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Octonion Internal Space Algebra for the Standard Model †

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Abstract: This paper surveys recent progress in our search for an appropriate internal space algebra for the standard model (SM) of particle physics. After a brief review of the existing approaches, we start with the Clifford algebras involving operators of left multiplication by octonions. A central role is played by a distinguished complex structure that implements the splitting of the octonions $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$, which reflect the lepton-quark symmetry. Such a complex structure on the 32-dimensional space $\mathcal{S}$ of $C\ell_{10}$ Majorana spinors is generated by the $C\ell_6 (\subset C\ell_{10})$ volume form, $\omega_6 = \gamma_1 \cdots \gamma_6$, and is left invariant by the Pati–Salam subgroup of $Spin(10)$, $G_{PS} = Spin(4) \times Spin(6)/\mathbb{Z}_2$. While the $Spin(10)$ invariant volume form $\omega_{10} = \gamma_1 \dots \gamma_{10}$ of $C\ell_{10}$ is known to split $\mathcal{S}$ on a complex basis into left and right chiral (semi)spinors, $\mathcal{P} = \frac{1}{2}(1 - i\omega_6)$ is interpreted as the projector on the 16-dimensional *particle subspace* (which annihilates the antiparticles).The standard model gauge group appears as the subgroup of G_{PS} that preserves the sterile neutrino (which is identified with the Fock vacuum). The $\mathbb{Z}_2$-graded internal space algebra $\mathcal{A}$ is then included in the projected tensor product $\mathcal{A} \subset \mathcal{P} C\ell_{10} \mathcal{P} = C\ell_4 \otimes C\ell_6^0$. The Higgs field appears as the scalar term of a superconnection, an element of the odd part $C\ell_4^1$ of the first factor. The fact that the projection of $C\ell_{10}$ only involves the even part $C\ell_6^0$ of the second factor guarantees that the color symmetry remains unbroken. As an application, we express the ratio $\frac{m_H}{m_W}$ of the Higgs to the W boson masses in terms of the cosine of the *theoretical* Weinberg angle.

Keywords: Clifford algebra; composition algebra; triality; Jordan algebra; complex structure; superselection rules; Higgs mass; superconnection; fermion doubling

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1. Introduction

The elaboration of the standard model (SM) of particle physics was completed in the early 1970s. To quote John Baez [1], "years trying to go beyond the Standard Model hasn't yet led to any clear success". The present survey belongs to an equally long—albeit less fashionable—effort to clarify the algebraic (or geometric) roots of the SM or, more specifically, to find a natural framework featuring its internal space properties. After discussing some old explorations, we provide an updated exposition of recent developments (particularly, of [2]) while clarifying the meaning and role of complex structures, and we concentrate on one structure associated with a Clifford algebra (in our case, $C\ell_6$) pseudoscalar.

Most ideas on the natural framework of the SM originate in the 1970s, the first decade of its existence (There are two exceptions: the Jordan algebras were introduced and classified in the 1930s [3,4]; and the non-commutative geometry approach was born in the late 1980s [5–7] and is still vigorously developed by Connes and collaborators and followers [8–12]).

First, early in 1973, the ultimate division algebra, the octonions[1], were introduced by Gürsey[2] and his student Günaydin [14,15] for the description of quarks and their $SU(3)$ color symmetry. The idea was taken up and extended to incorporate all four division algebras by G. Dixon (see [16,17] and the earlier work cited there) and was further developed by Furey [18–24]. Dubois-Violette (D-V) [25,26] emphasizes the lepton-quark correspondence and unimodularity of the color group $SU(3)_c$ as a physical motivation for introducing the octonions; they come equipped with a complex structure preserved by subgroup $SU(3)$ of the automorphism group G_2 of $\mathbb{O}$:

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 . \tag{1}$$

1.1. (Split) Octonions as Composition Algebras

One can in fact provide a basis-free definition of the octonions, starting with the splitting (1). To this end, one uses the skew symmetric vector product and standard inner product on $\mathbb{C}^3$ to define a non-commutative and non-associative distributive product xy on $\mathbb{O}$ and a real-valued, non-degenerate symmetric bilinear form $\langle x, y\rangle = \langle y, x\rangle$, such that the quadratic norm $N(x) = \langle x, x\rangle$ is multiplicative:

$$N(xy) = N(x)N(y) \quad \text{for} \quad N(x) = \langle x, x\rangle \tag{2}$$

(cf. [25,27,28]). Furthermore, by defining the real part of $x \in \mathbb{O}$ by $\mathrm{Re}\, x = \langle x, 1\rangle$ and the octonionic conjugation by $x \to x^* = 2\langle x, 1\rangle - x$, we have

$$xx^* = N(x)\mathbb{1} \Leftrightarrow x^2 - 2\langle x, 1\rangle x + N(x)\mathbb{1} = 0 . \tag{3}$$

A unital algebra with a non-degenerate quadratic norm obeying (2) is called a *composition algebra*.

Another basis-free definition of the octonions $\mathbb{O}$ and their split version $\widetilde{\mathbb{O}}$ can be provided in terms of quaternions using the Cayley–Dickson construction. We represent the quaternion as scalars plus vectors

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3,\ x = u + U,\ y = v + V,\ u, v \in \mathbb{R},\ U, V \in \mathbb{R}^3,$$

$$xy = uv - \langle U, V\rangle + uV + Uv + U \times V \tag{4}$$

with the vector product $U \times V \in \mathbb{R}^3$ satisfying

$$U \times V = -V \times U,\ (U \times V) \times W = \langle U, W\rangle V - \langle V, W\rangle U . \tag{5}$$

The product (4) is clearly non-commutative, but one verifies that it is associative. The Cayley–Dickson construction defines the octonions $\mathbb{O}$ and split octonions $\widetilde{\mathbb{O}}$ in terms of a pair of quaternions and a new "imaginary unit" ℓ is defined as:

$$x = u + U + \ell(v + V),\ \ell(v + V) = (v - V)\ell ,$$

$$\ell^2 = \begin{cases} -1 & \Rightarrow \quad x \in \mathbb{O} \\ \ \ 1 & \Rightarrow \quad x \in \widetilde{\mathbb{O}} . \end{cases} \tag{6}$$

We shall encounter the split octonions as generators of $C\ell(4, 2)$ in Section 3.1 below.

1.2. Jordan Algebras; Guts; Clifford Algebras

Studying quantum field theory, it appears natural to replace classical observables (real-valued functions) by an algebra of functions on space-time with values in a *finite dimensional Euclidean Jordan algebra*[3]. As a particularly attractive choice that incorporates the idea of lepton-quark symmetry, D-V proposes [25] the exceptional Jordan or *Albert* algebra of 3×3 Hermitian octonionic matrices,

$$J_3^8 = \mathcal{H}_3(\mathbb{O}), \tag{7}$$

which is the only irreducible one that does not admit an associative envelope [32]. Further progress was achieved in [27,28,30,33–35] by considering the Clifford algebra envelope of its non-exceptional subalgebra J_2^8, which fits one generation of fundamental fermions. In these papers, as well as in the present one, we are effectively working with associative algebras that should be viewed as an internal space counterpart of Haag's *field algebra* (see [36]).

A second development, *grand unified theory* (GUT), anticipated again in 1973 by Pati and Salam [37], became mainstream for a time[4]. Fundamental chiral fermions fit the complex spinor representation of $Spin(10)$, which was introduced as a GUT group by Fritzsch and Minkowski and by Georgi. A preferred symmetry breaking yields the maximal rank semisimple Pati–Salam subgroup

$$G_{\text{PS}} = \frac{Spin(4) \times Spin(6)}{\mathbb{Z}_2} \subset Spin(10),$$
$$Spin(4) = SU(2)_L \times SU(2)_R, \; Spin(6) = SU(4). \tag{8}$$

We note that G_{PS} is the only GUT group that does not predict a gauge triggered proton decay; it is also encountered in the non-commutative geometry approach to the SM [8,10]. In general, GUTs provide a nice home for the fundamental fermions, as displayed by the two 16-dimensional complex conjugate "Weyl spinors" of $Spin(10)$. Their other representations, however, such as the 45-dimensional adjoint representation of $Spin(10)$, are much too big and involve hypothetical particles such as lepto-quarks which cause difficulties.

The Clifford algebra[5] $C\ell_{10}$, on the other hand, like the Clifford algebra of any even-dimensional Euclidean vector space, has a unique irreducible representation (IR); in the case of[6] $C\ell_{10} \cong \mathbb{R}[2^5]$, it is the 32-component real (Majorana) spinor. Viewed as a representation of $Spin(10)$, it splits upon complexification into two 16-dimensional (complex) IRs that can be naturally associated to the left and right chiral fundamental (anti)fermions of one generation:

$$\mathbf{32} = \mathbf{16}_L + \mathbf{16}_R. \tag{9}$$

Clifford algebras were also applied to the SM in the 1970s—see [40] and the references therein. An essential difference in our approach is the use of octonions with a preferred complex structure in $C\ell_{8+\nu}$, $\nu = 0, 1, 2$ to restrict the corresponding gauge group (another new point, the use of the $\mathbb{Z}_2$ grading of $C\ell_{10}$ to define the Higgs field, will be discussed in Section 1.3 below).

The Pati–Salam subgroup of $Spin(10)$ is singled out as the stabilizer of the $C\ell_6 (\subset C\ell_8 \subset C\ell_{10})$ pseudoscalar

$$\omega_6 = \gamma_1 \dots \gamma_6 \;\text{ for }\; \gamma_\alpha = \sigma_0 \otimes \epsilon \otimes L_\alpha, \gamma_8 = \sigma_0 \otimes \sigma_1 \otimes \mathbb{1}_8 (\in C\ell_{10}),$$

$$\sigma_0 = \mathbb{1}_2, \epsilon = i\sigma_2, \; L_\alpha = L_{e_\alpha}, \; [L_\alpha, L_\beta]_+ = -2\delta_{\alpha\beta}\mathbb{1}_8, \; \alpha, \beta = 1, \cdots, 7. \tag{10}$$

Here, L_x is the operator of left multiplication in the eight-dimensional real vector space of the octonions, $L_x y = xy$ for $x, y \in \mathbb{O}$. The action of the operators $L_\alpha \in \mathbb{R}[8]$ on the octonion units will be made explicit in Section 2.2 (Equation (22)). The group G_{PS} (8) in fact preserves each factor in the graded tensor product representation of $C\ell_{10}$:

$$C\ell_{10} = C\ell_4 \hat{\otimes} C\ell_6 \tag{11}$$

Introduced earlier by Furey [20,21] and exploited in [2], the complex structure $J \in SO(10)$ generated by ω_6 will be displayed, and the physical interpretation of the mutually orthogonal projection operators

$$\mathcal{P} = \frac{1}{2}(1 - i\omega_6),\ \mathcal{P}' = \frac{1}{2}(1 + i\omega_6) \tag{12}$$

will be revealed in Section 2.3.

1.3. Main Message and Organization of the Paper

The present survey focuses on ongoing attempts to answer two questions:

(1) Why is the arbitrarily looking gauge group of the SM

$$G_{\text{SM}} = S(U(2) \times U(3)) = \frac{SU(2) \times SU(3) \times U(1)}{\mathbb{Z}_6}, \tag{13}$$

and what dictates its highly reducible representation for fundamental fermions?

(2) How do we put together the Higgs field with the gauge bosons? Can we explain their mass ratios?

1. Most physicists accept GUT as an answer to the first question. One has the intriguing result of Baez and Huerta [38], where G_{SM} appears as the intersection of two popular GUT subgroups of $Spin(10)$:

$$G_{\text{SM}} = SU(5) \cap G_{\text{PS}} \subset Spin(10).$$

A top-down approach starting with $Spin(10)$, however, should involve the maximal rank subgroup $U(5)$ instead of $SU(5)$, in line with the philosophy of Borel-de Siebenthal [41], yielding an extra $U(1)$ factor in the intersection.

The minority that are trying to go further includes, besides the fans of octonions and the already cited enthusiasts of almost commutative real spectral triples, Holger Nielsen, whose more than two decades of musing over the problem are reviewed in [42]. Our approach exploits the complex structure and particle projector $\mathcal{P}$ (12) associated with the Clifford pseudoscalar ω_6 (10); it permeates the entire paper (Sections 2.3, 3.2–3.4 and 5.1. . .).

