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Mission Target: Exotic Multiquark Hadrons—Sharpened Blades

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Abstract: Motivated by recent experimental progress in establishing the likely existence of (variants of) exotic hadrons, predicted to be formed by the strong interactions, various proposed concepts and ideas are compiled in an attempt to draft a coherent picture of the achievable improvement in the theoretical interpretation of exotic hadrons in terms of the underlying quantum field theory of strong interactions.

Keywords: exotic hadron states; multiquark adequacy; QCD sum rule; large-N_c limit; $1/N_c$ expansion

1. Significance of Fundamental Diverseness of Ordinary Hadrons and Multiquark States

Within the framework of (relativistic) quantum field theories, all strong interactions are described—at a fundamental level—by quantum chromodynamics (QCD), a renormalizable gauge theory, invariant under local transformations forming a representation of the compact non-Abelian Lie group SU(3). Two sorts of particles constitute the (basic) dynamical degrees of freedom of QCD: massless vector gauge bosons labeled gluons, transforming (inevitably) according to the eight-dimensional adjoint representation **8** of SU(3), and spin-$\frac{1}{2}$ fermions q_a, labeled quarks, each distinguished from all others by some quark flavor degree of freedom

$$a \in \{u, d, s, c, b, t(, \ldots ?)\} \tag{1}$$

and transforming according to the three-dimensional fundamental representation **3** of SU(3). The (few) fundamental parameters characterizing QCD are the masses m_a of the quarks q_a as well as the strong coupling g_s, frequently adopted in the form of a strong fine-structure coupling

$$\alpha_s \equiv \frac{g_s^2}{4\pi} \,. \tag{2}$$

This designation as quantum *chromo*dynamics derives from the fact that the quark and gluon degree of freedom affected by their gauge-group transformation is referred to as their color.

Among others, QCD features the phenomenon of color *confinement*: not the (colored) quarks and gluons but *exclusively* their color-singlet *hadron* bound states [1] invariant under the action of the QCD gauge group are, in the form of isolated states, experimentally observable. Closer inspection reveals that the hadron states *have to be* divided into two *disjoint* categories:

- **Conventional** (ordinary) hadrons include all mesons that consist of only a pair of quark and antiquark, as well as all baryons that consist of three quarks or of three antiquarks.
- **Exotic** hadrons are characterized by *non-conventional* quark and/or gluon compositions comprising *multiquark* states (tetraquarks, pentaquarks, hexaquarks, heptaquarks, etc.), "hybrid" quark–gluon bound states, or pure-gluon bound states (nick) named glueballs.

There is a (crucial) *fundamental* difference between conventional hadrons and exotic hadrons, based on a (more or less) trivial observation: any color-singlet multiquark arrangement of a number of quarks and/or antiquarks may be decomposed (in one or more ways) into a set of states that are also color singlets but consist of lesser numbers of quarks and/or antiquarks.

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Therefore, an (initially) tightly bound, "compact" *multiquark* hadron may reconfigure to molecular-type clusters of (ultimately) *conventional* hadrons, loosely bound by some residual forces [2]. In view of this, trustworthy attempts to describe exotic hadrons should (struggle to) also take into account the potential mixing of these two "phases" of multiquark hadrons.

The present note *recalls* a collection of recently proposed procedures and considerations, the application of which *might facilitate* gaining a theoretical understanding of (experimentally established) multiquark states. Both the origin and prospects of these tools are illustrated for the, hopefully, easiest case: the kind of tetraquarks presumably least plagued by complications of a technical nature given by (compact) bound states of two quarks and two antiquarks carrying four unequal flavor quantum numbers. (These tools' transfer to other cases seems evident.) In particular, a brief glance at the related present *experimental* situation [3–10] (Section 2) will be followed by a recapitulation of insights gained upon basing the strong interactions' gauge symmetry tentatively on special unitary groups of *higher* dimension [11–24] (Section 4) and a sketch of the advantages of trimming a popular technique for the nonperturbative analytical discussion of QCD bound states to fit the needs of multiquark hadrons [25–32] (Section 5).

2. Tetraquark Mesons—The Example of Multiquark Exotic Hadron States Par Excellence

All tetraquark mesons T are bound states of two antiquarks $\bar{q}_a, \bar{q}_c$ and two quarks q_b, q_d,

$$T = [\bar{q}_a\, q_b\, \bar{q}_c\, q_d] \,, \qquad a,b,c,d \in \{u,d,s,c,b\} \,, \tag{3}$$

henceforth calling the masses of the four (anti-) quarks constituting such state m_a, m_b, m_c, m_d. On group-theoretical grounds, the presence of these mesons in the hadron spectrum without coming into conflict with *confinement* of color is rendered possible by the appearance of two SU(3) singlet representations **1** in the (appropriate) tensor product of two fundamental SU(3) representations **3** as well as two complex-conjugate fundamental SU(3) representations $\overline{\mathbf{3}}$, as this product's decomposition into the *irreducible* SU(3) representations $\mathbf{1}, \mathbf{8}, \mathbf{10}, \overline{\mathbf{10}}, \mathbf{27}$ reveals:

$$q_b\, q_d\, \bar{q}_a\, \bar{q}_c \sim \mathbf{3} \otimes \mathbf{3} \otimes \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \mathbf{81} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \,. \tag{4}$$

As far as its *flavor* degrees of freedom are concerned, the four quark constituents of any tetraquark state (3) may contribute, at most, four different quark flavors and, trivially, carry at least one, the *same* for all the four (anti-) quarks. Owing to such simultaneous involvement of both quarks and antiquarks, however, the latters' hadron bound states need not feature all of the available quark flavors. Table 1 presents the listing [20] of conceivable quark-flavor arrangements in the tetraquark state (3), with respect to both the number of *different* flavors $a \neq b \neq c \neq d$ provided by two quarks and two antiquarks as well as the number of flavors exhibited by the related *hadron*, which might differ from the former number either because of mutual flavor–antiflavor compensations or because of quark-flavor double occurrences.

