigenvalue parameter here, therefore it generally contains parameters from the original superfluid equations, as discussed at the end of Section 3.2.

Furthermore, if the coupling function contains an additive constant (it usually does [13]), i.e., $b(\mathbf{x}, t) = b_0 + [b](\mathbf{x}, t)$, then the lapse function (45) can be rewritten in the form

$$N^2 = c_{(0)}^2 + 2\Phi, \tag{46}$$

where we denoted

$$\Phi \sim -\frac{b}{4m} \ln\left(|\Psi|^2/\rho_0\right) - \frac{[b]}{2m}, \tag{47}$$

$$c_{(0)}^2 \sim \frac{\mathcal{D}\,\omega}{2} - \frac{b_0}{2m} = \frac{1}{2m}(\hbar\omega - b_0), \tag{48}$$

where we used the notations "$\sim$" to emphasize the ambiguities discussed below, and $[b]$ is the truncated coupling function with any additive constant (denoted by b_0) removed. If we correctly guessed the physical frame (defined by the conformal factor of the metric tensor and set of correct coordinates), then Φ and $c_{(0)}$ can be interpreted, respectively, as the classical gravitational potential and speed of light in the Minkowski limit, written in the conformal Newtonian gauge.

It is important to clarify here the terminology 'gravitational potential'. From Equation (47), it is obvious that no potential exists *per se*, but many-body quantum-mechanical effects in the background superfluid act as what we perceive as gravity. The logarithmic term is directly related to quantum information entropy of the superfluid, cf. [8,31,32]; therefore, it is the change of the entropy of background superfluid that induces the "thermodynamic" force and associated "potential". One can imagine a classical analogue of this phenomenon: in diffusive systems, suspended particles move from regions of higher to lower concentrations, as if they were driven by some macroscopic potential, but in reality it is just that the total system tries to find a state with a minimum free energy.

It is worth summarizing the built-in ambiguities:

(i) the physical metric is derived up to a conformal factor, due to the remaining choice of a physical frame (units),
(ii) for a given conformal frame, values Φ and $c_{(0)}$ are defined up to a factor, due to time coordinate transformation,
(iii) the coupling b is defined up to a factor, due to the $U(1)$ symmetry of the original wave equation, metric signature choice and coordinate transformations, and

(iv) values Φ and $c_{(0)}^2$ are defined up to, respectively, additive function and additive constant, due to some terms being neglected because of various small density perturbation approximations.

For these reasons, we regard the physical interpretation of values (47) and (48) as gravitational potential and speed of light as the 'simplest', while keeping in mind that it is not unique until the above-mentioned ambiguities (i)–(iv) are fixed.

Finally, it is instructive to compare the derived 'simplest' form (47) of gravitational potential with its semi-heuristic analogue from [8,33], $\Phi = \frac{b}{m}\ln\left(|\Psi|^2/\rho_0\right)$, assuming vanishing external potential in both cases. The two potentials are identical up to the coupling b's rescaling and additive function $[b]/2m = (b - b_0)/2m$. The rescaling obviously cannot affect the phenomenological outcomes of [33], for it only requires the rescaling of some fitting parameters. The additive function does not affect galaxy rotation curves either, because, for the choice of b made in [8,33], it would yield the extra term proportional to the inverse radius squared, which can be safely neglected on astronomical scales.

5. Conclusions

In this paper, we discussed the emergence mechanism of four-dimensional spacetime from the dynamics of three-dimensional quantum Bose liquid in Euclidean space, while adopting the logarithmic superfluid model motivated in our earlier papers on the theme. This mechanism lies at the heart of the theory of superfluid vacuum, which is a viable candidate for a theory of classical and quantum gravity and relativistic particles.

According to this theory, relativistic symmetry is an approximate symmetry of Nature, which is robust in a low-momentum small-fluctuations limit of the quantum dynamics of the logarithmic superfluid; the latter thus plays the role of vacuum or non-removable background. Although such superfluid is defined in three-dimensional Euclidean space, the latter is unobservable by an observer dealing with small fluctuations of the superfluid, who, instead, detects relativistic particles moving along geodesics in curved four-dimensional spacetime.

We derived the metric tensor of this emergent spacetime and studied some of its special cases and limits, such as linear-phase flow and the linearized gravity limit. The latter was particularly interesting to us due to the correspondence principle and connection with the relativistic theory.