2. The second problem has been universally recognized (see, e.g., the popular account [43]). We follow the superconnection approach anticipated by Ne'eman and Fairlie—for a concise review and references, see Section 4.1. We exploit the restricted particle projector $\mathcal{P}_r$, which annihilates the sterile neutrino (Section 3.3) to deform the Fermi oscillators in the lepton sector into the odd generators of the simple Lie superalgebra postulated in [44,45]. The resulting difference between the flavor spaces of leptons and colored quarks allows one to compute the mass ratio m_H/m_W in agreement with the experiment (Section 4.2).

The paper aims to be self-contained and combines our contribution in a single narrative with a review of the background material. Section 2.1 provides a summary of the known triality realization of $Spin(8)$. Section 2.2 and Appendix A spell out the relation between left and right multiplication using imaginary octonion units, which is applied in Section 3.4 to display the stabilizer of R_7. We would like to single out two messages from Section 2.3: (1) the indirect connection between the $C\ell_6$ pseudoscalar and the complex structure $J \in SO(8)$ (33) and (35); (2) the observation that the Lie subalgebra of $so(8)$ that commutes with ω_6 and the electric charge operator Q (45) is the rank four subalgebra $su(3)_c \oplus u(1)_Q \oplus u(1)_{B-L}$. Section 3.1 contains, along with a glance on the equivalence class of Clifford algebras involving $C\ell(3,1), C\ell_{-6}(= C\ell(0,6)), C\ell_{10}$, the observation that the conformal Clifford algebra $C\ell(4,2)$ of this class is generated by the split octonions and gives rise to their isometry group $SO(4,4)$. Section 3.2 contains one of the main messages of the paper: the SM gauge group (13) is the subgroup of G_{PS} (8) that leaves the sterile neutrino invariant (Proposition 1). Section 3.3 discusses superselection rules and the superselection of the weak hypercharge. Section 3.4 reviews and comments on recent work [24,46] on the complex structure associated with the right action of octonion units as well as the derivation of the gauge group for the SM [35] and its left-right symmetric extension [47].

The Dirac operator $\gamma^\mu(\partial_\mu + A_\mu)$ anticommutes with the chirality γ_5 and hence intertwines the left and right fermions; so does the Higgs field, which substitutes a mass term in the fermionic Lagrangian. This inspired Connes and coworkers [5,6,9] to identify the Higgs field with the internal space part of the Dirac operator. This idea finds a natural implementation in the Clifford algebra approach for the SM *superconnection* (reviewed in Section 4.1). The concise exposition in Section 4 emphasizes our assumptions and some delicate points, referring the reader for calculational details to the preceding publication [2].

We recapitulate our convoluted route to $C\ell_{10}$ in Section 5.1. In Section 5.2, we compare our solution of the fermion doubling problem with the approach of [23]. A summary of the main results of the paper is given in Section 5.3, which also cites existing (inconclusive) attempts to understand why are there exactly three generations of fundamental fermions.

2. Triality Realization of $Spin(8)$: $C\ell_{-6}$

2.1. The Action of Octonions on Themselves

The group $Spin(8)$, the double cover of the orthogonal group $SO(8) = SO(\mathbb{O})$, can be defined (see [48,49]) as the set of triples $(g_1, g_2, g_3) \in SO(8) \times SO(8) \times SO(8)$ such that

$$g_2(xy) = g_1(x)\, g_3(y) \text{ for any } x, y \in \mathbb{O}\,. \tag{14}$$

If u is a unit octonion, $u^*u = 1$; then, the left and right multiplications by u are examples of isometries of $\mathbb{O}$

$$|L_u\, x|^2 = \langle ux, ux\rangle = \langle x, x\rangle,\ |R_u\, x|^2 = \langle xu, xu\rangle = \langle x, x\rangle \text{ for } \langle u, u\rangle = 1\,. \tag{15}$$

Using the *Moufang identity*,[7]

$$u(xy)u = (ux)(yu) \text{ for any } x, y, u \in \mathbb{O}\,, \tag{16}$$

One verifies that the triple $g_1 = L_u$, $g_2 = L_u\, R_u$, $g_3 = R_u$ satisfies (14) and hence belongs to $Spin(8)$. It turns out that triples of this type generate $Spin(8)$ (see [48] or Yokota's book [49] for a proof).

The mappings $x \to L_x$ and $x \to R_x$ are, of course, not algebra homomorphisms, as L_x and R_y each generate an associative algebra, while the algebra of octonions is non-associative. They do, however, preserve the quadratic relation $xy^* + yx^* = 2\langle x, y\rangle 1$:

$$L_x L_{y^*} + L_y L_{x^*} = 2\langle x, y\rangle \mathbb{1} = R_x R_{y^*} + R_y R_{x^*}\ (L_x^* = L_{x^*})\,. \tag{17}$$

Equation (17) applied to the span of the first six imaginary octonion units e_j, $j = 1, \cdots, 6$ and setting $L_{e_j} =: L_j$, $R_{e_j} =: R_j$ becomes the defining relation of the Clifford algebra $C\ell_{-6}$:

$$L_j L_k + L_k L_j = -2\delta_{jk} = R_j R_k + R_k R_j,\ j, k = 1, \cdots, 6\,. \tag{18}$$

2.2. $C\ell_{-6}$ as a Generating Algebra of $\mathbb{O}$ and $so(\mathbb{O})$

The octonions appear in any of the nested Clifford algebras $C\ell_8 \subset C\ell_9 \subset C\ell_{10}$. In fact, the minimal realization of $\mathbb{O}$ is provided by $C\ell_{-6}$, generated by the left multiplication L_α by six of the seven imaginary octonion units e_α. In general, $L_x L_y \neq L_{xy}$ (and similarly for R), but remarkably, as noted in [20], the relation $e_1(e_2(e_3(e_4(e_5(e_6\, e_a)))))) = e_7\, e_a$ is satisfied for all $a = 1, \cdots, 8\, (e_8 = 1)$ so that

$$L_1 L_2 \cdots L_6 = L_{e_7} =: L_7,\ R_1 R_2 \cdots R_6 = R_{e_7} =: R_7\,. \tag{19}$$

While $L_x R_x = R_x L_x$ (for $x \in \mathbb{O}$), the non-associativity of the algebra of octonions is reflected in the fact that for $x \neq y$, L_x and R_y in general, do not commute. The Lie algebra

$so(8)$ is spanned by the elements of the negative square of $C\ell_{-6}$. If we denote the exterior algebra on the span of $L_1, \cdots, L_6$ by

$$\Lambda^* \equiv \Lambda^* C\ell_{-6} = \Lambda^0 + \Lambda^1 + \cdots + \Lambda^6 \left(\Lambda^1 = \underset{1\le j\le 6}{\text{Span}}\, L_j,\ \Lambda^6 = \{\mathbb{R}\, L_7\}\right)$$

then $so(8) = \Lambda^1 + \Lambda^2 + \Lambda^5 + \Lambda^6$ (accordingly, the 28-dimensional adjoint representation of $so(8)$ splits into four irreducible representations of $so(6) : \mathbf{28} = \mathbf{6} + \mathbf{15} + \mathbf{6}^* + \mathbf{1}$. In particular, $\Lambda_5 = Span\{L_{\alpha 7}, \alpha = 1, \ldots, 6\}$ for $L_{\alpha\beta}$ defined below). A basis of the Lie algebra, given by

$$L_{\alpha 8} = \tfrac{1}{2}\, L_\alpha,\ L_{\alpha\beta} = -\tfrac{1}{4}\,[L_\alpha, L_\beta],\ \alpha, \beta = 1, \cdots, 7 \tag{20}$$

obeys the standard commutation relations (CRs) for $so(n)$ (herein $n = 8$):

$$[L_{ab}, L_{cd}] = \delta_{bc}\, L_{ad} - \delta_{bd}\, L_{ac} + \delta_{ad}\, L_{bc} - \delta_{ac}\, L_{bd},$$

$$L_{ab} = \tfrac{1}{4}(L_a L_b^* - L_b L_a^*),\ a, b, c, d = 1, 2, \cdots, 8\, (L_\alpha^* = -L_\alpha, L_8^* = L_8) \tag{21}$$

(and similarly for R_{ab}). Each element of $so(8)$ of square -1 defines a *complex structure* (see Section 2.3). Following [24], we shall single out the *Clifford pseudoscalars* L_7 and R_7 (19) (called *volume forms* in the highly informative lectures [51] and Coxeter elements in [52]). We shall use the (mod 7) multiplication rules of [13] for the imaginary octonion units

$$L_i e_j (= e_i e_j) = -\delta_{ij} + f_{ijk}\, e_k,\ f_{ijk} = 1$$

$$\text{for } (i, j, k) = (1,2,4)(2,3,5)(3,4,6)(4,5,7)(5,6,1)(6,7,2)(7,1,3) \tag{22}$$

and f_{ijk} is fully antisymmetric within each of the above seven triples. The Clifford pseudoscalar is naturally associated with the Cartan subalgebra of $so(6)$, spanned by

$$(L_{13}, L_{26}, L_{45}) \text{ as } L_7(e_1, e_2, e_4) = (e_3, e_6, e_5)\,. \tag{23}$$

We can write

$$L_7 = 2^3 L_{13} L_{26} L_{45} \text{ (as } 2L_{13} = L_1 L_3^* = -L_1 L_3 \text{ etc.).} \tag{24}$$

The infinitesimal counterpart of (14) reads

$$T_\alpha(x, y) = (L_\alpha\, x)y + x(R_\alpha\, y) \text{ for } \alpha, x, y \in \mathbb{O},\ \alpha^* = -\alpha\,,$$

$$\text{i.e.,} \qquad T_\alpha = L_\alpha + R_\alpha\,. \tag{25}$$

There is an involutive outer automorphism π of the Lie algebra $so(8)$ such that

$$\pi(L_\alpha) = T_\alpha,\ \pi(R_\alpha) = -R_\alpha,\ \pi(T_\alpha) = L_\alpha\ (\pi^2 = id)\,. \tag{26}$$

As proven in Appendix A,

$$\pi(L_{ab}) = E_{ab},\ \text{where } E_{ab}\, e_c = \delta_{bc}\, e_a - \delta_{ac}\, e_b\ (a, b, c = 1, 2, \cdots, 8,\ e_8 = 1). \tag{27}$$

$(L_{ab}), (E_{ab})$, and (R_{ab}) provide three bases of $so(8)$, each obeying the CRs (21). They are expressed by each other in terms of the involution π:

$$L_{ab} = \pi(E_{ab}),\ E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8},\ \alpha = 1, \cdots, 7\,. \tag{28}$$

We find, particularly (see Appendix A):

$$L_7 = 2L_{78} = E_{78} - E_{13} - E_{26} - E_{45} = 2E_{78} - R_7;\ 2L_{13} = E_{13} - E_{26} - E_{45} - E_{78},$$

$$2L_{26} = E_{26} - E_{13} - E_{45} - E_{78},\ 2L_{45} = E_{45} - E_{13} - E_{26} - E_{78}. \quad (29)$$

While $L_{78} = 4L_{13}L_{26}L_{45}$ (24) commutes with the entire Lie algebra $spin(6) = su(4)$ and the $u(1)$ generator (whose physical meaning is revealed by (45)).

$$C_1 = L_{13} + L_{26} + L_{45} \text{ centralizes } u(3) = u(1) \oplus su(3) \subset su(4) \quad (30)$$

where the second summand is the unbroken color Lie algebra $su(3) = su(3)_c$.

2.3. Complex Structure and Symmetry Breaking in $C\ell_n$

The algebra $C\ell_8$ is generated by two-by-two Hermitian matrices whose elements involve the operators L_a of the left multiplication by octonion units:

$$\gamma_a = \begin{pmatrix} 0 & L_a \\ L_a^* & 0 \end{pmatrix},\ L_a = L_{e_a},\ L_a^* = L_{e_a^*},\ a = 1, \ldots, 8. \quad (31)$$

Here, $e_8 = 1(= e_8^*), L_8 = \mathbb{1}_8$ is the unit operator in $\mathbb{R}^8$; $L_\alpha^* = -L_\alpha$ for $\alpha = 1, \ldots, 7$ so that the $C\ell_{10}$ generators γ_α (10) are obtained from the above (for $\alpha = 1, \ldots, 7$) through tensoring with the 2×2 unit matrix σ_0.