Needless to say, at least from the experimental point of view it may be more satisfactory if the exotic nature of a (suspected) multiquark is established already by its observed content of quark flavors. The corresponding species of multiquarks may be told apart by relying on

Definition 1. *A multiquark hadron is termed* ***flavor-exotic*** *if it exhibits more open quark flavors than the corresponding category of conventional hadrons does, which means at least three open quark flavors in the case of mesonic states or at least four open quark flavors in the case of baryonic states. By contrast, a multiquark hadron is called* ***flavor-cryptoexotic*** *if it does not meet this requirement.*

For the quark-flavor arrangements of tetraquarks, Table 1 offers several options to meet the requirement of being considered flavor-exotic: of course, there can exist merely one flavor arrangement that incorporates four mutually different quark flavors; however, there exist a few self-evident options for *flavor-exotic* tetraquarks to comprise not more than two or three different quark flavors by involving one or even two *double* appearances of a given flavor.

Table 1. Tetraquark states (3): Classification by different vs. open quark-flavor content $a \neq b \neq c \neq d$, *open-flavor number* referring to all flavors not counterbalanced by their antiflavors. (From Ref. [20]).

Number of Different Quark Flavors Involved	Quark Composition $\bar{q}_\square\, q_\square\ \bar{q}_\square\, q_\square$	Number of Open Quark Flavors Involved
4	$\bar{q}_a\, q_b\ \bar{q}_c\, q_d$	4
3	$\bar{q}_a\, q_b\ \bar{q}_c\, q_b$	4
	$\bar{q}_a\, q_b\ \bar{q}_a\, q_c$	4
	$\bar{q}_a\, q_b\ \bar{q}_b\, q_c$	2
	$\bar{q}_a\, q_b\ \bar{q}_c\, q_c$	2
2	$\bar{q}_a\, q_b\ \bar{q}_a\, q_b$	4
	$\bar{q}_a\, q_a\ \bar{q}_a\, q_b$	2
	$\bar{q}_a\, q_a\ \bar{q}_b\, q_a$	2
	$\bar{q}_a\, q_b\ \bar{q}_b\, q_a$	0
	$\bar{q}_a\, q_a\ \bar{q}_b\, q_b$	0
1	$\bar{q}_a\, q_a\ \bar{q}_a\, q_a$	0

Quite recently, various candidates for tetraquark states that are manifestly *flavor-exotic* by exhibiting (in accordance with Definition 1) four open quark flavors have been observed by experiment. Regarding the flavor compositions of these candidates, there are states each encompassing exactly one of all four lightest quarks [5,6,9,10] and "doubly flavored" ones containing only three different flavors but one of these twice [7,8] (see the summary in Table 2).

Table 2. Flavor-exotic tetraquark states: Experimental candidates, in the naming convention of LHCb [4].

Candidate Tetraquark Meson	(Minimal) Quark-Flavor Content	References
$T_{cs0}(2900)^0$	$c\bar{d}s\bar{u}$	[5,6]
$T_{cs1}(2900)^0$	$c\bar{d}s\bar{u}$	[5,6]
$T_{cc}(3875)^+$	$cc\bar{u}\bar{d}$	[7,8]
$T^a_{c\bar{s}0}(2900)^0$	$c\bar{s}d\bar{u}$	[9,10]
$T^a_{c\bar{s}0}(2900)^{++}$	$c\bar{s}u\bar{d}$	[9,10]

3. Correlation Functions of Hadron Interpolating Operators: Application to Multiquarks

For descriptions of hadronic states in terms of QCD, a pivotal contact point between the realm of QCD and the realm of hadrons is established by the concept of hadron interpolating operators. For a fixed hadron H under consideration, its—not necessarily unique—hadron interpolating operator, generically called $\mathcal{O}$, is a gauge-invariant local operator composed of the QCD dynamical degrees of freedom, the quark and gluon field operators, that betrays its nonzero overlap with the hadron $|H\rangle$ by the nonvanishing matrix element emerging from its getting sandwiched between the hadronic state $|H\rangle$ and the QCD vacuum $|0\rangle$: $\langle 0|\mathcal{O}|H\rangle \neq 0$. In all subsequent implementations of hadron interpolating operators, features such as parity or spin degrees of freedom can be safely ignored; they, therefore, get notationally suppressed.

For a *conventional* meson consisting of a quark of flavor b and an antiquark of flavor a, the most evident option for its interpolating operator is the quark–antiquark bilinear current

$$j_{\bar{a}b}(x) \equiv \bar{q}_a(x)\, q_b(x)\ . \tag{5}$$

For *exotic* hadrons belonging to the subset of *tetraquark* mesons characterized in Equation (3), the search for appropriate tetraquark interpolating operators, specifically named θ, is greatly facilitated by the observation [33] that (by means of suitable Fierz transformations [34]) *every* color-singlet operator that is composed of two quarks and two antiquarks can be expressed by a linear combination of only two different products of color-singlet conventional-meson interpolating operators of quark-bilinear-current shape (5). Thus, this “operator basis” reads

$$\theta_{\bar{a}b\bar{c}d}(x) \equiv j_{\bar{a}b}(x)\, j_{\bar{c}d}(x)\,, \qquad \theta_{\bar{a}d\bar{c}b}(x) \equiv j_{\bar{a}d}(x)\, j_{\bar{c}b}(x)\,. \tag{6}$$

Moreover, taking into account some useful identities recalled, for instance, by Equations (32) and (36) of Reference [24] or Equations (1) and (2) of Reference [31] *may* be regarded either as a kind of shortcut to or as explicit verification of these findings. The tetraquark interpolating operators (6) will provide some kind of playground for (most of) the ensuing considerations.

