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# Optical Soliton Perturbation with Parabolic Law Nonlinearity

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**Abstract:** This paper recovers a broad spectrum of optical solitons for the perturbed nonlinear Schrödinger's equation having a dual-power law of nonlinearity. The perturbation terms are from inter-modal dispersion and self-frequency shift. The integration scheme is the improved extended tanh function approach. The parameter constraints that naturally emerge are also enumerated.

**Keywords:** solitons; parabolic; constraints



**Citation:** Arnous, A.H.; Samir, I.; Biswas, A.; González-Gaxiola, O.; Moraru, L.; Iticescu, C.; Moldovanu, S.; Alghamdi, A.A. Optical Soliton Perturbation with Parabolic Law Nonlinearity. *Universe* **2023**, *9*, 155. https://doi.org/10.3390/universe9030155

Academic Editor: Lorenzo Iorio

Received: 7 December 2022
Revised: 16 March 2023
Accepted: 17 March 2023
Published: 21 March 2023



## 1. Introduction

Optical solitons are a research trove in the field of Quantum Optics. One of its everlasting areas is the study of soliton dynamics through optical fibers and metamaterials, as well as other forms of waveguide [1–10]. The standard governing model is the familiar nonlinear Schrödinger's equation (NLSE) that comes with various forms of self-phase modulation (SPM) structures. In addition, several forms of perturbation terms are typically taken into account that are of Hamiltonian as well as non-Hamiltonian type, thus making the model integrable or non-integrable, accordingly. The model that will be addressed in the paper today is the NLSE with the parabolic law of nonlinearity that carries a couple of Hamiltonian perturbations, and thus, the model is rendered integrable. The perturbation terms stem from intermodal dispersion and self-frequency shift. The integration tool is the improved extended tanh function approach. This would lead to the retrieval of a full spectrum of 1-soliton solutions, namely, the bright, dark, and singular solitons. The results are exhibited after a quick intro to the model and a succinct re-visitation to the integration algorithm. Also described are the restrictions for the occurrence of such soliton solutions.

*Governing Model*

The governing model is NLSE with the parabolic law of SPM, with intermodal dispersion and self-frequency shift,

$$i\psi_t = \beta\psi_{xx} + \sigma|\psi|^2\psi + \epsilon|\psi|^4\psi + i\left[\lambda\psi_x + \mu|\psi|^2\psi_x\right]. \tag{1}$$

In Equation (1), $\beta$ is the chromatic dispersion (CD), while SPM comes from $\sigma$ and $\epsilon$. Moreover, $x$ and $t$, which represent spatial and temporal coordinates, respectively, are the independent variables, whereas $\psi$ is the dependent variable. The perturbation terms stem from $\lambda$ and $\mu$ that emerge from intermodal dispersion and self-frequency shift, respectively. Finally, $i = \sqrt{-1}$.

In recent times, significant works have been published, in which the NLSE is also examined with cubic-quintic nonlinearities, with some even applying fractional calculus. See [11–15] and references therein.

## 2. Improved Extended Tanh-Function Approach (Succinct Recapitulation)

Suppose that we have the following nonlinear evolution equation:

$$F(u, u_t, u_x, u_{xx}, u_{xt}, \dots) = 0, \tag{2}$$

Here, $u = u(x,t)$ is an unknowable function, $F$ is a polynomial in the variable $u$ and its partial derivatives $u_t$, $u_x$ with respect to $t, x$, in which the highest order derivatives and nonlinear terms are included.
Step 1 : Use the traveling wave transformation

$$u(x,t) = U(\xi), \;\; \xi = k(x - vt), \tag{3}$$

where $k, v$ are constants that will be determined later. Then, Equation (2) is now a nonlinear ordinary differential equation of the type

$$P\Big(U, -kvU', kU', k^2U'', \dots\Big) = 0, \tag{4}$$

where $P$ is a polynomial in $U(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.
Step 2 : We assume that the solution of Equation (2) can be expressed in the form

$$U(\xi) = \sum_{i=0}^{N} \alpha_i \Phi^i + \sum_{i=1}^{N} \beta_i \Phi^{-i}, \tag{5}$$

where $\Phi$ satisfies

$$\Phi' = \varepsilon\sqrt{a_0 + a_1\Phi + a_2\Phi^2 + a_3\Phi^3 + a_4\Phi^4}, \tag{6}$$