A compact way to identify the particle states in a Clifford algebra $C\ell_{2n}$ is to introduce a complex structure which, as we shall demonstrate, gives rise to a fermionic Fock space in $C\ell_{2n}$.

$$(JX, JY) = (X, Y),\ (JX, Y) = -(X, JY),\ \forall X, Y \in E_{2n}. \quad (32)$$

For a non-zero vector X and a complex structure J, the vector $Y = JX$ is orthogonal to X (and has the same norm):

$$Y = JX \Rightarrow (X, Y) = 0\ ((X, X) = (Y, Y) > 0).$$

It follows that for each complex structure J in E_{2n}, there exists an orthonormal basis of the form $(\gamma_1, \ldots, \gamma_n, J\gamma_1, \ldots, J\gamma_n)$ in $C\ell_{2n}$. Then, $a_j = \frac{1}{2}(\gamma_j - iJ\gamma_j)$ and $a_j^* = \frac{1}{2}(\gamma_j + iJ\gamma_j)$, $j = 1, \ldots, n$ each span the image in $C\ell_{2n}(= C\ell_{2n}(\mathbb{C}))$ of a maximal isotropic subspace of the complexification of E_{2n}. Together, they yield a realization of the *canonical anticommutation relations* (CAR). Fermionic oscillators have been used in the present context in [21,53]. The complex structure in $so(2n)$ involves a distinguished maximal (rank n) Lie subalgebra (a notion studied in [41]), $u(n) \subset so(2n)$, which is generated by the products $b_j b_k^*$. It also selects two distinguished $u(n)$ singlet states in $C\ell_{2n}$, the *vacuum*, which is annihilated by all b_j and the antipode, which is annihilated by the b_j^*. Both singlets are annihilated by the simple part $su(n)$ of $u(n)$.

Complex structures have been studied in relation to spinors by Élie Cartan (since 1908), Veblen (1933), and Chevalley (1954). For a carefully written survey with historical highlights - see [54]. For a concise modern exposition that connects them to the states in a fermionic Fock space, see Dubois-Violette [55]. We have also been influenced by their use (in $SO(9)$) by Krasnov [46] and by relating them to Clifford pseudoscalars in [24].

The pseudoscalar ω_6 of $C\ell_6$ belongs to $C\ell_8$ but only defines a complex structure through its action on the octonion units. More precisely, taking the basic relations (22) and the identity $\epsilon^2 = -\sigma_0$ into account, we can write

$$\omega_6 = -\sigma_0 \otimes L_7\,, \text{ where } -L_7 e_a = \sum_b J_{ab} e_b,\ J_{\alpha\beta} = -f_{7\alpha\beta}, \alpha, \beta = 1, \ldots, 6,$$

$$J_{ab} = -J_{ba},\ J_{13} = J_{26} = J_{45} = -J_{78} = -1. \tag{33}$$

Warning: due to the non-associativity of $\mathbb{O}$, $-L_7 e_3 = e_1$ does not imply $-L_7 L_3 = L_1$, etc.

We shall see that each of the subgroups $Spin(n)^{\omega_6}$ of the spin groups of $C\ell_n$ for $n = 8, 9, 10$ that leaves ω_6 invariant is relevant for the particle physics:

$$Spin(8)^{\omega_6} = U(4),\ Spin(9)^{\omega_6} = SU(4) \times SU(2),\ Spin(10)^{\omega_6} = G_{PS} \tag{34}$$

The $U(1)$ factor in the $U(4)$ of $Spin(8)^{\omega_6}$ and the $SU(2)$ in $Spin(9)^{\omega_6}$ are generated by all three components of the "total weak isospin" $\mathbf{I} = \mathbf{I}^L + \mathbf{I}^R$, as will be made explicit in Section 3.2.

We shall define *a complex structure J corresponding to ω_6* by extrapolating (33) to a transformation of the gamma matrices of $C\ell_8$:

$$J : \gamma_a \to \sum_b J_{ab}\gamma_b = \omega_J \gamma_a \omega_J^*,\ \omega_J = \frac{1}{4}(1+\gamma_{13})(1+\gamma_{26})(1+\gamma_{45})(1-\gamma_{78}). \tag{35}$$

We can extend the basis (31) of $C\ell_8$ to $C\ell_{10}$, setting (cf. (10)):

$$\gamma_\alpha = \sigma_0 \otimes \epsilon \otimes L_\alpha \text{ for } \alpha = 1, \ldots, 7,$$

$$\gamma_8 = \sigma_0 \otimes \sigma_1 \otimes \mathbb{1}_8,\ \gamma_9 = \sigma_2 \otimes \sigma_3 \otimes \mathbb{1}_8,\ \gamma_{10} = \sigma_1 \otimes \sigma_3 \otimes \mathbb{1}_8. \tag{36}$$

In particular, $C\ell_9 = \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ is generated by the 32×32 matrices $\gamma_1, \ldots, \gamma_9$, which commute with $\omega_9 = \gamma_1\gamma_2 \ldots \gamma_9 = \sigma_2 \otimes \mathbb{1}_{16}$. The Lie subalgebra of $so(n)$ of the derivations of $C\ell_n, n = 8, 9, 10$ that commute with ω_6 (10) is $so(6) \oplus so(n-6)$. For $n = 10$, it is the Lie algebra $\mathfrak{g}_{PS}$ of the Pati–Salam group (8) that respects the tensor product representation (11) of $C\ell_{10}$.

We now proceed to give meaning to the projection operator

$$\mathcal{P} = \frac{1 - i\omega_6}{2} \quad (\mathcal{P}^2 = \mathcal{P}),\ tr\,\mathcal{P} = tr(1-\mathcal{P}) = 2^{\ell-1} \text{ for } n = 2\ell (= 6, 8, 10). \tag{37}$$

To begin with, we introduce the isotropic elements

$$2b_1 = (1-iJ)\gamma_1 = \gamma_1 + i\gamma_3,\ 2b_2 = (1-iJ)\gamma_2 = \gamma_2 + i\gamma_6, 2b_3 = (1-iJ)\gamma_4 = \gamma_4 + i\gamma_5. \tag{38}$$

These correspond to the projected octonion units $\frac{1}{2}(1+iL_7)e_\ell$, $\ell = 1, 2, 4$. Together with their conjugates b_j^* ($b_1^* = \frac{1}{2}(\gamma_1 - i\gamma_3)$, etc.), they realize the CAR

$$[b_j, b_k]_+ = 0,\ [b_j, b_k^*]_+ = \delta_{jk},\ j, k = 1, 2, 3. \tag{39}$$

The annihilation operators b_j span the (maximal) three-dimensional isotropic subspace $\mathcal{H}^{(1,0)}$ of the six-dimensional complex vector space $\mathbb{C}E_6$, while b_j^* span its orthogonal complement $\mathcal{H}^{(0,1)}$; we have:

$$J(1 \mp iJ)\gamma_\ell = \pm i(1 \mp iJ)\gamma_\ell \text{ for } \ell = 1, 2, 4. \tag{40}$$

The commuting Hermitian elements $i\gamma_{\ell 3\ell(mod 7)}$, $\ell = 1, 2, 4$, which span a Cartan subalgebra of the comlexified $so(6)$, can be expressed as commutators of b_j^* and $b_j, j = 1, 2, 3$ or as differences of the associated projection operators $p_j' - p_j$:

$$i\gamma_{\ell 3\ell} = [b_\ell^*, b_\ell] = p_\ell' - p_\ell,\ \ell = 1, 2,\ i\gamma_{45} = [b_3^*, b_3] = p_3' - p_3,$$

$$p_j = b_j b_j^*, p_j' = b_j^* b_j = 1 - p_j, p_j p_j' = 0,\ j = 1, 2, 3. \tag{41}$$

In terms of these operators, the $C\ell_6$ pseudoscalar and the projector $\mathcal{P}$ assume the form:

$$i\omega_6 = (p_1' - p_1)(p_2' - p_2)(p_3' - p_3) = \mathcal{P}' - \mathcal{P},\ \mathcal{P}' = 1 - \mathcal{P},$$
$$\mathcal{P} = \ell + q,\ \ell = p_1 p_2 p_3,\ q = q_1 + q_2 + q_3,\ q_j = p_j p_k' p_\ell'. \tag{42}$$

The triple (j,k,ℓ) being a permutation of $(1,2,3)$.

We shall identify the generators (of the comlexification $s\ell(3,\mathbb{C})$) of $su(3)$ with the traceless part of the matrix $(b_j b_k^*)$, whose elements belong to $\mathcal{H}^{(1,1)}$. Then, the splitting (42) of $\mathcal{P}$ into the $su(3)$ singlet ℓ and the triplet q implements the lepton-quark splitting anticipated by its image (1) on the octonions. We shall thus interpret the one-dimensional projectors ℓ and q_j as describing the lepton and the colored quark *states* in $C\ell_6$. The states ℓ and q_j are mutually orthogonal idempotents, with ℓ playing the role of the Fock vacuum in $C\ell_6$:

$$\ell^2 = \ell,\ \ell q_j = 0,\ q_j q_k = \delta_{jk} q_j,\ \ b_j \ell = 0 = \ell b_k^*. \tag{43}$$

Remark 1. *We shall argue in Section 3.3 that the identification of $\mathcal{P}$ as a particle subspace projector (adopted in [34]) would only be justified if we have a clear distinction between particles and antiparticles. This can be claimed for the 30 fundamental (anti)fermions of the $C\ell_{10}$ multiplet* **32** *(9), which have different quantum numbers with respect to the gauge Lie algebra $\mathfrak{g}_{SM}$ of the SM. It fails in the two-dimensional subspace of sterile neutrinos annihilated by $\mathfrak{g}_{SM}$; ν_R and $\bar{\nu}_L$ are allowed to form a coherent superposition—a Majorana spinor. In Sections 3.3 and 4, we shall adopt the restricted projector ℓ_r on the (three- rather than four-dimensional) lepton subspace excluding the sterile neutrino.*

In order to extend the Fock space picture to $C\ell_8$, we shall set

$$i\gamma_7 = a^* - a,\ \gamma_8 = a + a^*\ \Rightarrow i\gamma_{78} = [a^*, a] \tag{44}$$

where the pair (a^*, a) describes another Fermi oscillator ($[a, a^*]_+ = 1$) anticommuting with b_j, b_k^*. We shall fix the physical interpretation of $[a^*, a]$ by postulating that the *electric charge operator* is given by

$$Q := \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - a^* a = \frac{1}{2}(B - L - [a^*, a]),$$
$$\text{where } B - L = \frac{2i}{3}(L_{13} + L_{26} + L_{45}) = \frac{1}{3}\sum_j [b_j^*, b_j], \tag{45}$$

and stands for the difference between the baryon and the lepton numbers. $B - L$ takes the eigenvalues $\pm\frac{1}{3}$ for (anti)quarks and ∓ 1 for (anti)leptons. Demanding that the gauge Lie algebra within $so(8)$ commutes with both ω_6 and Q, we shall further reduce it from $so(6) \oplus so(2)$ to the rank four Lie subalgebra

$$\mathfrak{g}_4 = su(3)_c \oplus u(1)_Q \oplus u(1)_{B-L} = \{X \in u(4), [X, Q] = 0\}. \tag{46}$$

The knowledge of the charges $Q, B - L$ along with the color Lie algebra allows us to identify the primitive idempotents of $C\ell_8$, given by ℓ, q_j and multiplied by aa^* or a^*a, with the fundamental fermions:

$$\ell aa^* = \nu,\ \ell a^* a = e,\ \ q_j aa^* = u_j,\ q_j a^* a = d_j. \tag{47}$$

The "isotopic doublets" (ν, e) and (u_j, d_j) stand for neutrino/electron and up/down colored quarks. We see, in particular, that the Fock vacuum in $C\ell_8$ that is associated with the complex structure (33) is identified with the neutrino (as it has no charge and $a\nu = 0 = b_j \nu$). Note that the subalgebra of $\mathfrak{g}_4$ that annihilates ν is the known unbroken gauge Lie algebra $u(3)$ of the SM:

$$u(3)_{SM} = su(3)_c \oplus u(1)_Q = \{X \in \mathfrak{g}_4; X\nu = 0\}. \tag{48}$$

This picture ignores chirality, which will find its place in $C\ell_{10}$ (Section 3.2).