This pleasing observation [33] points out a promising route on how to reasonably proceed. Namely, the enabled basic two-current structure (6) of the tetraquark interpolating operators θ suggests starting (envisaged) analyses of tetraquarks from *correlation functions*—in general, defined by vacuum expectation values of time-ordered products, symbolized by T, of chosen field operators—of four quark-bilinear operators (5). If tolerated by the involved dynamics, tetraquark states in appropriate four-point correlation functions of such kind should become manifest by their contributions in the form of intermediate-state poles. Momentarily focusing on only essential aspects, all these four-current correlation functions are of the general structure

$$\left\langle \mathrm{T}\Big(j(y)\, j(y')\, j^\dagger(x)\, j^\dagger(x')\Big)\right\rangle . \tag{7}$$

Upon the application of well-understood procedures, the correlation functions (7) also entail the amplitudes encoding scatterings of two conventional mesons into two conventional mesons. Because of the two-current structure (6), contact with tetraquark states, in the form of correlation functions involving tetraquark interpolating operators θ, can be established by identification or contraction of configuration-space coordinates of *proper* quark-bilinear currents j, forming

- *twice* configuration-space contracted *two-point* correlation functions of *two* operators (6)

$$\left\langle \mathrm{T}\Big(\theta(y)\, \theta^\dagger(x)\Big)\right\rangle = \lim_{\substack{x'\to x\\ y'\to y}} \left\langle \mathrm{T}\Big(j(y)\, j(y')\, j^\dagger(x)\, j^\dagger(x')\Big)\right\rangle ; \tag{8}$$

- *once* contracted *three-point* correlation functions of *one* operator (6) and *two* operators (5)

$$\left\langle \mathrm{T}\Big(j(y)\, j(y')\, \theta^\dagger(x)\Big)\right\rangle = \lim_{x'\to x} \left\langle \mathrm{T}\Big(j(y)\, j(y')\, j^\dagger(x)\, j^\dagger(x')\Big)\right\rangle . \tag{9}$$

An immediate implication of the mere conceptual nature of unconventional multiquark states is, as already stressed in Section 1, their potential to undergo *clustering* without getting into conflict with color confinement [2]. For the correlation-function underpinned analyses of tetraquark properties, this finding should be regarded as a strong hint that, presumably or even very likely, not all QCD-level contributions to some correlation function are, in general, of relevance for such formation of a tetraquark pole. It appears opportune to distinguish any contribution that may play a rôle in tetraquark studies even by nomenclature, this is achieved in

Definition 2. *A QCD contribution to a correlation function (7) is termed* ***tetraquark-phile*** *[17,22] if it is (potentially) capable of supporting the formation of a tetraquark-related intermediate-state pole.*

As guidance through the process of filtering all of the QCD-level contributions as implicitly requested by Definition 2, a self-evident, easy-to-implement criterion may be devised [16,18]:

Proposition 1. *For a given four-point correlation function (7) with external momenta in initial state p_1, p_2 and external momenta in final state q_1, q_2, considered as a function of the Mandelstam variable*

$$s \equiv (p_1 + p_2)^2 = (q_1 + q_2)^2 , \tag{10}$$

a QCD-level contribution is supposed to be tetraquark-phile if it exhibits a nonpolynomial dependence on s and if it develops an intermediate-state four-quark-related branch cut starting at the branch point

$$\hat{s} \equiv (m_a + m_b + m_c + m_d)^2 . \tag{11}$$

For any contribution to a correlation function, the capability of supporting the formation of a tetraquark pole by satisfying all requirements in Proposition 1 may be straightforwardly and unambiguously decided by consulting the related Landau equations [35]: the existence of an appropriate solution to (the relevant set of) those Landau equations indicates the presence of an expected branch cut. References [18,24,31] show some examples worked out in all details.

As announced in Section 1, the benefit of implementing such a program is exemplified for the meanwhile even experimentally observed [5,6,9,10] *subset* of all those flavor-exotic tetraquarks that exhibit not less than (the feasible maximum of) four unequal quark flavors:

Definition 3. *The quark-flavor composition of a tetraquark (3) is called* ***definitely flavor-exotic*** *if it comprises four mutually different quark flavors $a \neq b \neq c \neq d$, that is, if this state is of the kind*

$$T = [\bar{q}_a \, q_b \, \bar{q}_c \, q_d] , \qquad a, b, c, d \in \{u, d, s, c, b\} , \qquad a \neq b \neq c \neq d . \tag{12}$$

At least for the case of the definitely flavor-exotic tetraquarks (12), there exist two definitely distinguishable quark-flavor distributions in (from the point of view of intermediate states) incoming and outgoing states of a correlation function (7): the quark-flavor arrangements in initial and final state might be either identical or different. These two possibilities were given the names:

Definition 4. *A definitely flavor-exotic correlation function (7) of four interpolating currents (5) is*

- ***flavor-preserving*** *[19] for equal quark-flavor distributions of incoming and outgoing states,*
- ***flavor-rearranging*** *[19] for unlike incoming- and outgoing-state quark-flavor distributions.*

For the two categories of correlation functions (7), it is straightforward yet worthwhile (since instructive) to investigate their contributions of lowest orders to the perturbative expansions in the power of the strong fine-structure constant (2). Representative examples of contributions are given, for flavor-preserving cases, in Figures 1 and 2 and, for flavor-rearranging cases, in Figures 3 and 4. (In the plots, internal gluon exchanges are depicted in the form of curly lines.) As expected, such considerations disclose differences in analyses but similarities in outcomes:

- For flavor-preserving correlation functions, the line of argument proves to be, more or less, evident. *All* the contributions of the type of Figure 1a or of the type of Figure 1b, involving at most one gluon exchange, are doubtlessly disconnected. The contributions that involve a single gluon exchange between their two (otherwise disconnected) quark loops *vanish* identically, due to the vanishing of the sum over color degrees of freedom of each of the two quark loops. Phrased *slightly* more technically, this can be traced back to the tracelessness of all generators of a special unitary group, governing the couplings of quarks and gluons. Consequently, *exclusively* contributions that involve, at least, two gluon exchanges of an appropriate topology may be viewed as tetraquark-phile. These insights are, of course, corroborated by identifying these tetraquark-phile contributions according to Proposition 1 by explicit inspection [16] by way of their Landau equations. Replacing any double contraction (8) in Figure 1 by a single contraction (9) confirms the tetraquark-phile nature of contributions of the type of Figure 2 or related higher orders.

- For flavor-rearranging correlation functions, simple optical guidance in this analysis is, beyond doubt, hardly imaginable: already the lowest-order contributions turn out to be connected. Rather, one has to gladly accept any assistance offered by the tool called Landau equations. For the three lowest-order contributions exemplified in Figure 3, the usage of this formalism is demonstrated, in full detail, in Appendix A of Reference [18], in the Appendix of Reference [31], as well as in Section 4 of Reference [24]. For this kind of analysis, it might prove advantageous to recast the encountered plots into a box shape, by "unfolding" all these plots [14,18,24,31]. These efforts' outcome is that contributions of the type of Figure 3a or of the type of Figure 3b, being characterized by no or only one internal gluon exchange, do not incorporate the requested four-quark singularities. The involvement of this feature starts not before the level of two gluon exchanges of *suitable* positioning, which then holds, of course, likewise for the single contractions (9) in Figure 4.