where $\varepsilon = \pm 1$. This equation gives various kinds of fundamental solutions [16].
Step 3 : Calculate the positive integer $N$ in Equation (5) by balancing the order of the highest derivative and the largest power of the nonlinear component in Equation (2).
Step 4 : Substitute (5) into (4), along with (6). As a result of this substitution, we obtain a polynomial of $\Phi$. By collecting all terms of the same power in this polynomial and equating them to zero, we have an overdetermined system of algebraic equations that may be solved by Maple or Mathematica to obtain the unknown parameters $k$, $v$, $\alpha_0$, $\alpha_i$, and $\beta_i (i = 1, 2, \dots)$. Consequently, we obtain the exact solutions of (2).

## 3. Application to the Model

The application of the improved extended tanh-function approach to perturbed NLSE with the parabolic law of nonlinearity structure is hypothesized with the solution structure:

$$\psi(x,t) = U(\xi)e^{i(-\kappa x + \omega t + \phi(\xi))}, \tag{7}$$

where the wave variable $\xi$ is given by

$$\xi = k(x - vt). \tag{8}$$

The soliton speed is given by $v$, while $\omega$ and $\kappa$ represent the frequency shift and propagation constant, respectively; they are real parameters.

Plugging (1) into Equation (1) and separating the imaginary and real components

$$\begin{aligned}&\beta k^2 U'' - \beta k^2 U \phi'^2 - k(-2\beta\omega + \lambda + v)U\phi' + (-k\mu\phi' + \mu\omega + \sigma)U^3 \\ &+(\kappa + \lambda\omega - \beta\omega^2)U + \epsilon U^5 = 0,\end{aligned} \tag{9}$$

and

$$2\beta k^2 \phi' U' + \beta k^2 U \phi'' + kU'(-2\beta\omega + \lambda + v) + 3k\mu U^2 U' = 0. \tag{10}$$

Equation (9) may be integrated after being multiplied by $U$ to arrive at

$$\phi' = -\frac{G}{\beta k^2 U^2} + \frac{2\beta\omega - \lambda - v}{2\beta k} - \frac{3\mu U^2}{4\beta k} \tag{11}$$

where $G$ is an integration constant. Now, substituting Equation (11) in Equation (10) results in

$$\begin{aligned}&-16G^2 + 4kU^4\left(k\left(4\beta\kappa + \lambda^2 + v^2 - 4\beta v\omega + 2\lambda v\right) - 2G\mu\right) + 16\beta^2 k^4 U^3 U'' + \\ &8k^2 U^6 (2\beta\sigma + \lambda\mu + \mu v) + k^2 U^8 \left(16\beta\epsilon + 3\mu^2\right) = 0.\end{aligned} \tag{12}$$

Using the transformation $U^2 = V$, Equation (12) can be written as

$$g_1 V^4 + g_2 V^3 + g_3 V^2 + g_4 V + g_5 + V'^2 = 0, \tag{13}$$

where

$$\begin{cases} g_1 = \frac{16\beta\epsilon + 3\mu^2}{12\beta^2 k^2}, \\ g_2 = \frac{2\beta\sigma + \mu(\lambda + v)}{\beta^2 k^2}, \\ g_3 = \frac{k\left(4\beta(\kappa - v\omega) + (\lambda + v)^2\right) - 2G\mu}{\beta^2 k^3}, \\ g_4 = 4F, \quad g_5 = \frac{4G^2}{\beta^2 k^4}, \end{cases} \tag{14}$$

where $F$ is the constant of integration. These newly introduced parameters $g_j$ for $1 \leq j \leq 5$ reduce the ODE to a compact form, as visible in (13). The only purpose of the paper is to address the integrability of the model to locate its soliton solutions using the algorithm that is adopted.