3. The Internal Space Subalgebra of $C\ell_{10}$

3.1. Equivalence Class of Lorentz-like Clifford Algebras

Nature appears to select real Clifford algebras $C\ell(s,t)$ of the equivalence class of $C\ell(3,1)$ (with a Lorentz signature in four dimensions) in Élie Cartan's classification (which involves[8] the signs, $\omega^2(s,t)$ and $(-1)^{s-t}$):

$$C\ell(s,t) = \mathbb{R}[2^n], \text{ for } s-t = 2(\mathrm{mod}\, 8),\ s+t = 2n\,. \tag{49}$$

They act on 2^n dimensional *Majorana spinors* that irreducibly transform under the *real* 2^n dimensional representation of the spin group $Spin(s,t)$. If $\gamma_1, \cdots, \gamma_{2n}$ is the image in $C\ell(s,t)$ of an orthonormal basis of the underlying vector space $\mathbb{R}^{s,t}$, then the Clifford pseudoscalar defines a complex structure

$$\omega_{s,t} = \gamma_1 \cdots \gamma_{2n},\ 2n = s+t,\ \omega_{s,t}^2 = -1\,, \tag{50}$$

which commutes with the action of $Spin(s,t)$. Upon complexification, the resulting *Dirac spinor* splits into two *inequivalent* 2^{n-1} dimensional complex *Weyl* (or *chiral*) *spinor* representations, which are irreducible *over* $\mathbb{C}$ under $Spin(s,t)$. The corresponding projectors Π_L and Π_R on the left and right spinors are given in terms of the chirality χ, which involves the imaginary unit i:

$$\Pi_L = \tfrac{1}{2}(1-\chi),\ \Pi_R = \tfrac{1}{2}(1+\chi),\ \chi = i\omega_{s,t}\,,$$

$$\chi^2 = \mathbb{1} \Leftrightarrow \Pi_L^2 = \Pi_L,\ \Pi_R^2 = \Pi_R,\ \Pi_L\Pi_R = 0,\ \Pi_L + \Pi_R = \mathbb{1}\,. \tag{51}$$

Another interesting example of the same equivalence class (also with indefinite metric) is the *conformal Clifford algebra* $C\ell(4,2)$ (with isometry group $O(4,2)$). We shall demonstrate that just as $C\ell_{-6}$ was viewed (in Section 2.2) as the *Clifford algebra of the octonions*, $C\ell(4,2)$ plays the role of the *Clifford algebra of the split octonions* (also appearing in bitwistor theory [56]):

$$x = v + V + l(w+W),\ v,w \in \mathbb{R},\ V = iV_1 + jV_2 + kV_3,\ W = iW_1 + jW_2 + kW_3$$

$$i^2 = j^2 = k^2 = ijk = -1, l^2 = 1,\ Vl = -lV\,. \tag{52}$$

Indeed, defining the mapping (cf. (6)),

$$i \to \gamma_{-1},\ j \to \gamma_0,\ l \to \gamma_1,\ jl \to \gamma_2,\ \ell k \to \gamma_3,\ \ell i \to \gamma_4$$

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}\mathbb{1},\ \eta_{11} = \eta_{22} = \eta_{33} = \eta_{44} = 1 = -\eta_{-1,-1} = -\eta_{00} \tag{53}$$

we find that the missing split-octonion (originally, quaternion) imaginary unit $k\,(= ij = -ji)$ can be identified with the $C\ell(4,2)$ pseudoscalar:

$$\omega_{4,2} = \gamma_{-1}\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 \leftrightarrow k,\ \omega_{4,2}^2 = -1,\ [w_{4,2}, \gamma_\nu]_+ = 0\,. \tag{54}$$

The conjugate to the split octonion x (52) and its norm are

$$x^* = v - V - \ell(w+W),\ N(x) = xx^* = v^2 + V^2 - w^2 - W^2,$$

so that the isometry group of $\widetilde{\mathbb{O}}$ is $O(4,4)$ (in particular, the maximal compact subalgebra $so(4) \oplus so(4) \subset so(4,4)$ is spanned by $\gamma_{jk}, j,k = 1,\ldots,4$ and by $\omega_{4,2}, \gamma_\alpha$, $\alpha = -1, 0$, and their commutators. The remaining 16 non-compact generators of $so(4,4)$ involve the square-one matrices $\gamma_j, \gamma_\alpha\gamma_j, \gamma_j\omega_{4,2}$).

As we are interested in the geometry of the internal space of the SM acted upon by a compact gauge group, we shall work with (positive or negative) definite Clifford algebras $C\ell_{2\ell}$, $\ell = 1 (\mathrm{mod}\, 4)$. The algebra $C\ell_{-6}$ considered in Section 2 belongs to this family (with $\ell = -3$). For $\ell = 1$, we obtain the Clifford algebra of the two-dimensional conformal field theory; the one-dimensional Weyl spinors correspond to analytic and antianalytic functions. Here, we shall argue that for the next allowed value, $\ell = 5$, the algebra $C\ell_{10} = C\ell_4 \widehat{\otimes} C\ell_6$ (11) fits the internal space of the SM beautifully if we associate the two factors to the color and flavor degrees of freedom, respectively. We shall strongly restrict the physical interpretation of the generators $\gamma_{ab}\left(= \frac{1}{2}[\gamma_a, \gamma_b],\ a, b = 1, \cdots, 10\right)$ of the derivations of $C\ell_{10}$ by demanding that the splitting (11) of $C\ell_{10}$ into $C\ell_4$ and $C\ell_6$ is preserved. This amounts to selecting a first step of symmetry breakings of the GUT group $Spin(10)$, which leads to the semi-simple Pati–Salam group $(Spin(4) \times Spin(6))/\mathbb{Z}_2$ (8). Each summand of $\mathfrak{g}_{PS}$, $so(4)$ and $so(6)$, expressed in terms of Fermi creation and annihilation operators, has a distinguished Lie subalgebra, $u(2)$ and $u(3)$, that belongs to $\mathcal{H}^{1,1}$. We identify the leptons and quarks with $u(3)$ singlets and triplets. This identification implements the lepton-quark symmetry alluded to by (1).

3.2. $\mathfrak{g}_{SM}$ as Annihilator of Sterile Neutrino

We proceed to extend the complex structure J (33) and (35) to $C\ell_{10}$, expressing, in particular, the electroweak gauge group generators in terms of the fermionic oscillators corresponding to the $C\ell_4$ factor in (11). To this end, we complement the definition (38) of b_j by

$$2a_1 = (1 - iJ)\gamma_{10} = \gamma_{10} - i\gamma_9,\ 2a_2 = \gamma_8 - i\gamma_7 \Rightarrow i\gamma_{78} = [a_2^*, a_2], i\gamma_{910} = [a_1^*, a_1] \tag{55}$$

(where γ_a are given by (36)). In particular, (a_2, a_2^*) coincide with the unique flavor fermionic oscillator (a, a^*) (44) of $C\ell_8$. They allow us to define two pairs of complementary projectors.

$$\pi_\alpha = a_\alpha a_\alpha^*,\ \pi'_\alpha = a_\alpha^* a_\alpha = 1 - \pi_\alpha,\ \alpha = 1, 2,\ \pi_\alpha \pi'_\alpha = 0,\ \pi_\alpha + \pi'_\alpha = 1. \tag{56}$$

The three pairs of color $(p_j, p'_j, j = 1, 2, 3)$ and two pairs of flavor $(\pi_\alpha, \pi'_\alpha, \alpha = 1, 2)$ projectors give rise to a ($2^5 = 32$-dimensional) maximal abelian subalgebra of $C\ell_{10}$ of the commuting observables. The flavor gauge Lie algebra gnerators, the left and right chiral isospin components, are expressed in terms of $a_\alpha^{(*)}$:

$$I_+^L = a_1^* a_2,\ I_-^L = a_2^* a_1,\ [I_+^L, I_-^L] = 2I_3^L = \pi'_1\pi_2 - \pi_1\pi'_2 = \pi'_1 - \pi'_2;$$

$$I_+^R = a_1 a_2,\ I_-^R = a_2^* a_1^*,\ [I_+^R, I_-^R] = 2I_3^R = \pi_1\pi_2 - \pi'_1\pi'_2 = \pi_2 - \pi'_1. \tag{57}$$

The *chirality operator* $\chi = \Pi_R - \Pi_L$ is expressed in terms of the $C\ell_{10}$ pseudoscalar (as implied by (51) for $s = 10, t = 0$):

$$\chi = i\omega_{10} = i\omega_6\gamma_{78}\gamma_{910} = (\mathcal{P}' - \mathcal{P})[a_1^*, a_1][a_2, a_2^*] = (\mathcal{P}' - \mathcal{P})(P_1 - P'_1),$$

$$P_1 = (2I_3^L)^2 = \pi'_1\pi_2 + \pi_1\pi'_2,\ P'_1 = (2I_3^R)^2 = \pi_1\pi_2 + \pi'_1\pi'_2 = 1 - P_1, \tag{58}$$

so that $\Pi_L = \mathcal{P}P_1 + \mathcal{P}'P'_1$, $\Pi_R = \mathcal{P}P'_1 + \mathcal{P}'P_1$. Within the particle subspace $\mathcal{P}$, the operator P_1 projects on the left chiral and P'_1 on the right chiral fermions.

The sum $I_3 = I_3^L + I_3^R$ coincides with the total isospin projection that generates the commutant $u(1)$ of $su(4)$ in $u(4)$—see the discussion after Equation (34) in Section 2.3. Conversely, I_3^L, I_3^R appear as chiral projections of I_3:

$$2I_3 = 2I_3^L + 2I_3^R = [a_2, a_2^*] = \pi_2 - \pi'_2 = 2Q - (B - L),$$

$$(2I_3)^2 = 1,\ I_3^L = P_1 I_3 P_1,\ I_3^R = P'_1 I_3 P'_1\ (I_3^L I_3^R = 0). \tag{59}$$

The identification of the vacuum vector $a_1a_2b_1b_2b_3$ (annihilated by all a_α, b_j) becomes consequential if we demand that this ket-vector is a singlet with respect to the gauge group of the SM. The fact that the left and right isospin cannot vanish simultaneously (because $(2(I_3^L + I_3^R))^2 = 1$) implies that the Lie algebra $\mathfrak{g}_{SM}$ of the SM should be chiral:

$$\mathfrak{g}_{SM} \subset u(2) \oplus u(3), \text{where } u(2) = su(2)_L \oplus u(1)_{I_3^R},\ u(3) = su(3)_c \oplus u(1)_{B-L}. \tag{60}$$

It is therefore rewarding that we can identify the Fock space vacuum in $C\ell_{10}$ (given by ν of (47) for $C\ell_8$) with the (right handed, hypothetical) *sterile neutrino* (in fact, ν_R and its antipode $\bar{\nu}_L$ do not interact with the gauge bosons):

$$\nu_R = \pi_1\pi_2\ell,\ a_\alpha\nu_R = 0 (= \nu_R a_\alpha^*),\ b_j\nu_R = 0 \Leftrightarrow \bar{\nu}_L = \pi_1'\pi_2'\ell',\ a_\alpha^*\bar{\nu}_L = 0 \,\text{etc.} \tag{61}$$

The role of the electric charge Q (45) that breaks the $u(4)$ symmetry of ω_6 in $so(8)$ to $u(3) \oplus u(1)_Q$ is played by the *weak hypercharge* Y in $so(10)$:

$$\frac{1}{2}Y = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - \frac{1}{2}\sum_{\alpha=1}^{2} a_\alpha^* a_\alpha = \frac{1}{2}\sum_{\alpha=1}^{2} a_\alpha a_\alpha^* - \frac{1}{3}\sum_{j=1}^{3} b_j b_j^*. \tag{62}$$

They both annihilate the respective vacuum state as well as its antipode. This is made obvious by the two forms of Y in Equation (62) as sums of the normal and antinormal products. By definition, Y belongs to the center of the broken symmetry subalgebra of $\mathfrak{g}_{PS}$. As pointed out in [2] and as will be discussed below in Section 3.3, it gives rise to a superselection rule in the SM.