As an overall summary of the two classes of definitely flavor-exotic correlation functions (7) identified by Definition 4, the systematic scrutiny of their lowest-order contributions shows that tetraquark-phile contributions (an essential ingredient, since providing the singularities that, upon summation, *may* support the development of intermediate-state tetraquark poles) will not emerge before the next-to-next-to-lowest order in a series expansion in powers of the strong fine-structure constant (2), that is, in terms of α_s, have to be at least of the order $O(\alpha_s^2)$.

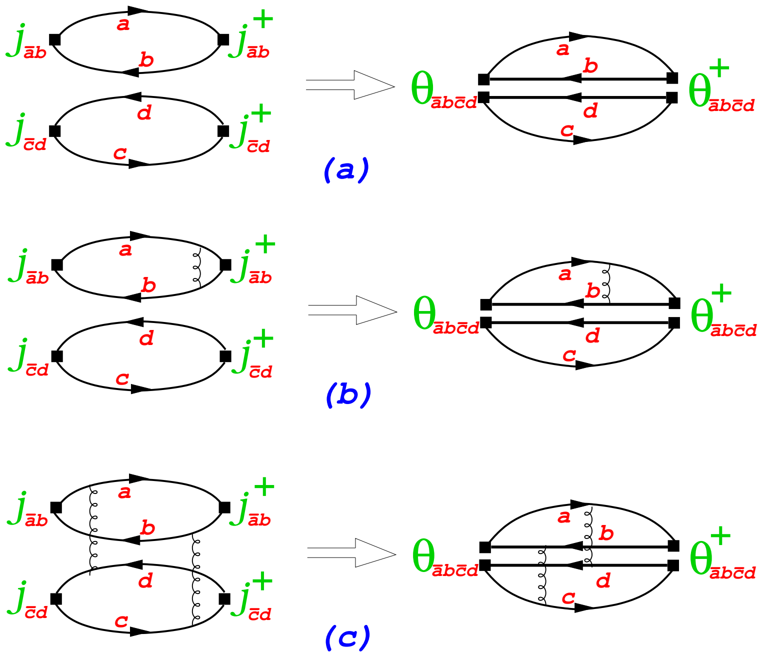

Figure 1. Definitely flavor-exotic four-current correlation function (7) of flavor-preserving type (**left**) and (**right**) its contraction (8) to two-point correlation function of tetraquark interpolating operators (6) [30,32]. Representative contributions of lowest perturbative orders: (**a**) $O(\alpha_s^0)$, (**b**) $O(\alpha_s)$ and (**c**) $O(\alpha_s^2)$.

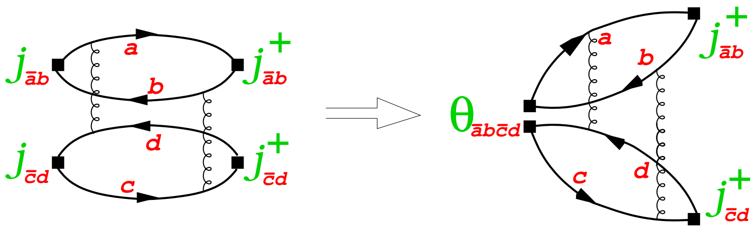

Figure 2. Definitely flavor-exotic four-current correlation function (7) of flavor-preserving type (**left**) and (**right**) contraction (9) to a correlation function of one tetraquark interpolating operator (6) and two quark-bilinear currents (5) [30]: typical contribution of *lowest tetraquark-phile* perturbative order $O(\alpha_s^2)$.

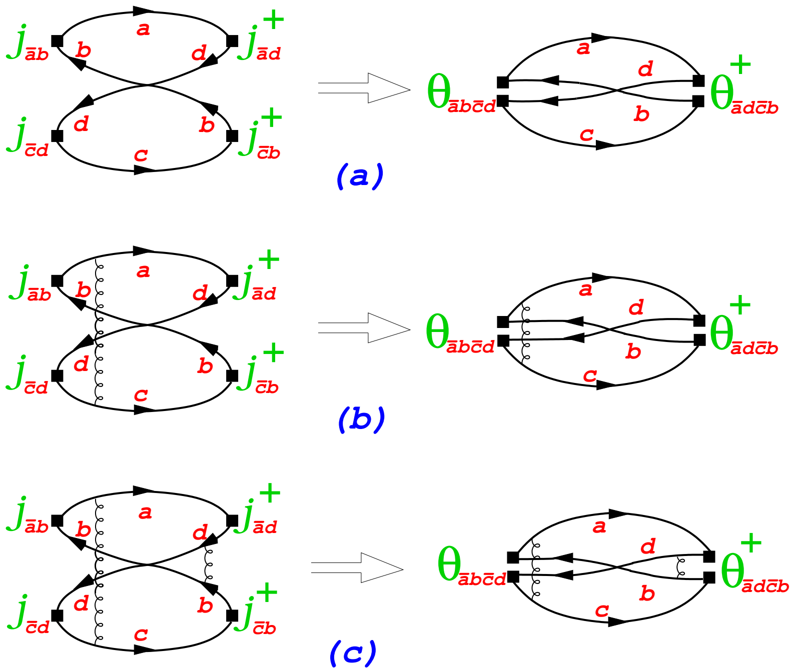

Figure 3. Definitely flavor-exotic four-current correlation function (7) of flavor-*rearranging* type (**left**) and (**right**) its contraction (8) to two-point correlation function of tetraquark interpolating operators (6) [30,31]. Representative contributions of lowest perturbative orders: (**a**) $O(\alpha_s^0)$; (**b**) $O(\alpha_s)$ and (**c**) $O(\alpha_s^2)$.