Balancing $V'^2$ with $V^4$ in Equation (13) gives $N = 1$ . Consequently, we reach

$$V(\xi) = \alpha_0 + \alpha_1 \Phi(\xi) + \frac{\beta_1}{\Phi(\xi)}. \tag{15}$$

Substituting Equation (15) with the set of Equation (7), collecting all the terms of the form $\Phi$ together, and equating each coefficient to zero yields a set of algebraic equations which can be solved using some software such as Maple or Mathematica to obtain:

**Result (1):** If we set $a_0 = a_1 = a_3 = 0$, we obtain

$$\begin{aligned}&\alpha_0 = -\frac{\Delta_1 + g_3}{2g_2}, \quad \beta_1 = 0, \quad g_1 = \frac{g_2(g_3 - \Delta_1)}{4g_4}, \\ &g_5 = \frac{-2\Delta_1 g_3^2 + 5\Delta_1 g_2 g_4 - 2g_3^3 + 7g_2 g_4 g_3}{16 g_2^2}, \\ &a_4 = \frac{\alpha_1^2 g_2(\Delta_1 - g_3)}{4g_4}, \quad a_2 = \frac{1}{4}(3\Delta_1 - g_3), \quad \Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}.\end{aligned} \tag{16}$$

Then, the solution corresponding to (1) are bright soliton solutions with $3\Delta_1 - g_3 > 0$.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} + \sqrt{\frac{g_4(g_3 - 3\Delta_1)}{g_2(\Delta_1 - g_3)}} \operatorname{sech}\left[\frac{1}{2}\sqrt{3\Delta_1 - g_3}(x - vt)\right] \right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{17}$$

**Result (2):**
**Case (1):** If we set $a_1 = a_3 = 0$ and $a_0 = \frac{a_2^2}{4a_4}$, we obtain
(i)

$$\alpha_0 = -\frac{\Delta_1 + g_3}{2g_2}, \quad \beta_1 = 0, \quad g_1 = \frac{g_2(g_3 - \Delta_1)}{4g_4}, \quad g_5 = \frac{g_4(g_3 - \Delta_1)}{4g_2},$$
$$a_4 = \frac{\alpha_1^2 g_2(\Delta_1 - g_3)}{4g_4}, \quad a_2 = \frac{1}{4}(3\Delta_1 - g_3). \tag{18}$$

Then, the solution corresponding to (1) are bright soliton solutions with $3\Delta_1 - g_3 < 0$.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} \pm \sqrt{-\frac{g_4(3\Delta_1 - g_3)}{2g_2(\Delta_1 - g_3)}} \tanh\left[\frac{1}{2}\sqrt{-\frac{1}{2}(3\Delta_1 - g_3)}(x - vt)\right] \right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{19}$$

(ii)

$$\alpha_0 = -\frac{\Delta_1 + g_3}{2g_2}, \quad \alpha_1 = 0, \quad g_1 = \frac{g_2(g_3 - \Delta_1)}{4g_4}, \quad g_5 = \frac{g_4(g_3 - \Delta_1)}{4g_2},$$
$$a_4 = \frac{-2\Delta_1 g_3^2 + 9\Delta_1 g_2 g_4 - 2g_3^3 + 3g_2 g_4 g_3}{16\beta_1^2 g_2^2}, \quad a_2 = \frac{1}{4}(3\Delta_1 - g_3),$$
$$\Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}. \tag{20}$$

Then, the solution corresponding to (1) are singular soliton solutions, with $3\Delta_1 - g_3 > 0$.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} \pm \frac{1}{g_2}\sqrt{\frac{2\Delta_1 g_3^2 - 9\Delta_1 g_2 g_4 + 2g_3^3 - 3g_2 g_4 g_3}{2(3\Delta_1 - g_3)}} \times \coth\left[\frac{1}{2}\sqrt{-\frac{1}{2}(3\Delta_1 - g_3)}(x - vt)\right] \right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{21}$$

**Case (2):** If we set $a_1 = a_3 = 0$ and $a_0 = \frac{a_2^2 m^2 (1 - m^2)}{a_4 (2m^2 - 1)^2}$, we obtain

$$\alpha_0 = -\frac{\Delta_1 + g_3}{2g_2}, \quad \alpha_1 = \sqrt{-\frac{2a_4(\Delta_1 + g_3)}{g_2^2}}, \quad g_1 = \frac{g_2(g_3 - \Delta_1)}{4g_4}, \quad a_2 = \frac{1}{4}(3\Delta_1 - g_3),$$
$$\Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}, \quad \beta_1 = 0,$$
$$g_5 = \frac{1}{16g_2^2(1 - 2m^2)^2} \times \Big( -2\Delta_1 g_3^2 \Big(8m^4 - 8m^2 + 1\Big) + \Delta_1 g_2 g_4 \Big(56m^4 - 56m^2 + 5\Big) - 2g_3^3 \Big(8m^4 - 8m^2 + 1\Big) + g_2 g_4 g_3 \Big(40m^4 - 40m^2 + 7\Big)\Big). \tag{22}$$