The significance of choosing the sterile neutrino as a Fock vacuum is summarized by the following:

Proposition 1. *The Lie subgroup of G_{PS} (8) that leaves the Fock vacuum ν_R (61) invariant is the SM gauge group (13).*

Proof. We shall first complete the argument that the maximal Lie subalgebra of $\mathfrak{g}_{PS}$ annihilating the sterile neutrino is $\mathfrak{g}_{SM}$. We have already noted that the Lie subalgebra of $\mathfrak{g}_{PS}$ for which the vacuum transforms as a singlet is $u(2) \oplus u(3)$ (60). This follows from the observation that generators involving $a_1^*a_2^*$ and $b_j^*b_k^*$ transform ν_R into a right-handed electron e_R and an up quark u_R, respectively. It remains to analyze the two-dimensional center $u(1)_{B-L} + u(1)_{I_3^R}$ of this extended algebra. ν_R and $\bar{\nu}_L$ are eigenvectors of both generators with eigenvalues of opposite sign; only multiples of Y annihilate the sterile neutrino:

$$(2I_3^R - 1)\nu_R = 0 = (B - L + 1)\nu_R,\ Y = B - L + 2I_3^R \Rightarrow Y\nu_R = 0 = Y\bar{\nu}_L. \tag{63}$$

This establishes the characterization of the Lie algebra $\mathfrak{g}_{SM}$ as the annihilator of the sterile neutrino. It will be straightforward to extend the result to the SM gauge group (13) after displaying the quantum numbers of the fundamental fermions in the following subsection. □

3.3. Superselection Rules: Restricted Particle Subspace

The weak hypercharge (62) and (63) generates the $u(1)$ center of the gauge Lie algebra of the SM and hence commutes with all gauge transformations. It is not only conserved in the observed micro processes but even in hypothetical ones, such as a possible proton decay (with a conserved $B - L$), or in the presence of a Majorana neutrino (a coherent superposition of ν_R and $\bar{\nu}_L$) that would break $B - L$ by two units. The weak hypercharge was proposed in [2] as a *superselection rule*, assuming that Y commutes with all observables. The Jordan algebra of the 32-dimensional space of internal observables of one generation splits into 11 superselection sectors corresponding to the 11 different eigenvalues of Y (see Appendix to [2]).

Superselection rules (SSR) were introduced by Wick, Wightman, and Wigner [57,58] in 1952. The superselection of the electric charge has been thoroughly discussed in [58] and the review [59]; for more references and a historical survey addressed to philosophers, see [60]. The charge Q (45) is superselected by the exact symmetry of the SM (otherwise, $I^L_\pm$ do not commute with it). SSRs are also related to measurement theory [52]. SSR and superselection sectors are an essential part of the Doplicher–Haag–Roberts reconstruction of quantum fields from the algebra of observables—see [36].

For all we know, the exact symmetry of the SM is given by the rank four unbroken Lie algebra (obtained from $\mathfrak{g}_4$ (46) by the substitution $B - L \to Y$):

$$\mathfrak{a}_4 = su(3)_c \oplus u(1)_Y \oplus u(1)_Q,\ Q = \frac{1}{2}Y + I^L_3 = \frac{1}{3}\sum_{j=1}^{3} p'_j - \pi'_2. \tag{64}$$

The states of the fundamental (anti)fermions are given by the primitive idempotents of $C\ell_{10}$, represented by the $2^5 = 32$ different products of the five pairs of basic projectors $\pi^{(')}_\alpha, p^{(')}_j$ (56) (41). The 16 particles can be labeled by the eigenvalues of the pair of superselected charges (Q, Y):

$$\begin{aligned} (\nu_R) = \ell\,\pi_1\pi_2 = (0,0) = |\nu_R\rangle\langle\nu_R|,\ (\nu_L) = \ell\,\pi'_1\pi_2 = (0,-1) = |\nu_L\rangle\langle\nu_L|, \\ (e_L) = \ell\,\pi_1\pi'_2 = (-1,-1) = |e_L\rangle\langle e_L|,\ (e_R) = \ell\,\pi'_1\pi'_2 = (-1,-2) = |e_R\rangle\langle e_R|; \\ \ell = (\nu_L) + (e_L) + (\nu_R) + (e_R) = p_1\,p_2\,p_3,\ \ell^2 = \ell,\ \mathrm{tr}\,\ell = 4. \end{aligned} \tag{65}$$

$$\begin{aligned} (u^j_L) = q_j\,\pi'_1\pi_2 = (\tfrac{2}{3},\tfrac{1}{3}) = |u^j_L\rangle\langle u^j_L|,\ (d^j_L) = q_j\,\pi_1\pi'_2 = (-\tfrac{1}{3},\tfrac{1}{3}) = |d^j_L\rangle\langle d^j_L|, \\ (u^j_R) = q_j\,\pi_1\pi_2 = (\tfrac{2}{3},\tfrac{4}{3}) = |u^j_R\rangle\langle u^j_R|,\ (d^j_R) = q_j\,\pi'_1\pi'_2 = (-\tfrac{1}{3},-\tfrac{2}{3}) = |d^j_R\rangle\langle d^j_R|; \\ q_j = (u^j_L) + (d^j_L) + (u^j_R) + (d^j_R) = p_j\,p'_k\,p'_\ell,\ q_i\,q_j = \delta_{ij}q_j,\ \mathrm{tr}\,q_j = 4 \end{aligned} \tag{66}$$

$(j,k,\ell) \in \mathrm{Perm}(1,2,3)$, $q = q_1 + q_2 + q_3 = q^2$, $\mathrm{tr}\,q = 12$ (as the color is unobservable, we do not bother to assign to it eigenvalues of the diagonal operators $i\gamma_{13}, i\gamma_{26}, i\gamma_{45}$ that would replace the index j). Note that chirality in the particle subspace $\mathcal{P}\chi = \chi\mathcal{P}$ is determined by the hypercharge:

$$\mathcal{P}\chi = \mathcal{P}(\Pi_R - \Pi_L) = \mathcal{P}(-1)^{3Y}. \tag{67}$$

The charges (Q, Y) for the corresponding antiparticles have the opposite sign. The spectrum of Y and of $2I^L_3 = 2Q - Y$, together with the analysis of [38], allow us to complete the group theoretic version of Proposition 1.

Remark 2. *The factorization of the primitive idempotents (65) and (66) into bra and kets involves choices. We demand, following [2], that they are Hermitian conjugate elements of $C\ell_{10}$, homogeneous in $a^{(*)}_\alpha$ and $b^{(*)}_j$ such that the kets corresponding to a left(right) chiral particle contains an odd (or even) number of factors. Choosing then $|\nu_R\rangle = a_1 a_2 \ell$, $|\nu_L\rangle = a^*_1|\nu_R\rangle$, we find:*

$$\begin{aligned} \langle\nu_R| &= \ell a^*_2 a^*_1\ \Rightarrow (\nu_R) = \pi_1\pi_2\ell,\ |\nu_L\rangle = \pi'_1 a_2\ell\,, \\ |e_L\rangle &= I^L_-|\nu_L\rangle = -a_1\pi'_2\ell\,,\ |e_R\rangle = -a^*_1|e_L\rangle = \pi'_1\pi'_2\ell = I^R_-|\nu_R\rangle; \\ |d^j_L\rangle &= \pi_1 a^*_2\,q_j\,,\ |u^j_L\rangle = I^L_+|d^j_L\rangle = a^*_1\pi_2\,q_j\,, \\ |d^j_R\rangle &= a^*_1|d^j_L\rangle = a^*_1 a^*_2\,q_j\,,\ u^j_R = a_1|u^j_L\rangle = \pi_1\pi_2\,q_j\,, \end{aligned} \tag{68}$$

$q_j = p_j\,p'_k\,p'_\ell$, $j,k,\ell \in \mathrm{Perm}(1,2,3)$, i.e. $q_1 = p_1\,p'_2\,p'_3 = p_1\,p'_3\,p'_2$, etc. We note that all above kets as well as all primitive idempotents (65) (66) obey a system of five equations (specific for each particle), $a_\alpha|\nu_R\rangle = 0 = b_j|\nu_R\rangle$, $a^*_1|\nu_L\rangle = a_2|\nu_L\rangle = 0 = b_j|\nu_L\rangle$, $\alpha = 1,2$, $j = 1,2,3$, etc., so that they are minimal right ideals in agreement with the philosophy of Furey [20].

The fact that $\nu_R, \bar{\nu}_L$ are not distinguished by the superselected charges has a physical implication; one can consider their coherent superposition as in the now popular theory of a (hypothetical) Majorana neutrino. This suggests the introduction of a *restricted* 15-dimensional *particle subspace* with projector

$$\mathcal{P}_r = \mathcal{P} - (\nu_R) = q + \ell_r\,, \quad \ell_r = \ell(1 - \pi_1\pi_2)\,. \tag{69}$$

Theories whose field algebra is a tensor product of a Dirac spinor bundle on a space-time manifold with a finite dimensional internal space usually encounter the problem of fermion doubling [61] (still discussed over 20 years later, [62]). It was proposed in [34] as a remedy to consider the algebra $\mathcal{P}C\ell_{10}\mathcal{P}$, where $\mathcal{P}$ is the projector (42) on the 16 dimensional particle subspace (including the hypothetical right-handed sterile neutrino). It is important—and will be essential in the treatment of the Higgs field (Section 4)—that the operators $a_\alpha^{(*)}$ and $b_j^{(*)}$ behave quite differently under particle projection. While $a_\alpha^{(*)}$ commute with $\mathcal{P}$ so that

$$\mathcal{P}a_\alpha^{(*)}\mathcal{P} = a_\alpha^{(*)}\mathcal{P} = \mathcal{P}a_\alpha^{(*)},\ [\mathcal{P}a_\alpha^*, \mathcal{P}a_\beta]_+ = \mathcal{P}\delta_{\alpha\beta}, \tag{70}$$

the 2-sided particle projection of $b_j^{(*)}$ vanishes:

$$\mathcal{P}b_j\mathcal{P} = 0 = \mathcal{P}b_j^*\mathcal{P}. \tag{71}$$

Accordingly, while the generators (57) of the (electroweak) flavor "left-right symmetry" $su(2)_L \oplus su(2)_R$ just get multiplied by $\mathcal{P}$, the particle subspace projections of the $su(3)_c$ generators take a modified form:

$$\begin{gathered}\mathcal{P}b_jb_k^*\mathcal{P} = b_jb_k^*p'_\ell =: B_{jk} \text{ for } (j,k,\ell) \in Perm(1,2,3),\ B_{jj} - B_{kk} := q_j - q_k;\\ T_a = \tfrac{1}{2}B_{jk}\lambda_a^{kj}\,,\ \lambda_a \in \mathcal{H}_3(\mathbb{C}),\ \mathrm{tr}\,\lambda_a = 0,\ \mathrm{tr}\,\lambda_a\lambda_b = 2\delta_{ab},\ a,b = 1,\cdots,8\,,\end{gathered} \tag{72}$$

but still obey the same CR. It makes sense to separately consider the gauge Lie algebra in the lepton and quark sectors (or the factors $C\ell_4$ and $C\ell_6$ in $C\ell_{10}$), noting that $\mathcal{P}(B-L) = -1$ for leptons and $\mathcal{P}(B-L) = \frac{1}{3}$ for quarks. It is particularly appropriate to treat the lepton sector by itself when using the restricted particle space as it is there that the flavor oscillators $a_\alpha^{(*)}$ are also modified:

$$\ell_r a_1^{(*)}\ell_r = a_1^{(*)}\pi'_2 =: A_1^{(*)},\ \ell_r a_2^{(*)}\ell_r = a_2^{(*)}\pi'_1 =: A_2^{(*)}. \tag{73}$$

The operators $A_\alpha^{(*)}$ provide a realization of the four odd generators of the smallest simple Lie superalgebra, $s\ell(2|1)$, whose even part is $su(2)_L \oplus u(1)_Y$ (for a detailed identification with the standard definition of $s\ell(2|1)$, see Section 3 of [2]). The non-vanishing anticommutators of $A_\alpha^{(*)}$ are:

$$\begin{gathered}[A_1, A_1^*]_+ = \pi'_2 = -Q,\ [A_2, A_2^*]_+ = \pi'_1 = Q - Y,\\ [A_1^*, A_2]_+ = a_2a_1^* = -I_+,\ [A_1, A_2^*]_+ = -I_-;\ [I_+, I_-] = 2I_3 = 2Q - Y\end{gathered} \tag{74}$$

(where we are omitting the superscript L on I_a). We shall apply the odd generators $A_\alpha^{(*)}$ in defining the Higgs part of a superconnection in Section 4. The *minimal associative envelope* $\mathcal{A}_\ell$ of $s\ell(2|1) \subset C\ell_4$ is nine-dimensional; it contains on top of $A_\alpha^{(*)}$ and their anticommutators (74) the projector $A_1^*A_1 = A_2^*A_2 = \pi'_1\pi'_2 \in C\ell_4$. The resulting *internal space algebra* that ignores the sterile neutrinos is the direct sum

$$\mathcal{A} = \mathcal{A}_\ell \otimes \ell \oplus C\ell_4 \otimes \mathcal{A}_q,\ \mathcal{A}_q = qC\ell_6^0 q. \tag{75}$$

Here, $\mathcal{A}_q$ is effectively the nine-dimensional associative envelope of $u(3) \subset C\ell_6^0$.