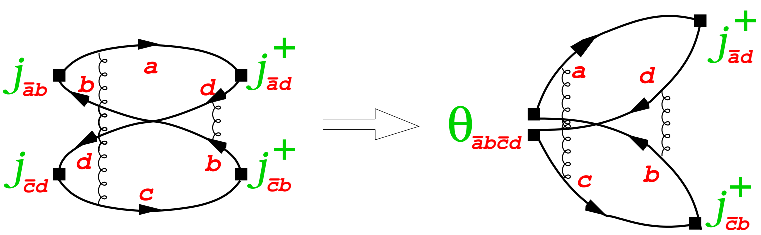

Figure 4. Definitely flavor-exotic four-current correlation function (7) of flavor-*rearranging* type (**left**) and (**right**) contraction (9) to a correlation function of one tetraquark interpolating operator (6) and two quark-bilinear currents (5) [31]: typical contribution of *lowest tetraquark-phile* perturbative order $O(\alpha_s^2)$.

4. Number of Color Degrees of Freedom, Unfixed: Large-N_c Limit and $1/N_c$ Expansion

Quite generally, first insights, even if only of qualitative nature, may be gained from the reduction in the complexity of QCD, enacted by the increase in the number of color degrees of freedom and, in parallel, the decrease in the strength of the strong-interaction coupling g_s. In some more detail, that simplification of QCD [11,12] proceeds along the following moves:

- Generalize QCD to the gauge theories invariant under a non-Abelian Lie group SU(N_c). The dynamical degrees of freedom of each of the latter quantum field theories hence are its gauge bosons, still retaining their designation as gluons and transforming according to the (N_c^2-1)-dimensional, adjoint representation of SU(N_c), and its fermionic quarks that transform according to the N_c-dimensional, fundamental representation of SU(N_c).
- Allow the number of color degrees of freedom, N_c, to increase from $N_c = 3$ to infinity:

$$N_c \to \infty \,. \tag{13}$$

- For the strong coupling strength g_s, demand the *related* decrease, with rising N_c, to zero:

$$g_s \propto \frac{1}{\sqrt{N_c}} = O(N_c^{-1/2}) \xrightarrow[N_c\to\infty]{} 0 \,. \tag{14}$$

Clearly, for the strong fine-structure coupling α_s this requirement implies the behavior

$$\alpha_s \propto \frac{1}{N_c} = O(N_c^{-1}) \xrightarrow[N_c\to\infty]{} 0 \,. \tag{15}$$

Therefore, in the large-N_c limit, the product $N_c\,\alpha_s$ approaches a meaningful finite value. Only by establishing a careful balance between the growth of N_c and the vanishing of α_s, the latter requirement allows for both reasonable generalization of QCD to its large-N_c limit and exploitation of any corresponding $1/N_c$ expansion, that is, the expansion in powers of $1/N_c$.

According to the above characterization of large-N_c QCD, for each QCD contribution to a correlation function its behavior in the large-N_c limit is determined by two ingredients:

- the number of *closed* loops of the color degrees of freedom carried by quarks or gluons,
- the number of either the strong couplings (14) or the strong fine-structure constants (15).

Keeping this in mind, the large-N_c behavior of arbitrary correlation functions will be found. In particular, for the tetraquark-phile (and therefore tetraquark-pole relevant) contributions, indicated by the subscript "tp", to definitely flavor-exotic correlation functions (7), one obtains

- for any flavor-preserving contribution of the type employed by Figure 1c or Figure 2,

$$\left\langle \mathrm{T}\Big(j_{\bar{a}b}(y)\,j_{\bar{c}d}(y')\,j^\dagger_{\bar{a}b}(x)\,j^\dagger_{\bar{c}d}(x')\Big)\right\rangle_{\rm tp} = O(N_c^2\,\alpha_s^2) = O(N_c^0)\,, \tag{16}$$

$$\left\langle \mathrm{T}\Big(j_{\bar{a}d}(y)\,j_{\bar{c}b}(y')\,j^\dagger_{\bar{a}d}(x)\,j^\dagger_{\bar{c}b}(x')\Big)\right\rangle_{\rm tp} = O(N_c^2\,\alpha_s^2) = O(N_c^0)\,, \tag{17}$$

- for each flavor-rearranging contribution of the kind adopted by Figure 3c or Figure 4,

$$\left\langle \mathrm{T}\Big(j_{\bar{a}b}(y)\,j_{\bar{c}d}(y')\,j^\dagger_{\bar{a}d}(x)\,j^\dagger_{\bar{c}b}(x')\Big)\right\rangle_{\rm tp} = O(N_c\,\alpha_s^2) = O(N_c^{-1})\,. \tag{18}$$

This general discrepancy between the large-N_c behavior of the flavor-preserving and of the flavor-rearranging four-point correlation functions expressed, for all contributions of any tetraquark-phile type, by Equations (16) and (17), on the one hand, and by Equation (18), on the other hand, has a startling or even disturbing implication for the spectra of tetraquark mesons to be expected in the large-N_c limit. In the scattering of a pair of *conventional* mesons,

$$M_{\bar{a}b} = [\bar{q}_a\,q_b]\,, \qquad a,b \in \{u,d,s,c,b,t(,\dots?)\}\,, \tag{19}$$

a tetraquark T betrays its existence by contributing in form of an intermediate-state pole. Its couplings to conventional mesons are governed by *transition amplitudes* $A(T \longleftrightarrow M_{\bar{a}b}\,M_{\bar{c}d})$.

Given the discrepancy between those classes of contributions for large N_c, consistency in the large-N_c limit turns out [16,18] to impose constraints on any involved transition amplitudes.

The QCD predictions for the large-N_c behavior of the correlation functions introduced in Section 3 cannot be matched, at hadron level, by the presence of merely a single tetraquark state [21]. Rather, fulfillment of the large-N_c behavior requested by Equations (16)–(18) by the tetraquark-pole contributions necessitates the pairwise occurrence of tetraquarks, that is to say, of a minimum of two (corresponding) tetraquarks [16,18]. The two tetraquarks, generically denoted by T_A and T_B, have to exhibit *unequal* N_c dependences of their transition amplitudes to the two possible quark-flavor divisions among the two conventional mesons in initial and final states; their dominant decay channels, however, exhibit the same large-N_c behavior. Thus, in the large-N_c limit their total decay widths, Γ, behave in a similar fashion,

$$\Gamma(T_A) = O(N_c^{-2}) = \Gamma(T_B)\,, \tag{20}$$

and the large-N_c interrelationships of the four involved transition amplitudes are of the kind

$$\underbrace{A(T_A \longleftrightarrow M_{\bar{a}b}\,M_{\bar{c}d}) = O(N_c^{-1})}_{\Longrightarrow \qquad \Gamma(T_A) = O(N_c^{-2})} \quad \overset{\boxed{N_c \text{ order}}}{>} \quad A(T_A \longleftrightarrow M_{\bar{a}d}\,M_{\bar{c}b}) = O(N_c^{-2})\,, \tag{21}$$

$$A(T_B \longleftrightarrow M_{\bar{a}b}\,M_{\bar{c}d}) = O(N_c^{-2}) \quad \overset{\boxed{N_c \text{ order}}}{<} \quad \underbrace{A(T_B \longleftrightarrow M_{\bar{a}d}\,M_{\bar{c}b}) = O(N_c^{-1})}_{\Longrightarrow \qquad \Gamma(T_B) = O(N_c^{-2})}\,. \tag{22}$$