Then, the corresponding solution of (1) is Jacobi's elliptic cosine function.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} + \frac{1}{g_2}\sqrt{\frac{2a_2 m^2(\Delta_1 + g_3)}{(2m^2-1)}}\mathrm{cn}\left[\frac{1}{2}\sqrt{\frac{3\Delta_1 - g_3}{(2m^2-1)}}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{23}$$

We achieve a bright soliton solution with $3\Delta_1 - g_3 > 0$ if we select $m = 1$.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} + \frac{\sqrt{2a_2(\Delta_1 + g_3)}}{g_2}\mathrm{sech}\left[\frac{\sqrt{3\Delta_1 - g_3}}{2}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{24}$$

**Case (3):** If we set $a_1 = a_3 = 0$ and $a_0 = \frac{a_2^2(1-m^2)}{a_4(2-m^2)^2}$, we obtain

$$\alpha_0 = \frac{\Delta_1 - g_3}{2g_2}, \quad \alpha_1 = \sqrt{\frac{2a_4(\Delta_1 - g_3)}{g_2^2}}, \quad \beta_1 = 0, \quad g_1 = \frac{g_2(\Delta_1 + g_3)}{4g_4}, \quad a_2 = \frac{1}{4}(-3\Delta_1 - g_3)$$
$$\Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}, \quad g_5 = \frac{2\Delta_1 g_3^2 m^4 - 2g_3^3 m^4 + \Delta_1 g_2 g_4\left(-5m^4 - 16m^2 + 16\right) + g_2 g_3 g_4\left(7m^4 - 16m^2 + 16\right)}{16g_2^2(m^2-2)^2}. \tag{25}$$

Then, the solution corresponding to (1) is Jacobi's elliptic function of the third kind.

$$\psi(x,t) = \left\{ \frac{\Delta_1 - g_3}{2g_2} + \frac{1}{g_2}\sqrt{-\frac{2a_2 m^2(\Delta_1 - g_3)}{(2-m^2)}}\mathrm{dn}\left[\frac{1}{2}\sqrt{\frac{-3\Delta_1 - g_3}{(2-m^2)}}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{26}$$

We achieve a bright soliton solution with $3\Delta_1 + g_3 < 0$ if we select $m = 1$.

$$\psi(x,t) = \left\{ \frac{\Delta_1 - g_3}{2g_2} + \frac{1}{g_2}\sqrt{2a_2(g_3 - \Delta_1)}\mathrm{sech}\left[\frac{\sqrt{-(3\Delta_1 + g_3)}}{2}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{27}$$

**Case (4):** If we set $a_1 = a_3 = 0$ and $a_0 = \frac{a_2^2 m^2}{a_4(m^2+1)^2}$, we obtain

(i)

$$\alpha_0 = -\frac{\Delta_1 + g_3}{2g_2}, \quad \alpha_1 = \sqrt{-\frac{2a_4(\Delta_1 + g_3)}{g_2^2}}, \quad \beta_1 = 0, \quad g_1 = \frac{g_2(g_3 - \Delta_1)}{4g_4},$$
$$a_2 = \frac{1}{4}(3\Delta_1 - g_3), \quad \Delta_1 = \sqrt{g_3^2 - 2g_2 g_4},$$
$$g_5 = \frac{-2\Delta_1 g_3^2\left(m^2-1\right)^2 - 2g_3^3\left(m^2-1\right)^2 + \Delta_1 g_2 g_4\left(5m^4 - 26m^2 + 5\right) + g_2 g_4 g_3\left(7m^4 + 2m^2 + 7\right)}{16g_2^2(m^2+1)^2}. \tag{28}$$

Then, the solutions corresponding to (1) are Jacobi's elliptic sine function.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} \pm \frac{1}{g_2}\sqrt{\frac{2a_2 m^2(\Delta_1 + g_3)}{(m^2+1)}}\mathrm{sn}\left[\frac{1}{2}\sqrt{\frac{g_3 - 3\Delta_1}{(m^2+1)}}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{29}$$