It turns out that the value of the speed of light, which is a fundamental parameter in the theory of relativity, is a derived notion in superfluid vacuum theory. Its value is a combination of the Planck constant and the original parameters of the background superfluid. The whole theory thus contains only two essential fundamental constants: the Planck constant and the mass of a constituent particle of the background superfluid.

Furthermore, we considered the linearized limit of the emergent spacetime and derived effective gravitational potential, subject to a number of conditions. It turns out that gravity, which is regarded as a fundamental interaction in the Newtonian or Einstein's theories of gravity, can be viewed as a result of the dynamics of quantum excitations inside the background superfluid being projected onto a measurement apparatus of a relativistic observer, which confirms the picture [7,8]. The specific form of the gravitational potential is determined by quantum information entropy density, which is (implicitly) present in logarithmic nonlinear wave equations.

We compared the result with the potential semi-heuristically derived in our earlier paper [8]. For our standard choice of the coupling function b, two potentials are shown to be empirically equivalent at the macroscopical scale [33], because their difference is the Reissner-Nördstrom-type term (proportional to the inverse radius-vector squared), which tends to zero at spatial infinity much faster than the Newtonian (Schwarzschild) term.

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Abbreviations

The following abbreviations are used in this manuscript:

ADM	Arnowitt–Deser–Misner
BEC	Bose–Einstein condensate
F-observer	Full observer
R-observer	Relativistic observer
SV	Superfluid vacuum
SVT	Superfluid vacuum theory