3.4. Complex Structure Associated with R_7: A Comment

Inspired by [24,46,63], we shall display and discuss the symmetry subalgebras of $C\ell_n, n = 8, 9, 10$ of the complex structure generated by the Clifford pseudoscalar ω_6^R corresponding to the right action of the octonions:

$$\omega_6^R = \gamma_1^R \cdots \gamma_6^R \text{ for } \gamma_\alpha^R = \epsilon \otimes R_\alpha \quad \alpha = 1, \cdots, 7\,. \tag{76}$$

Written in terms of the color projectors p_j and p'_j, the Hermitian pseudoscalar $i\omega_6^R$ assumes the form:

$$i\omega_6^R = \frac{1}{2}(\mathcal{P}' - \mathcal{P} - 3(B-L)) = \ell + q' - \ell' - q\,,$$
$$\ell' = p'_1 p'_2 p'_3, q' = \sum_{j=1}^{3} q'_j,\ q'_1 = p'_1 p_2 p_3, q'_2 = p_1 p'_2 p_3, q'_3 = p_1 p_2 p'_3 \tag{77}$$

where we have used

$$L = \ell - \ell'\,,\ 3B = q - q'\,. \tag{78}$$

While the term $\mathcal{P}' - \mathcal{P}$ (42) commutes with the entire derivation algebra $spin(6) = su(4)$ of $C\ell_6$, the centralizer of $B - L$ in $su(4)$ is $u(3)$—see Proposition A2 in Appendix A. It follows that the commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$, while its centralizer in $so(9)$ is the gauge Lie algebra $\mathfrak{g}_{\rm SM} = su(3) \oplus su(2) \oplus u(1)$ of the SM; finally, in $so(10)$, ω_6^R is invariant under the left-right symmetric extension of $\mathfrak{g}_{\rm SM}$ [24,63],

$$\mathfrak{g}_{\rm LR} = su(3)_c \oplus su(2)_L \oplus su(2)_R \oplus u(1)_{B-L}\,. \tag{79}$$

Furthermore, as proven in [46], the subgroup of $Spin(9)$ that leaves ω_6^R invariant is precisely the gauge group[9] $G_{\rm SM} = S(U(2) \times U(3))$ (13) of the SM (with the appropriate $\mathbb{Z}_6$ factored out). One is then tempted to assume that $C\ell_9$, the associative envelope of the Jordan algebra $J_2^8 = \mathcal{H}_2(\mathbb{O})$, may play the role of the internal algebra of the SM, corresponding to one generation of fundamental fermions, with $Spin(9)$ as a GUT group [27,28,33]. We shall demonstrate that although $G_{\rm SM}$ appears as a subgroup of $Spin(9)$, its representation, obtained by restricting the (unique) spinor IR **16** of $Spin(9)$ to $S(U(2) \times U(3))$ only involves $SU(2)$ doublets, so it has no room for $(e_R), (u_R), (d_R)$ (65) (66). We shall see how this comes about when restricting the realization (57) of $\mathbf{I}^L$ and $\mathbf{I}^R$ to $Spin(9) \subset C\ell_9$. It is clear from (57) that only the sum $a_1 + a_1^* = \gamma_9$ (not a_1 and a_1^* separately) belongs to $C\ell_9$. So, the $su(2)$ subalgebra of $spin(9)$ corresponds to the diagonal embedding $su(2) \hookrightarrow su(2)_L \oplus su(2)_R$:

$$I_+ = I_+^L + I_+^R = (a_1^* + a_1)\, a_2 = \gamma_9\, a_2,\ I_- = I_-^L + I_-^R = a_2^* \gamma_9$$
$$2I_3 = 2I_3^L + 2I_3^R = [a_2, a_2^*] = \pi_2 - \pi'_2\,. \tag{80}$$

In other words, the spinorial IR **16** of $Spin(9)$ is an eigensubspace of the projector $P_1 = (2I_3^L)^2$. It consists of four $SU(2)_L$ particle doublets and their right chiral antiparticles. More generally, the only simple orthogonal groups with a pair of inequivalent complex that conjugate fundamental IRs are $Spin(4n+2)$ (see, e.g., [64], Proposition 5.2, p. 571). They include $Spin(10)$ but not $Spin(9)$.

A direct description of the IR $\mathbf{16}_L$ of $Spin(10)$ acting on $\mathbb{CH} \otimes \mathbb{CO}$ is given in [23] (Here, $\mathbb{CH}$ and $\mathbb{CO}$ are a short hand for the complexified quaternions and octonions: $\mathbb{CH} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$). The right action of $\mathbb{CH}$ on elements of $\mathbb{CH} \otimes \mathbb{CO}$, which commutes with the left-acting $spin(10)$, is interpreted in [23] as the Lorentz $(SL(2, \mathbb{C}))$ transformation of the (unconstrained) two-component Weyl spinors.

The left-right symmetric extension $\mathfrak{g}_{\rm LR}$ (78) of $\mathfrak{g}_{\rm SM}$ has a long history, starting with [65] and vividly (with an admitted bias) told in [66]; it has been recently invigorated in [67,68]. The group $G_{\rm LR}$ was derived by Boyle [47] starting with the group E_6 of determinant preserving linear automorphisms of the complexified Albert algebra $\mathbb{C}J_3^8$ and following the procedure of [35].

4. Particle Subspace and the Higgs Field

4.1. The Higgs as a Scalar Part of a Superconnection

The space of differential forms $\Lambda^* = \Lambda^0 + \Lambda^1 + \Lambda^2 + \ldots$ can be viewed as a $\mathbb{Z}_2$ graded setting $\Lambda_{ev} = \Lambda_0 + \Lambda_2 + \ldots, \Lambda_{od} = \Lambda_1 + \Lambda_3 + \ldots$. Let $M = M_0 + M_1$ be a $\mathbb{Z}_2$ graded matrix algebra. A superconnection in the sense of Quillen [69,70] is an element of $\Lambda_{ev} \otimes M_1 + \Lambda_{od} \otimes M_0$, the odd part of the tensor product $\Lambda^* \otimes M$; a critical review of the convoluted history of this notion and its physical implications is given in Section 4 of [71] (one should also mention the neat exposition of [72] in the context of the Weinberg–Salam model with two Higgs doublets).

Let D be the Yang–Mills connection one-form of the SM,

$$D = dx^\mu(\partial_\mu + A_\mu(x)),$$

$$iA_\mu = W_\mu^+ I_+^L + W_\mu^- I_-^L + W_\mu^3 I_3^L + \frac{N}{2} Y B_\mu + G_\mu^a T_a, \tag{81}$$

where $Y, \mathbf{I}^L$, and T_a are given by (62), (57), and (72), respectively; G_μ^a is the gluon field, and $\mathbf{W}_\mu$ and B_μ provide an orthonormal basis of electroweak gauge bosons; the normalization constant N will be fixed in Equation (93) below. Then, one defines a superconnection $\mathbb{D}$ in [34] involving the chirality χ (60) by

$$\mathbb{D} = \chi D + \Phi, \quad \Phi = \sum_\alpha (\varphi_\alpha a_\alpha^* - \overline{\varphi}_\alpha a_\alpha) \in \mathcal{P}C\ell_{10}^1\mathcal{P} = \mathcal{P}C\ell_4^1 \tag{82}$$

(we omit, for the time being, the projector $\mathcal{P}$ in A_μ and Φ). The last equation follows from (71); the projection on the particle subspace kills the odd part of $C\ell_6$, thus ensuring that the quarks' color symmetry remains unbroken. The factor χ (first introduced in this context in [71]) ensures the anticommutativity of Φ and $\chi\mathcal{D}$ without changing the Yang–Mills curvature $D^2 = (\chi D)^2$.

The projector $\mathcal{P}$ (42) on the 16-dimensional particle subspace that includes the hypothetical right chiral neutrino (and is implicit in (82)) was adopted in [34]. By contrast, particles are only distinguished from antiparticles in [2] if they have different quantum numbers in the Lie algebra of the SM

$$\mathfrak{g}_{SM} = su(3)_c \oplus su(2)_L \oplus u(1)_Y. \tag{83}$$

Thus, in [2], $\mathcal{P}$ is replaced by the 15-dimensional projector $\mathcal{P}_r = q + \ell_r$ (69). We have seen that the projected odd operators $A_\alpha^{(*)} = \ell_r a_\alpha^{(*)} \ell_r$ give rise to a realization of the four odd elements of the eight-dimensional simple Lie superalgebra $s\ell(2|1)$ whose even part is the four-dimensional Lie algebra $u(2)$ of the Weinberg–Salam model of the electroweak interactions. It is precisely the Lie superalgebra that was proposed in 1979 independently by Ne'eman and by Fairlie [44,45] (and denoted by them $su(2|1)$) in their attempt to unify $su(2)_L$ with $u(1)_Y$ (and explain the spectrum of the weak hypercharge). Let us stress that the representation space of $s\ell(2|1)$ consists of the observed left and right chiral leptons (rather than of bosons and fermions like in the popular speculative theories in which the superpartners are hypothetical). Note that the trace of Y on negative chirality leptons (ν_L, e_L) is equal to its eigenvalue on the unique positive chirality state (e_R) (equal to -2) so that only the supertrace of Y vanishes on the lepton (as well as on the quark) space. This observation is useful in the treatment of anomaly cancellation (cf. [73]).

We shall sketch the main steps in the application of the superconnection (82) to the bosonic sector of the SM, emphasizing specific additional hypotheses used on the way (for detailed calculations, see [2]).

The canonical curvature form

$$\mathbb{D}^2 = D^2 + \chi[D,\Phi] + \Phi^2,\ [D,\Phi] = dx^\mu(\partial_\mu\Phi + [A_\mu,\Phi]) \tag{84}$$

satisfies the *Bianchi identity*

$$\mathbb{D}\mathbb{D}^2 = \mathbb{D}^2\mathbb{D}\ (\Rightarrow \chi(d\Phi^2 + [A,\phi^2] + [\Phi, D\Phi]_+) = 0)\,, \tag{85}$$

equivalent to the (super) Jacobi identity of our Lie superalgebra. It is important that the Bianchi identity, which is needed for the consistency of the theory, still holds if we add to $\mathbb{D}^2$ a constant matrix term with a similar structure. Without such a term, the Higgs potential would be a multiple of $\mathrm{Tr}\,\Phi^4$ and would only have a trivial minimum at $\Phi = 0$, yielding no symmetry breaking. The projected form of Φ (82) and hence the admissible constant matrix addition to Φ^2 depends on whether we use the projector $\mathcal{P}$ (as in [34]) or P_r (as in [2]). In the first case we just replace $a_\alpha^{(*)}$ with $a_\alpha^{(*)}\mathcal{P}$. In the second, however, the odd generators for leptons and quarks differ, and we set:

$$\Phi = \ell(\phi_1 A_1^* - \overline{\phi}_1 A_1 + \phi_2 A_2^* - \overline{\phi}_2 A_2) + \rho q \sum_{\alpha=1}^{2}(\phi_\alpha a_\alpha^* - \overline{\phi}_\alpha a_\alpha)\,, \tag{86}$$

where ρ (like N in (81)) is a normalization constant that will be fixed later. Recalling that ℓ and q are mutually orthogonal ($\ell q = 0 = q\ell,\ \ell + q = \mathcal{P}$), we find

$$\Phi^2 = \ell(\phi_1\overline{\phi}_2\, I_+^L + \overline{\phi}_1\phi_2\, I_-^L - \phi_1\overline{\phi}_1\pi_2' - \phi_2\,\overline{\phi}_2\pi_1')$$

$$-\rho^2 q(\phi_1\overline{\phi}_1 + \phi_2\,\overline{\phi}_2)\ (\phi_\alpha = \phi_\alpha(x))\,. \tag{87}$$

This suggests defining the SM field strength (the extended curvature form) as

$$\mathbb{F} = i(\mathbb{D}^2 + \widehat{m}^2)\,,\quad \widehat{m}^2 = m^2(\ell(1-\pi_1\,\pi_2) + \rho^2 q) \tag{88}$$

(while $\widehat{m}^2 = m^2\mathcal{P}$ for the 16-dimensional particle subspace of [34]).