We achieve a dark soliton solution with $3\Delta_1 - g_3 < 0$ if we select $m = 1$.

$$\psi(x,t) = \left\{ -\frac{\Delta_1 + g_3}{2g_2} \pm \frac{1}{g_2}\sqrt{2a_2(\Delta_1 + g_3)}\tanh\left[\frac{1}{2}\sqrt{\frac{g_3 - 3\Delta_1}{2}}(x - vt)\right]\right\}^{\frac{1}{2}} \times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{30}$$

(ii)

$$\alpha_0 = \tfrac{\Delta_1 - g_3}{2g_2}, \quad \alpha_1 = 0, \quad \beta_1 = \sqrt{\tfrac{m^2\left(2\Delta_1 g_3^2 - 9\Delta_1 g_2 g_4 - 2g_3^3 + 3g_2 g_4 g_3\right)}{4a_4 g_2^2 (m^2+1)^2}}, \quad g_1 = \tfrac{g_2(\Delta_1 + g_3)}{4g_4},$$
$$a_2 = \tfrac{1}{4}(-3\Delta - g_3), \quad \Delta_1 = \sqrt{g_3^2 - 2g_2 g_4},$$
$$g_5 = \tfrac{2\Delta_1 g_3^2\left(m^2-1\right)^2 - 2g_3^3\left(m^2-1\right)^2 + \Delta g_2 g_4\left(-5m^4+26m^2-5\right) + g_2 g_4 g_3\left(7m^4+2m^2+7\right)}{16g_2^2(m^2+1)^2}. \tag{31}$$

Then, the solution corresponding to (1) is

$$\psi(x,t) = \left\{ \frac{\Delta_1 - g_3}{2g_2} \pm \frac{1}{2g_2}\sqrt{\frac{2\Delta_1 g_3^2 - 9\Delta_1 g_2 g_4 - 2g_3^3 + 3g_2 g_4 g_3}{a_2(m^2+1)}} \times \mathrm{ns}\left[\frac{1}{2}\sqrt{\frac{3\Delta_1 + g_3}{(m^2+1)}}(x - vt)\right]\right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{32}$$

We achieve a singular soliton solution with $3\Delta_1 + g_3 > 0$ if we select $m = 1$.

$$\psi(x,t) = \left\{ \frac{\Delta_1 - g_3}{2g_2} \pm \frac{1}{2g_2}\sqrt{\frac{2\Delta_1 g_3^2 - 9\Delta_1 g_2 g_4 - 2g_3^3 + 3g_2 g_4 g_3}{2a_2}} \times \coth\left[\frac{1}{2}\sqrt{\frac{3\Delta_1 + g_3}{2}}(x - vt)\right]\right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{33}$$

Jacobi elliptic functions are indicated by the symbols $\mathrm{sn}(\xi|m)$, $\mathrm{cn}(\xi|m)$, and $\mathrm{dn}(\xi|m)$, where $m$ is the elliptic modulus.
The elliptic Jacobian functions transform into hyperbolic ones when $m \to 1$:

$$\mathrm{cn}(\xi|1) \to \operatorname{sech}\xi \qquad \mathrm{sn}(\xi|1) \to \tanh\xi \qquad \mathrm{dn}(\xi|1) \to \operatorname{sech}\xi$$

The elliptic Jacobian functions transform into trigonometric ones when $m \to 1$:

$$\mathrm{cn}(\xi|0) \to \cos\xi \qquad \mathrm{sn}(\xi|0) \to \sin\xi \qquad \mathrm{dn}(\xi|0) \to 1$$

**Result (3):** If we set $a_2 = a_4 = 0$, we obtain

$$\alpha_0 = \frac{-3g_2 \pm \Delta_2}{12g_1}, \quad \alpha_1 = 0, \quad \beta_1 = \frac{3a_1(-3g_2 \pm \Delta_2)}{3g_2(-3g_2 \pm \Delta_2) + 24g_1 g_3},$$
$$g_5 = \frac{1}{6}\left(-3\alpha_0^3 g_2 - 5\alpha_0^2 g_3 - 6\alpha_0 g_4\right), \quad a_3 = \frac{-3\alpha_0^2 g_2 - 4\alpha_0 g_3 - 3g_4}{3\beta_1},$$
$$a_0 = \frac{\beta_1^2(3\alpha_0 g_2 + g_3)}{6\alpha_0^2}, \quad \Delta_2 = \sqrt{3\left(3g_2^2 - 8g_1 g_3\right)}. \tag{34}$$