References

1. Volovik, G.E. The Universe in a Helium Droplet. *Int. Ser. Monogr. Phys.* **2003**, *117*, 1–507.
2. Huang, K. *A Superfluid Universe*; World Scientific: Singapore, 2016; pp. 1–136.
3. Scott, T.C. From Modified Newtonian Dynamics to Superfluid Vacuum Theory. *Entropy* **2023**, *25*, 12. [CrossRef] [PubMed]
4. Ourabah, K. On the Collective Properties of Quantum Media. *Eur. Phys. J. Plus* **2023**, *138*, 55. [CrossRef]
5. Dirac, P.A.M. Is There an Aether. *Nature* **1951**, *168*, 906. [CrossRef]
6. Barceló, C.; Liberati, S.; Visser, M. Analogue Gravity. *Living Rev. Relativ.* **2005**, *8*, 12. [CrossRef]
7. Zloshchastiev, K.G. Spontaneous Symmetry Breaking and Mass Generation as Built-in Phenomena in Logarithmic Nonlinear Quantum Theory. *Acta Phys. Polon.* **2011**, *42*, 261. [CrossRef]
8. Zloshchastiev, K.G. An Alternative to Dark Matter and Dark Energy: Scale-dependent Gravity in Superfluid Vacuum Theory. *Universe* **2020**, *6*, 180. [CrossRef]
9. Zloshchastiev, K.G. Volume Element Structure and Roton-maxon-phonon Excitations in Superfluid Helium Beyond the Gross–Pitaevskii Approximation. *Eur. Phys. J. B* **2012**, *85*, 273. [CrossRef]
10. Zloshchastiev, K.G. Temperature-driven Dynamics of Quantum Liquids: Logarithmic Nonlinearity, Phase Structure and Rising Force. *Int. J. Mod. Phys. B* **2019**, *33*, 1950184. [CrossRef]
11. Scott, T.C.; Zloshchastiev, K.G. Resolving the Puzzle of Sound Propagation in Liquid Helium at Low Temperatures. *Low Temp. Phys.* **2019**, *45*, 1231. [CrossRef]
12. Zloshchastiev, K.G. Resolving the Puzzle of Sound Propagation in a Dilute Bose–Einstein Condensate. *Int. J. Mod. Phys. B* **2022**, *36*, 2250121. [CrossRef]
13. Zloshchastiev, K.G. On the Dynamical Nature of Nonlinear Coupling of Logarithmic Quantum Wave Equation, Everett-Hirschman Entropy and Temperature. *Z. Naturforsch. A* **2018**, *73*, 619. [CrossRef]
14. Boudjeriou, T. Global Well-posedness and Finite Time Blow-up for a Class of Wave Equation Involving Fractional p-Laplacian with Logarithmic Nonlinearity. *Math. Nachr.* **2023**, *296*, 938–956. [CrossRef]
15. Haining, F.; Ying, L.; Binlin, Z. Existence and Concentration of Ground State Solutions for a Logarithmic Schrödinger Equation with the Critical Exponent. *Sci. Sin. Math.* **2022**, *52*, 1377–1406. [CrossRef]
16. Cordeiro, S.M.S.; Pereira, D.C.; da Costa Baldez, C.A.; Raposo da Cunha, C.A. Global Existence and Asymptotic Behavior for a Timoshenko System with Internal Damping and Logarithmic Source Terms. *Arab. J. Math.* **2023**, *9*, 105–118. [CrossRef]
17. Durt, T. Testing de Broglie's Double Solution in the Mesoscopic Regime. *Found. Phys.* **2023**, *53*, 2. [CrossRef]
18. Carles, R. Logarithmic Schrödinger Equation and Isothermal Fluids. *EMS Surv. Math. Sci.* **2022**, *9*, 99–134. [CrossRef]
19. Peng, X.; Jia, G. Existence and Concentration Behavior of Solutions for the Logarithmic Schrödinger–Poisson System with Steep Potential. *Z. Angew. Math. Phys.* **2023**, *74*, 29. [CrossRef]
20. Kinach, M.P.; Choptuik, M.W. Dynamical Evolution of $U(1)$ Gauged Q-balls in Axisymmetry. *Phys. Rev. D* **2023**, *107*, 035022. [CrossRef]
21. Chiueh, T.; Hsu, Y.-H. Theory and Phenomenology of Stressed Wave-dark-matter Soliton. *Phys. Rev. D* **2023**, *107*, 063011. [CrossRef]
22. Fan, H.N.; Zhang, B.L. Fractional Schrödinger Equations with Logarithmic and Critical Nonlinearities. *Acta Math. Sin.-Engl. Ser.* **2023**, *39*, 285–325. [CrossRef]
23. Raposo, C.; Cattai, A.; Vera, O.; Gorain, C.G.; Pereira, D. Global Solution and Blow-up for a Thermoelastic System of p-Laplacian Type with Logarithmic Source. *Math. Sci. Appl. E-Notes* **2023**, *11*, 112–128. [CrossRef]
24. Chang, X.; Wang, R.; Yan, D. Ground States for Logarithmic Schrödinger Equations on Locally Finite Graphs. *J. Geom. Anal.* **2023**, *33*, 211. [CrossRef]
25. De Broglie, L. Sur la Possibilité de Relier les Phénomènes d'Interférence et de Diffraction à la Théorie des Quanta de Lumière. *Comptes Rendus* **1926**, *183*, 447–448.
26. Madelung, E. Quantentheorie in Hydrodynamischer Form. *Z. Phys.* **1927**, *40*, 322–326. [CrossRef]

27. Dunn, J.E.; Serrin, J.B. On the Thermomechanics of Interstitial Working. *Arch. Rat. Mech. Anal.* **1985**, *88*, 95–133. [CrossRef]
28. Zloshchastiev, K.G. Matrix Logarithmic Wave Equation and Multi-channel Systems in Fluid Mechanics. *J. Theor. Appl. Mech.* **2019**, *57*, 843–852. [CrossRef]
29. Schrödinger, E. Maxwell's and Dirac's Equations in the Expanding Universe. *Proc. R. Irish Acad. A Math. Phys. Sci.* **1940**, *46*, 25–47.
30. Infeld, L.; Schild, A. A New Approach to Kinematic Cosmology. *Phys. Rev.* **1945**, *68*, 250–272. [CrossRef]
31. Zloshchastiev, K.G. Logarithmic Nonlinearity in Theories of Quantum Gravity: Origin of Time and Observational Consequences. *Grav. Cosmol.* **2010**, *16*, 288–297. [CrossRef]
32. Avdeenkov, A.V.; Zloshchastiev, K.G. Quantum Bose Liquids with Logarithmic Nonlinearity: Self-sustainability and Emergence of Spatial Extent. *J. Phys. B At. Mol. Opt. Phys.* **2011**, *44*, 195303. [CrossRef]
33. Zloshchastiev, K.G. Galaxy Rotation Curves in Superfluid Vacuum Theory. *Pramana* **2023**, *97*, 2. [CrossRef]