Table 3 compares several available *expectations* for the large-N_c dependence of the total decay rates Γ of definitely exotic and cryptoexotic tetraquarks, indicating a few discrepancies likely resulting from differences in underlying assumptions or contributions considered as crucial.

Table 3. Tetraquark total decay widths: expected upper bounds on large-N_c behavior (from Ref. [20]).

Author Collective	**Decay Width Γ** Definitely Exotic Tetraquarks	**Decay Width Γ** Cryptoexotic Tetraquarks	**References**
Knecht, Peris	$O(1/N_c^2)$	$O(1/N_c)$	[13]
Cohen, Lebed	$O(1/N_c^2)$	—	[14]
Maiani, Polosa, Riquer	$O(1/N_c^3)$	$O(1/N_c^3)$	[15]
Lucha, Melikhov, Sazdjian	$O(1/N_c^2)$	$O(1/N_c^2)$	[16,18]

5. Multiquark-Adequate QCD Sum Rules Recognizing "Peculiarities" of Exotic Hadrons

From a mainly theoretical point of view, the description of any hadronic bound states of the fundamental degrees of freedom of QCD in a thoroughly analytical fashion appears to be most favorable; a promising approach complying with this intention, well-grounded in the framework of relativistic quantum field theories, is realized by the QCD sum rule formalism.

In the version *originally* devised by Shifman, Vainshtein, Zakharov [25], and others [26], a QCD sum rule embodies an analytical relationship between, on the one hand, properties of the hadron state (formed by the strong interactions) in the focus of one's current interest and, on the other hand, the (few) *basic* parameters of their underlying quantum field theory, QCD. In principle, the routine derivation of a QCD sum rule follows well-established procedures [27]. The starting point of the construction of a QCD sum rule is the evaluation of an appropriate correlation function—which clearly has to involve an operator interpolating the hadron under investigation—in parallel both at the phenomenological hadron level and at the fundamental QCD level, followed (of course) by equating both evaluations' outcomes:

- In the course of QCD-level evaluation, Wilson's *operator product expansion* [28] (enabling conversion of a nonlocal product of operators into a series of local operators) is invoked to separate nonperturbative and (to some extent calculable) perturbative contributions.
  - The *perturbative* contributions, identical to the lowest term of this operator product expansion, can be inferred in the form of a series in powers of the strong coupling (2).
  - The *nonperturbative* contributions involve, apart from derivable prefactors, *vacuum condensates*, i.e., the vacuum expectation values of products of quark and/or gluon field operators, which may be interpreted as a kind of effective parameters of QCD.
- In the course of hadron-level evaluation, the insertion of a complete set of hadron states guarantees that the hadron under study shows up by way of its intermediate-state pole.

By application of dispersion relations (and, if necessary, a sufficient number of subtractions), both perturbative QCD-level evaluation and hadron-level evaluation can be re-expressed (for the sake of convenience) in the form of dispersion integrals of appropriate spectral densities.

The predictive value and therefore usefulness of the QCD–hadron relations constructed in this manner is perceptibly increased by taking consecutively both the following measures:

1. Subject both sides of such a relation to a Borel transformation to another variable called Borel parameter τ. This results in the *entire* removal of any subtraction term introduced and the suppression of the hadron-level contributions above the hadronic ground state. Under a Borel transformation, all *vacuum condensates* in the nonperturbative QCD-level contributions are multiplied by powers of τ. So, these terms are called *power corrections*.
2. Rely on the assumption of quark–hadron duality, which postulates a (needless to stress, approximately realized) cancellation of all perturbative QCD-level contributions above suitably defined effective thresholds, $s_{\rm eff}$, against all higher hadron-level contributions, consisting of hadron excitations and hadron continuum. In implementing this concept, the problem of pinning down the nature of $s_{\rm eff}$ may be dealt with in two different ways:
   - Without knowing better, just a guessed *fixed* value of the parameter $s_{\rm eff}$ is adopted:
     $$s_{\rm eff} = \text{const} . \tag{23}$$
   - In contrast, slipping in *limited* information about a targeted hadron state opens the possibility [29] to work out the *expected* $s_{\rm eff}$ dependence on the Borel parameters τ:
     $$s_{\rm eff} = s_{\rm eff}(\tau) . \tag{24}$$

The roadmap for the construction of QCD sum rules sketched above has originally been drafted for analyses of *conventional* hadrons. Its unreflected application (in unchanged form) also to multiquark states seems, in view of the far-reaching discrepancies between the exotic and the conventional categories of hadrons, to be either too optimistic or a little bit too naïve. Rather, one should be open for (potentially favorable) modifications of the customary QCD sum-rule approach, modifications that might be capable of improving the achieved accuracy of the predictions of QCD sum rules for the class of multiquark exotic hadrons. In particular, upon performing necessary evaluations of correlation functions at the QCD level one might find it advantageous to take into account the QCD contributions' feature of being tetraquark-phile, in Definition 2 implied to be desirable and by Proposition 1 given its precise meaning, or not. With respect to the power corrections, in any QCD sum-rule derivation indispensable for its QCD-level evaluation, the problem of whether a given nonperturbative vacuum-condensate contribution is tetraquark-phile or not may be analyzed along the lines indicated in Section 3 (as has been demonstrated at the example of definitely flavor-exotic tetraquarks [23,30,31]).

Targeting *definitely flavor-exotic tetraquarks* (12), the versions of correlation functions (7) indicated in Definition 4 have to be discriminated and hence subjected to separate treatment.