Then, the solution corresponding to (1) is Weierstrass's elliptic function.

$$\psi(x,t) = \left\{ \frac{-3g_2 \pm \Delta_2}{12g_1} + \frac{3a_1(-3g_2 \pm \Delta_2)}{3g_2(-3g_2 \pm \Delta_2) + 24g_1g_3} \left( \frac{1}{\wp\left[\frac{\sqrt{a_3}}{2}(x-vt), l_2, l_3\right]} \right) \right\}^{\frac{1}{2}}$$
$$\times e^{i(\kappa x - \omega t + \phi(k(x-vt)))}, \tag{35}$$

where $l_2 = -4a_1/a_3$, and $l_3 = -4a_0/a_3$ are the Weierstrass's function invariants.

$$\begin{aligned}
\wp\left[\frac{\sqrt{a_3}}{2}(x-vt), l_2, l_3\right] = \quad & \sigma_3 + \frac{\sigma_1 - \sigma_3}{\mathrm{sn}^2\left[\sqrt{\frac{1}{2}a_3(\sigma_1-\sigma_3)}(x-vt)\right]} \\
&= \sigma_2 + (\sigma_1 - \sigma_3)\frac{\mathrm{dn}^2\left[\sqrt{\frac{1}{2}a_3(\sigma_1-\sigma_3)}(x-vt)\right]}{\mathrm{sn}^2\left[\sqrt{\frac{1}{2}a_3(\sigma_1-\sigma_3)}(x-vt)\right]} \\
&= \sigma_1 + (\sigma_1 - \sigma_3)\frac{\mathrm{cn}^2\left[\sqrt{\frac{1}{2}a_3(\sigma_1-\sigma_3)}(x-vt)\right]}{\mathrm{sn}^2\left[\sqrt{\frac{1}{2}a_3(\sigma_1-\sigma_3)}(x-vt)\right]},
\end{aligned} \tag{36}$$

where $\sigma_1 = \wp\left[\frac{l_2}{2}\right]$, $\sigma_2 = \wp\left[\frac{l_3}{2}\right]$, $\sigma_3 = \wp\left[\frac{l_2+l_3}{2}\right]$, and $\sigma_1 \neq \sigma_3$. We achieve a soliton solution with $a_3(\sigma_1 - \sigma_3) > 0$ if we select $a_1 = a_0 \neq 0$, which leads to $\sigma_1 = \sigma_2$. In this case, we have

$$\begin{aligned}
\wp\left[\frac{\sqrt{a_3}}{2}(x-vt), l_2, l_3\right] = \quad & \sigma_1 \coth^2\left[\sqrt{\frac{a_3(\sigma_1-\sigma_3)}{2}}(x-vt)\right] - \\
& \sigma_3 \,\mathrm{csch}^2\left[\sqrt{\frac{a_3(\sigma_1-\sigma_3)}{2}}(x-vt)\right].
\end{aligned} \tag{37}$$

**Result (4):** If we set $a_1 = a_2 = a_0 = 0$, we obtain

$$\begin{aligned}
&\alpha_0 = -\frac{\Delta_3 + 2g_3}{3g_2}, \quad \alpha_1 = \frac{9a_3g_4}{2\Delta_3g_3 - 4g_3^2 + 9g_2g_4}, \quad \beta_1 = 0, \quad \Delta_3 = \sqrt{4g_3^2 - 9g_2g_4}, \\
&g_1 = \frac{4\Delta_3g_3^2 - 9\Delta_3g_2g_4 - 8g_3^3 + 27g_2g_4g_3}{54g_4^2}, \quad g_5 = \frac{-4\Delta_3g_3^2 + 9\Delta_3g_2g_4 - 8g_3^3 + 27g_2g_4g_3}{54g_2^2}, \\
&a_4 = \frac{\alpha_1^2(-4\Delta_3g_3^2 + 9\Delta_3g_2g_4 + 8g_3^3 - 27g_2g_4g_3)}{54g_4^2}.
\end{aligned} \tag{38}$$