4.2. Higgs Potential and Mass Formulae

This yields the bosonic Lagrangian (setting $TrX = \frac{1}{4}trX$—see [2])

$$\mathcal{L}(x) = \mathrm{Tr}\left\{\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\Phi + [A_\mu,\Phi])(\partial^\mu\Phi + [A^\mu,\Phi])\right\} - V(\Phi) \tag{89}$$

where the Higgs potential $V(\Phi)$ is given by (noting that $Tr\ell_r = \frac{3}{4}$):

$$V(\Phi) = \mathrm{Tr}\,(\widehat{m}^2 + \Phi^2)^2 - \tfrac{1}{4}m^4 = \tfrac{1}{2}(1+6\rho^4)(\phi\overline{\phi} - m^2)^2\,. \tag{90}$$

Minimizing $V(\Phi)$ gives the expectation value of the square of $\phi = (\phi_1,\phi_2)$:

$$\langle\phi\,\overline{\phi}\rangle = \phi_1^m\,\overline{\phi_1^m} + \phi_2^m\,\overline{\phi_2^m} = m^2,\ \text{for } \Phi^m = \sum_{\alpha=1}^{2}\phi_a^m a_\alpha^*(\ell\pi_{3-\alpha}' + \rho q) + c\cdot c \tag{91}$$

(the superscript m indicates that ϕ_α takes a constant in x values depending on the mass parameter m). The mass spectrum of the gauge bosons is determined by the term $-\,\mathrm{Tr}\,[A_\mu,\Phi][A^\mu,\Phi]$ of the Lagrangian (89), with A_μ and Φ given by (81) and (86) for $\phi_\alpha = \phi_\alpha^m$. The gluon field G_μ does not contribute to the mass term as $C\ell_6^0$ commutes with $C\ell_4^1$. The resulting quadratic form is generally not degenerate, so it does not yield a massless photon. It does so, however, if we assume that Φ^m is electrically neutral (i.e., commutes with Q (64)):

$$[\Phi^m, Q] = 0 \Rightarrow \phi_2^m = 0\ (=\overline{\phi_2^m})\,. \tag{92}$$

The normalization constant N (=tg θ_w) is fixed by assuming that $2I_3^L$ and NY are equally normalized:

$$N^2 = \frac{\mathrm{Tr}\,(2I_3^L)^2}{\mathrm{Tr}\,Y^2} = \frac{3}{5}\left(= (\mathrm{tg}\,\theta_w)^2 \Leftrightarrow \sin^2\theta_w = \frac{3}{8}\right). \tag{93}$$

As $Y(\nu_R) = 0 = I_3^L(\nu_R)$, this result for the "Weinberg angle at unification scale" is independent of whether we use $\mathcal{P}$ or $\mathcal{P}_r$. If one takes the trace over the leptonic subspace, the result would have been $(\mathrm{tg}\,\theta_w)^2 = \frac{1}{3}(\Rightarrow \sin\theta_w = \frac{1}{2}$, [44]), which is closer to the measured low-energy value.

Demanding, similarly, that the leptonic contribution to Φ^2 is the same as that for a colored quark (which gives $\rho = 1$ for the projector $\mathcal{P}$), we find

$$\rho^2 = \frac{\mathrm{Tr}(\ell(1-\pi_1\pi_2)\Phi^2)}{\mathrm{Tr}\,q_j\,\Phi^2} = \frac{Tr(\pi_1'\pi_2'\,\phi\,\overline{\phi} + \pi_1'\pi_2\,\phi_2\,\overline{\phi}_2 + \pi_1\pi_2'\,\phi_1\,\overline{\phi}_1)}{4\,\phi\,\overline{\phi}} = \frac{1}{2}. \tag{94}$$

The ratio $\frac{m_H^2}{m_W^2}$, on the other hand, is found to be

$$\frac{m_H^2}{m_W^2} = 4\,\frac{1+6\rho^4}{1+6\rho^2} = \begin{cases} 4 \text{ for } \rho^2 = 1 \\ \frac{5}{2} \text{ for } \rho^2 = \frac{1}{2} \end{cases}. \tag{95}$$

The result of [2], which is much closer to the observed value, can also be written in the form $m_H^2 = 4\cos^2\theta_W\, m_W^2$, where θ_W is the theoretical Weinberg angle (93).

5. Outlook

5.1. Coming to $C\ell_{10}$

The search for an appropriate choice of a finite dimensional algebra suited to represent the internal space $\mathcal{F}$ of the SM is still ongoing. The road to the choice of $C\ell_{10}$, our first step to the restricted algebra $\mathcal{A}$ (75), has been convoluted.

In view of the lepton-quark correspondence that is embodied in the splitting (1) of the normed division algebra $\mathbb{O}$ of the octonions, the choice of Dubois-Violette [25] of the exceptional Jordan algebra $\mathcal{F} = \mathcal{H}_3(\mathbb{O})$ (7) appeared to be particularly attractive. We realized [27,28,35] that the simpler to work with subalgebra

$$J_2^8 = \mathcal{H}_2(\mathbb{O}) \subset \mathcal{H}_3(\mathbb{O}) = J_3^8 \tag{96}$$

corresponds to the observables of one generation of fundamental fermions. The associative envelope of J_2^8 is $C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16]$ with the associated symmetry group $Spin(9)$. It was proven in [35] that the SM gauge group G_{SM} (13) is the intersection of $Spin(9)$ with the subgroup of the automorphism group F_4 of J_3^8 that preserves the splitting (1); that is, the group $\frac{SU(3)\times SU(3)}{\mathbb{Z}_3} \subset F_4$.

We were thus inclined to identify $Spin(9)$ as the most economic GUT group. As demonstrated in Section 3.4, however, the restriction of the spinor IR **16** of $Spin(9)$ to its subgroup G_{SM} gives room to only half of the fundamental fermions, the $SU(2)_L$ doublets; the right chiral singlets, e_R, u_R, d_R, are left out. It was then recognized that the (octonionic) Clifford algebra $C\ell_{10}$ does the job. The particle interpretation of $C\ell_{10}$ is dictated by the choice of a (maximal) set of five commuting operators in the Pati–Salam Lie subalgebra of $so(10)$ that leaves our complex structure invariant. This led us to presenting all chiral leptons and quarks of one generation as mutually orthogonal idempotents (65) and (66).

Furey [21] arrived (back in 2018) at the tensor product $C\ell_4\hat{\otimes}C\ell_6$ (11) following the $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ road. In fact, Clifford algebras have arisen as an outgrowth of Grassmann algebras and quaternions[10]. The 32 products $e_a\varepsilon_\nu$ $(=\varepsilon_\nu e_a)$, $a = 1,\cdots,8$ $(e_8 = 1\!\!1)$, $\nu = 0,1,2,3$ of octonion and quaternion units may serve as components of a $Spin(10)$ Dirac (bi)spinor, acted upon by $C\ell_{10}$ (with generators (36) involving the operators L_α)—cf. [23].

5.2. Two Ways to Avoid Fermion Doubling

There are two inequivalent possibilities to avoid fermion doubling within $C\ell_{10}$. One, which was adopted in [2,34] and in Section 3.3 of the present survey, consists in projecting on the particle subspace, which incorporates four $SU(2)_L$ doublets and eight $SU(2)_L$ (right chiral) singlets, with projector (42)

$$\mathcal{P} = \ell + q = \frac{1 - i\omega_6}{2}\,,\ \ell = p_1\, p_2\, p_3\,,\ q = q_1 + q_2 + q_3 \tag{97}$$

(see (70)–(72)). Here, ω_6 is the $C\ell_6$ pseudoscalar, the distinguished complex structure used in [24] as a first step in the "cascade of symmetry breakings". The particle projector (97) is only invariant under the Pati–Salam subgroup (8) of $Spin(10)$. The more popular alternative, which was adopted in [23], projects on the left chiral fermions (four particle doublets and eight antiparticle singlets) with projector Π_L, which is defined in terms of the $C\ell_{10}$ chirality $\chi = i\omega_{10}$:

$$\Pi_L = \frac{1 - \chi}{2} = \mathcal{P}P_1 + \mathcal{P}'P_1' \quad (\mathcal{P} + \mathcal{P}' = 1 = P_1 + P_1')\,, \tag{98}$$

invariant under the entire $Spin(10)$; here, P_1 projects on $SU(2)_L$ doublets (cf. (58)). The components of the resulting $\mathbf{16}_L$ are viewed in [23] as Weyl spinors; the right action of (complexified) quaternions (that commutes with the left $spin(10)$ action) is interpreted as an $s\ell(2,\mathbb{C})$ (Lorentz) transformation.

The difference of the two approaches, which can be labeled by the projectors $\mathcal{P}$ and Π_L (on the left and right particles and on the left particles and antiparticles, respectively), has implications in the treatment of the generalized connection (including the Higgs) and anomalies. Thus, for the Π_L (anti)leptons (ν_L, e_L), $\bar{e}_L$, $\bar{\nu}_L$ we have vanishing trace of the hypercharge, $\operatorname{tr}\Pi_L Y = 0$. For $\mathcal{P}$ leptons, $(\nu_L, e_L), \nu_R, e_R$, the traces of the left and right chiral hypercharge are equal: $\operatorname{tr}(\mathcal{P}\Pi_L Y) = -2 = \operatorname{tr}(\mathcal{P}\Pi_R Y)$, so that, as noted in Section 4.1, only the supertrace vanishes in this case. The associated Lie superalgebra fits Quillen's notion of superconnection ideally. A real "physical difference" only appears under the assumption that the electroweak hypercharge is superselected and $\mathcal{P}$ is replaced by the restricted projector $\mathcal{P}_r$ on the 15-dimensional particle subspace (with the sterile neutrino ν_R and with the vanishing hypercharge excluded). Then, the leptonic (electroweak) part of the SM is governed by the Lie superalgebra $s\ell(2|1)$, whose four odd generators are given by third-degree monomials in $a_\alpha^{(*)}$, the $C\ell_4$ Fermi oscillators. The replacement of ℓ by ℓ_r breaks the quark-lepton symmetry; while each colored quark q_j appears in four flavors, the colorless leptons number just three. This yields a relative normalization factor between the quark and leptonic projection of the Higgs field and allows us to derive (in [2]) the relation

$$m_H^2 = \tfrac{5}{2}\, m_W^2 = 4\cos^2\theta_{\text{th}}\, m_W^2\,, \tag{99}$$

where θ_{th} is the *theoretical* Weinberg angle, such that $\text{tg}^2\,\theta_W = \frac{3}{5}$. The relation (99) is satisfied within 1% accuracy by the observed Higgs and $W^{\pm}$ masses.

5.3. Summary and Discussion; a Challenge

After the pioneering work of Feza Gürsey and the collaborators in the 1970s, Geoffrey Dixon devoted over 30 years to division algebras, which is followed by Cohl Furey since the 2010s. The Clifford algebra approach to unification, coupled with fermionic creation and annihilation operators, has also been pursued since the late 1970s by the Italian group around Roberto Caslbuoni. The notion of superconnection was anticipated and applied to the Weinberg–Salam model during the first decade of the creation of the SM as well. Thus, the basic ingredients of our endeavor have been with us for some 50 years. The pretended new features of the present survey concern certain details. Here belong:

- The interpretation of the Clifford pseudoscalar ω_6 as $i(\mathcal{P} - \mathcal{P}')$, where $\mathcal{P}$ and $\mathcal{P}'$ are the particle and antiparticle projection operators, respectively.

- The realization that the projected Clifford algebra

$$\mathcal{P}C\ell_{10}\mathcal{P} = C\ell_4 \otimes C\ell_6^0, \tag{100}$$

only involves the even part $C\ell_6^0$ of $C\ell_6$; coupled with the assignment of the Higgs field to the odd part, $C\ell_4^1$ of the first factor explains the symmetry breaking of the (electroweak) flavor symmetry while preserving the color gauge group.

- Exhibiting the role of the sterile neutrino (of the first generation of fundamental fermions) as the vacuum state of the theory. The gauge group of the SM is identified as the maximal subgroup of the Pati–Salam group G_{PS} (8) that leaves ν_R invariant.

- Singling out the *reduced* 15-dimensional *particle subspace* yields a relation between the Higgs and the W boson masses and the theoretical Weinberg angle satisfied within 1% accuracy.