- In the flavor-preserving case, one has to start from the four-point correlation functions

$$\left\langle \mathrm{T}\Big(j_{\bar{a}b}(y)\, j_{\bar{c}d}(y')\, j^\dagger_{\bar{a}b}(x)\, j^\dagger_{\bar{c}d}(x')\Big)\right\rangle , \qquad \left\langle \mathrm{T}\Big(j_{\bar{a}d}(y)\, j_{\bar{c}b}(y')\, j^\dagger_{\bar{a}d}(x)\, j^\dagger_{\bar{c}b}(x')\Big)\right\rangle . \tag{25}$$

Applying the *traditional* QCD sum-rule manipulations to twofold contractions (8) of the correlation functions (25) yields as an outcome of this enterprise a relationship, depicted in Figure 5, that incorporates a (vast) multitude of QCD-level and hadron-level quantities.

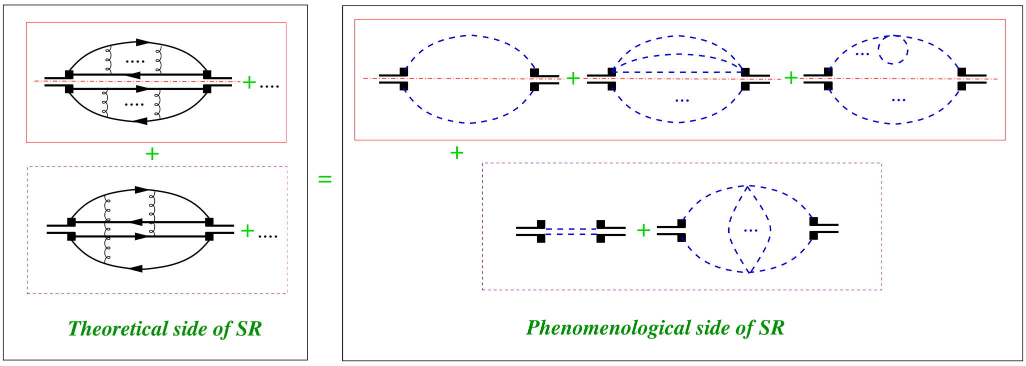

Figure 5. Aggregation of a pair of *unconnected* conventional-meson QCD sum rules of the kind recalled by Figure 6 (top row, separated by a red dot-dashed line) and (bottom row) the *tetraquark-adequate* QCD sum rule of generic structure as in Figure 7, potentially supporting tetraquark intermediate-state poles: outcome of the uncritical evaluation of correlation functions (25) still awaiting its disentanglement [30,32].

However, a more in-depth analysis [30] reveals that, already on diagrammatic grounds, this conglomerate decomposes, in fact, into two QCD sum rules for *conventional* mesons (Figure 6) and one further QCD sum rule that, potentially, supports the development of *tetraquark poles* and rightly deserves the label of being "tetraquark-adequate" (Figure 7). In the course of its QCD-level evaluation, this latter QCD sum rule receives, exclusively, tetraquark-phile contributions, in the sense of Proposition 1; all the perturbative among these enter in form of dispersion integrals of tetraquark-adequate spectral densities, ρ_{p}. An analogous reflection for single contractions (9) of the correlation functions (25) leads to similar QCD sum-rule findings, all perturbative tetraquark-phile QCD contributions being encoded, in dispersive formulation, in tetraquark-adequate spectral densities Δ_{p}.

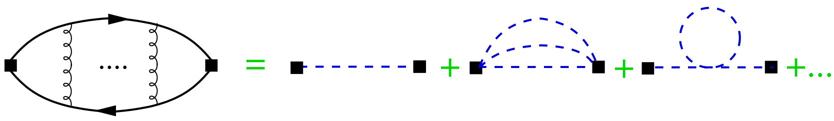

Figure 6. Schematical composition of QCD sum rules for conventional mesons (blue dashed lines) [30].

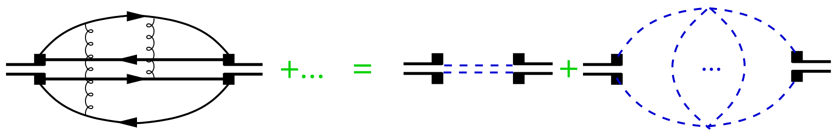

Figure 7. Schematical composition of a tetraquark-adequate QCD sum rule of flavor-preserving type: tetraquark-phile contributions at QCD level, at hadron level counterbalanced by non-separable meson contributions (blue dashed lines), and perhaps those of tetraquark poles (blue dashed double line) [30].

- In the flavor-rearranging case, one has to deal with the four-point correlation function

$$\left\langle \mathrm{T}\Big(j_{\bar{a}b}(y)\, j_{\bar{c}d}(y')\, j^\dagger_{\bar{a}d}(x)\, j^\dagger_{\bar{c}b}(x')\Big)\right\rangle . \tag{26}$$

Here, irrespective of (ultimately necessary) spatial contractions (8) and (9) of four-point correlation functions (7), the analysis is unfortunately not thus straightforward as in the flavor-preserving case: Within QCD-level evaluation, all *tetraquark-phile* contributions (defined by requiring them to satisfy the constraint formulated in Proposition 1) may be identified, case by case, by inspection of the solutions of the relevant Landau equations. Within hadron-level evaluation, that QCD-level characteristic of being tetraquark-phile or not is mirrored by the ability of any contributions at the hadron level to accommodate, in their s channel, two-meson intermediate states or not, in addition to a possible presence of tetraquark intermediate-state poles [31]. Hardly surprisingly, these insights translate the outcome of the QCD sum-rule formalism based on the correlation function (26) into a quark–hadron relation of (expected) two-component structure symbolically shown in Figure 8. All *perturbative* tetraquark-phile QCD-level contributions find their way into a tetraquark-adequate QCD sum rule arising from a *precursor* as in Figure 8b by spectral densities $\rho_{\rm r}$ in the double-contractions case (8) and $\Delta_{\rm r}$ in the single-contraction case (9).

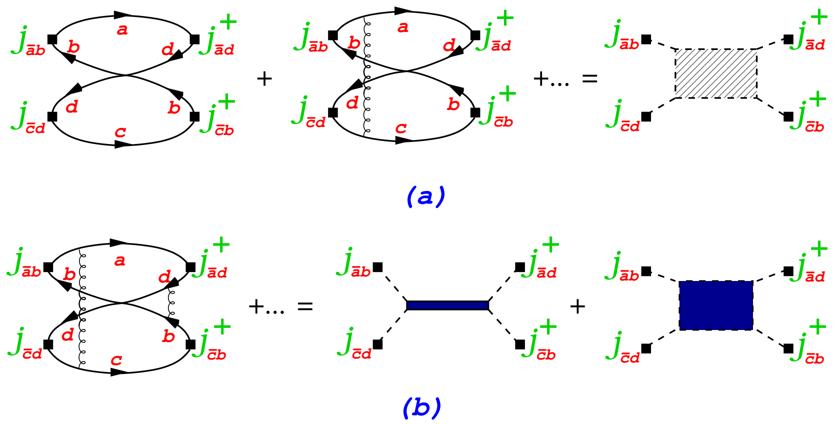

Figure 8. Outcome of application of established QCD sum-rule techniques to correlation functions (26), consisting of two uncorrelated quark–hadron relationships: (**a**) one equating the non-tetraquark-phile QCD contributions with hadron contributions not involving any two-meson s-channel cuts (subsumed by hatched rectangle); (**b**) the precursor of a tetraquark-adequate QCD sum rule, involving two-meson s-channel cuts (subsumed by a filled rectangle) and *maybe* tetraquark poles (blue horizontal bar) too [31].

For a definitely flavor-exotic tetraquark (12), the properties of foremost interest are mass M,

- decay constants $f_{\bar a b \bar c d}$ and $f_{\bar a d \bar c b}$, arising from the vacuum–tetraquark matrix elements of the two distinct operators (6) interpolating any definitely flavor-exotic tetraquark (12),

$$f_{\bar a b \bar c d} \equiv \langle 0|\theta_{\bar a b \bar c d}|T\rangle\,, \qquad f_{\bar a d \bar c b} \equiv \langle 0|\theta_{\bar a d \bar c b}|T\rangle\,; \tag{27}$$

- momentum-space amplitudes $A(T \to j_{\bar a b}\, j_{\bar c d})$ and $A(T \to j_{\bar a d}\, j_{\bar c b})$, Fourier-transformed vacuum–tetraquark matrix elements of appropriate pairs of quark bilinear currents (5),

$$\begin{aligned} \langle 0|\mathrm{T}[j_{\bar a b}(y)\, j_{\bar c d}(y')]|T\rangle &\xrightarrow[\text{transformation}]{\text{Fourier}} A(T \to j_{\bar a b}\, j_{\bar c d})\,,\\ \langle 0|\mathrm{T}[j_{\bar a d}(y)\, j_{\bar c b}(y')]|T\rangle &\xrightarrow[\text{transformation}]{\text{Fourier}} A(T \to j_{\bar a d}\, j_{\bar c b})\,. \end{aligned} \tag{28}$$

In terms of these hadronic properties, all effective-threshold improved multiquark-adequate QCD sum rules resulting from (once or twice) contracted four-point correlation functions (7) assume, for the example of definitely flavor-exotic tetraquarks, *symbolically* the form [30,31]

$$(f_{\bar{a}b\bar{c}d})^2 \exp(-M^2\,\tau) = \int\limits_{\hat{s}}^{s_{\rm eff}(\tau)} {\rm d}s\,\exp(-s\,\tau)\,\rho_{\rm p}(s) + \text{Borel-transformed power corrections}\,, \tag{29}$$

$$f_{\bar{a}b\bar{c}d}\,A(T\to j_{\bar{a}b}\,j_{\bar{c}d})\exp(-M^2\,\tau) = \int\limits_{\hat{s}}^{s_{\rm eff}(\tau)} {\rm d}s\,\exp(-s\,\tau)\,\Delta_{\rm p}(s) + \text{Borel-transformed power corrections}\,, \tag{30}$$

$$f_{\bar{a}b\bar{c}d}\,f_{\bar{a}d\bar{c}b}\exp(-M^2\,\tau) = \int\limits_{\hat{s}}^{s_{\rm eff}(\tau)} {\rm d}s\,\exp(-s\,\tau)\,\rho_{\rm r}(s) + \text{Borel-transformed power corrections}\,, \tag{31}$$

$$f_{\bar{a}d\bar{c}b}\,A(T\to j_{\bar{a}b}\,j_{\bar{c}d})\exp(-M^2\,\tau) = \int\limits_{\hat{s}}^{s_{\rm eff}(\tau)} {\rm d}s\,\exp(-s\,\tau)\,\Delta_{\rm r}(s) + \text{Borel-transformed power corrections}\,. \tag{32}$$

The general lesson to be learned from the above for both *perturbative and nonperturbative* QCD contributions to QCD sum-rule approaches applied to *any* type of multiquark hadrons: paying attention to deploy exclusively spectral densities and power corrections computed in a multiquark-phile manner should avoid or, at least, diminish the "contamination" of inferred QCD sum-rule predictions by input not related at all to the multiquark hadrons under study.

6. Summary, Conclusion, and Outlook—Multiquark-Instigated Theoretical Adaptations

The multiquark states among the conceivable exotic hadrons feature a characteristic not shared by any conventional hadrons, namely, *cluster reducibility* [2], that is to say, their ability to fragment into *color-singlet* bound states of lesser numbers of constituents, eventually into a set of conventional hadrons. A promising implication for various theoretical approaches to multiquarks is the advantage gained by pertinent modification of one's favored formalism.

Here, such improvements have been illustrated for the set of flavor-exotic tetraquarks. An analogous contemplation can be (and has been) performed for the class of flavor-cryptoexotic tetraquarks [16–20,24]. It goes without saying that there one gets confronted with additional complications: the potential mixing of these tetraquark states with conventional mesons that carry precisely the quantum numbers of those tetraquarks. Mutatis mutandis, these findings should be straightforwardly transferable to any other multiquark states, such as the likewise established [3] pentaquark baryons. The numerical impact of proposed changes may only be quantified by confronting (definite) multiquark predictions with experimental counterparts. All ideas did attract interest of *tetraquark and pentaquark* QCD sum-rule practitioners [36–42].

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Abbreviations

The following abbreviations are used in this manuscript:

LHCb	Large Hadron Collider beauty
OPE	operator product expansion
QCD	quantum chromodynamics

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