What is missing for completing the "Algebraic Design of Physics"—to quote from the title of the 1994 book by Geoffrey Dixon—is a true understanding of the *three generations* of fundamental fermions. None of the attempts in this direction [18,25,30,47,74] has brought a clear success. The exceptional Jordan algebra $J_3^8 = \mathcal{H}_3(\mathbb{O})$ (7) with its built-in triality was first proposed to this end in [25] (continued in [33]); in its straightforward interpretaton, however, it corresponds to the triple coupling of left and right chiral spinors with a vector in internal space rather than to three generations of fermions. As recalled in (Section 5.2 of) [30], any finite dimensional unital module over $\mathcal{H}_3(\mathbb{O})$ has the (disappointingly unimaginative) form of a tensor product of $\mathcal{H}_3(\mathbb{O})$ with a finite dimensional real vector space E. It was further suggested there that the dimension of E should be divisible by three, but the idea was not pursued any further. Boyle [47] proposed considering the complexified exceptional Jordan algebra, whose group of determinant preserving linear automorpghisms is the compact form of E_6. This led to a promising left-right symmetric extension of the gauge group of the SM, but the discussion has not yet shed new light on the three generation problem. Yet another development based on the study of indecomposable representations of Lie superalgebras can be traced back from [75], but only the mathematical machinery has been discussed so far.

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Appendix A

Appendix A.1. Notation for Clifford Algebras

$C\ell(s,t)$ stands for the Clifford algebra generated by γ_α satisfying

$$[\gamma_\alpha, \gamma_\beta]_+ = 2\eta_{\alpha\beta},\ \eta_{\alpha\alpha} = 1 \text{ for } \alpha = 1, \ldots, s, \eta_{\alpha\alpha} = -1 \text{ for } \alpha = s+1, \ldots, s+t$$

($\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$). Its automorphism group is the (non-compact for $st \neq 0$) orthogonal group $O(s,t) = O(t,s)$. As internal symmetries correspond to compact gauge groups, we are mainly working with (positive or negative) definite forms and use the abbreviated notation $C\ell_s = C\ell(s,0)$ and $C\ell_{-t} = C\ell(0,t)$ for the associated Clifford algebras. The even subalgebra $C\ell^0(s,t)$ is defined as the (closed under multiplication) span of products of an even number of γ matrices; The odd subspace $C\ell^1(s,t)$ is defined as the (real) span of products of odd numbers of γs (which is not closed under multiplication).

Appendix A.2. Interrelations between the L, E, and R Bases of $so(8)$

The imaginary octonion units $e_1, \cdots, e_7$ obey the anticommutation relations of $C\ell_{-7}$

$$[e_\alpha, e_\beta]_+ := e_\alpha e_\beta + e_\beta e_\alpha = -2\,\delta_{\alpha\beta}\,,\ \alpha, \beta = 1, \cdots, 7 \tag{A1}$$

and give rise to the seven generators $L_\alpha = L_{e_\alpha}$ of the Lie algebra $so(8)$:

$$L_{\alpha 8} := \tfrac{1}{2}\, L_\alpha =: -L_{8\alpha}\,,\ L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}] \in so(7) \subset so(8)\,. \tag{A2}$$

For $\alpha \neq \beta$, there is a unique γ such that

$$L_\alpha\, e_\beta = f_{\alpha\beta\gamma}\, e_\gamma = \pm e_\gamma\,,\ f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} = f_{\gamma\alpha\beta}\,. \tag{A3}$$

The *structure constants* $f_{\alpha\beta\gamma}$ (22) (which only take the values $0, \pm 1$) obey for different triples (α, β, γ) the relations

$$f_{\alpha\beta\gamma} = f_{\alpha+1\,\beta+1\,\gamma+2} = f_{2\alpha, 2\beta, 2\gamma} \pmod 7\,. \tag{A4}$$

The list (22) follows from $f_{124} = 1$ and the first Equation (A4), taking into account relations such as $f_{679} \equiv f_{672} \pmod 7$, etc. Note that $f_{\alpha\beta\gamma} \neq 0$ $f_{\alpha\beta\gamma}$ are the structure constants of a (quaternionic) $su(2)$ Lie algebra; they are *not* the structure constants of $so(7) \subset so(8)$.

Define the involutive outer automorphism π of the Lie algebra $so(8)$ by its action (26) on the left and right multiplication L_α and R_α of octonions by imaginary octonions $\alpha = -\alpha^*$:

$$\pi(L_\alpha) = L_\alpha + R_\alpha =: T_\alpha\,,\ \pi(R_\alpha) = -R_\alpha \Rightarrow \pi(T_\alpha) = L_\alpha\,. \tag{A5}$$

In the basis (A1) and (A3) of imaginary octonion units e_α ($\alpha = 1, \cdots, 7$), by setting $e_8 = \mathbb{1}$ and $L_{\alpha 8} = \frac{1}{2}\, L_\alpha$ (A2), $R_{\alpha 8} = \frac{1}{2}\, R_\alpha = -R_{8\alpha}$, we define E_{ab} by the second relation (27)

$$E_{ab}\, e_c := \delta_{bc}\, e_a - \delta_{ac}\, e_b\,,\ a, b, c = 1, \cdots, 8 \quad (e_8 = 1)\,. \tag{A6}$$

Proposition A1. *Under the above assumptions/definitions, we have*

$$\pi(L_{ab}) = E_{ab} \quad (\mathit{for}\ L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}]\,,\ L_{\alpha 8} = \tfrac{1}{2}\, L_\alpha = -L_{8\alpha})\,. \tag{A7}$$

Proof. From the first equation (A5) and from (A1), (A2), and (A6), it follows that

$$E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8} = \pi(L_{\alpha 8})\,. \tag{A8}$$

The proposition then follows from the relations

$$L_{\alpha\beta} = [L_{\alpha 8}, L_{8\beta}]\,, \quad E_{\alpha\beta} = [E_{\alpha 8}, E_{8\beta}] \tag{A9}$$

and from the assumption that π is a Lie algebra homomorphism. □

Corollary A1. *From (A7) and the involutive character of* π*, it follows that, conversely,*

$$\pi(E_{ab}) = L_{ab}\,. \tag{A10}$$

To each $\alpha = 1, \cdots, 7$*, there are three pairs* $\beta\gamma$ *such that* $L_{\beta\gamma}$ *and* $E_{\beta\gamma}$ *commute with* L_α *and among themselves and allow for the expression* $L_\alpha = 2L_{\alpha 8}$ *in terms of* $E_{\alpha 8}$ *and the corresponding* $E_{\beta\gamma}$*:*

$$
\begin{aligned}
L_1 &= 2\,L_{18} = E_{18} - E_{24} - E_{37} - E_{56}\,, \\
L_2 &= 2\,L_{28} = E_{28} + E_{14} - E_{35} - E_{67}\,, \\
L_3 &= 2\,L_{38} = E_{38} + E_{17} + E_{25} - E_{46}\,, \\
L_4 &= 2\,L_{48} = E_{48} - E_{12} + E_{36} - E_{57}\,, \\
L_5 &= 2\,L_{58} = E_{58} + E_{16} - E_{23} - E_{47}\,, \\
L_6 &= 2\,L_{68} = E_{68} - E_{15} + E_{27} - E_{34}\,, \\
L_7 &= 2\,L_{78} = E_{78} - E_{13} - E_{26} - E_{45}\,, \textit{ or } L_\alpha = E_{\alpha 8} - \sum_{\beta<\gamma} f_{\alpha\beta\gamma}\, E_{\beta\gamma}\,.
\end{aligned} \tag{A11}
$$

Recalling that $E_{ab} = \pi(L_{ab})$ (A8) and the fact that π is involutive, so that $\pi(E_{ab}) = L_{ab}$ (A10), we deduce, in particular,

$$
\begin{aligned}
2\,E_{78} &= L_{78} - L_{13} - L_{26} - L_{45}\,, \\
R_7 = 2\,E_{78} - 2\,L_{78} &= -L_{78} - L_{13} - L_{26} - L_{45}\,,
\end{aligned} \tag{A12}
$$

thus reproducing (29).

We now proceed to displaying the commutant of $i\omega_6$ and $i\omega_6^R$ in $so(7+j)$, $j = 1,2,3$.

Proposition A2. *While the Lie algebra $spin(6) = su(4)$ commutes with L_7, the commutant of R_7 (A12) in $su(4) \subset s\ell(4,\mathbb{C})$ is $u(3)(\subset s\ell(4,\mathbb{C}))$, given by*

$$
u(3) = \left\{ \sum_{j,k=1}^{3} C_{jk}[b_j^*, b_k]\,;\; C_{jk} \in \mathbb{C}\,,\; C_{kj} = -\overline{C_{jk}} \right\} \tag{A13}
$$

in the fermionic oscillator relalization of $C\ell_6(\mathbb{C})$ (the bar over C_{jk} standing for complex conjugation).

Proof. The fact that $L_7 = 2\,L_{78}$ commutes with the generators $L_{\alpha\beta}$ ($\alpha, \beta = 1, \cdots, 6$) of $so(6)$ and follows from (21). To find the commutant of R_7 (A12), it is convenient to use the fermionic realization of the complexification $s\ell(4,\mathbb{C})$ of $su(4)$, which is spanned by the nine commutators $[b_j^*, b_k]$ in (A13) and the six products

$$
b_j\, b_k = -b_k\, b_j\,,\; b_j^*\, b_k^* = -b_k^*\, b_j^*\,,\; j,k = 1,2,3,\; j \neq k\,. \tag{A14}
$$

The sum $L_{13} + L_{26} + L_{45}$ in (A12) is a multiple of $B - L$ (58), the Hermitian generator of the center of $g\ell(3,\mathbb{C})$,

$$
B - L\left(= \frac{i}{3}(\gamma_{13} + \gamma_{26} + \gamma_{45}) \right) = \frac{1}{3}\sum_{j=1}^{3}[b_j^*, b_j]\,. \tag{A15}
$$

The relations

$$
\begin{aligned}
&[B - L, b_j^*\, b_k^*] = \tfrac{2}{3}\, b_j^*\, b_k^*\,,\; [B - L, b_j\, b_k] = -\tfrac{2}{3}\, b_j\, b_k\,, \\
&\Big[\Big[B - L, [b_j^*, b_k]\Big]\Big] = 0\,,\; j,k = 1,2,3,\; j \neq k\,,
\end{aligned} \tag{A16}
$$

show that the commutant of $B - L$ (and hence of R_7) in $su(4)$ is $u(3)$. □

Corollary A2. *The commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$; the commutant of ω_6^R in $spin(9)$ is the gauge Lie algebra of the SM:*

$$
\mathcal{G}_{\mathrm{SM}} = \{a \in spin(9)\,;\; [a, \omega_6^R] = 0\} = u(3) \oplus su(2)\,. \tag{A17}
$$

Notes

1 For a pleasant-to-read review of octonions, their history, and applications, see [13].

2 I had the good fortune to know him personally. See Witten's eloquent characterization of his personality and work in the Wikipedia entry on Feza Gürsey (1921–1991).

3 These algebras are defined and classified in [3,4]; for a concise review, see [29], Section 3.2 of [25], and Section 2 of [30]; concerning Pascual Jordan (1902–1980), see [31].

4 For an enlightening review of the algebra of GUTs and some 40 references, see [38].

5 Aptly called *geometric algebra* by its inventor—see [39].

6 For any associative ring $\mathbb{K}$, particularly for the division rings $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we denote the algebra of $m \times m$ matrices with entries in $\mathbb{K}$ by $\mathbb{K}[m]$.

7 See [50] for a reader-friendly review of Moufang loops and for a glimpse of the personality of Ruth Moufang (1905–1971).

8 The 10-fold classification of $\mathbb{Z}_2$ graded Clifford algebras also involves signs coming from squaring two antiunitary charge conjugation operators—see [51] Chapter 13, pp. 87–125.

9 The group G_{SM} was earlier obtained in [35], starting with the Albert algebra J_3^8 (7).

10 The Dublin Professor of Astronomy William Rowan Hamilton (1805–1865) and the Stettin Gymnasium teacher Hermann Günter Grassmann (1809–1877) published their papers on quaternions and on "extensive algebras", respectively, in the same year of 1844. William Kingdom Clifford (1845–1879) combined the two in a "geometric algebra" in 1878, a year before his death, aged 33, referring to both of them.

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