Then, the solution corresponding to (1) is

$$\psi(x,t) = \left\{ -\frac{\Delta_3 + 2g_3}{3g_2} + \frac{9a_3^2g_4}{2a_4(2\Delta_3g_3 - 4g_3^2 + 9g_2g_4)} \exp\left[\pm\frac{a_3}{2\sqrt{-a_4}}(x-vt)\right] \right\}^{\frac{1}{2}}$$
$$\times e^{i(\kappa x - \omega t + \phi(k(x-vt)))}. \tag{39}$$

**Result (5):**

**Case (1):** If we set $a_3 = a_4 = 0$ and $a_0 = \frac{a_1^2}{4a_2}$, we obtain

$$\alpha_0 = \frac{\Delta_1 + \sqrt{2\left(-\Delta_1 g_3 + g_3^2 - 3g_2 g_4\right)} - g_3}{2g_2}, \quad \alpha_1 = 0, \quad \Delta_1 = \sqrt{g_3^2 - 2g_2 g_4},$$
$$\beta_1 = -\frac{\sqrt{2}a_1 \Delta_1 g_4}{\sqrt{-\Delta_1 g_3 + g_3^2 - 3g_2 g_4}\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)} \quad a_2 = \frac{\Delta_1 g_3 + 3g_3^2 - 6g_2 g_4}{2\Delta_1},$$
$$g_1 = \frac{g_2\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)}{4g_4\sqrt{g_3^2 - 2g_2 g_4}}, \quad g_5 = \frac{g_4\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)}{4g_2\sqrt{g_3^2 - 2g_2 g_4}}. \tag{40}$$

Then, the solution corresponding to (1) is

$$\psi(x,t) = \left\{ \frac{\Delta_1 + \sqrt{2\left(-\Delta_1 g_3 + g_3^2 - 3g_2 g_4\right)} - g_3}{2g_2} - \frac{\sqrt{2}a_1 \Delta_1 g_4}{\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)\sqrt{-\Delta_1 g_3 + g_3^2 - 3g_2 g_4}} \right.$$
$$\left. \times \left( \frac{1}{-\frac{a_1}{2a_2} + \exp\left[\pm\sqrt{\frac{\Delta_1 g_3 + 3g_3^2 - 6g_2 g_4}{2\Delta_1}}(x - vt)\right]} \right) \right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{41}$$

**Case (2):** If we set $a_1 = a_3 = a_4 = 0$, we obtain

$$\alpha_0 = \frac{\Delta_1 - g_3}{2g_2}, \quad \alpha_1 = 0, \quad \beta_1 = \pm\sqrt{\frac{4a_0 g_4}{\Delta_1 g_2 + g_3 g_2}}, \quad \Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}, \quad g_1 = \frac{g_2(\Delta_1 + g_3)}{4g_4},$$
$$g_5 = \frac{2\Delta_1 g_3^2 - 5\Delta_1 g_2 g_4 - 2g_3^3 + 7g_2 g_4 g_3}{16g_2^2}, \quad a_2 = \frac{1}{4}(-3\Delta_1 - g_3). \tag{42}$$

Then, the solution corresponding to (1) is

$$\psi(x,t) = \left\{ \frac{\sqrt{g_3^2 - 2g_2 g_4} - g_3}{2g_2} + \sqrt{\frac{g_4(3\Delta_1 + g_3)}{\Delta_1 g_2 + g_3 g_2}} \operatorname{csch}\left[\sqrt{\frac{1}{4}(3\Delta_1 + g_3)}(x - vt)\right] \right\}^{\frac{1}{2}}$$
$$\times e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{43}$$

**Result (6):**

**Case (1):** If we set $a_0 = a_1 = 0$, we obtain

$$\alpha_0 = \frac{2a_2 - \left(4g_3 \pm \sqrt{(4g_3 - 2a_2)^2 - 36g_2 g_4}\right)}{6g_2}, \quad \alpha_1 = \frac{a_3 \alpha_0^2}{2\alpha_0^2 g_2 + 2\alpha_0 g_3 + g_4}, \quad \beta_1 = 0,$$
$$g_5 = \frac{1}{4}\left(\alpha_0^3(-g_2) - 2\alpha_0^2 g_3 - 3\alpha_0 g_4\right), \quad a_4 = \frac{\alpha_1^2\left(3\alpha_0^2 g_2 + 2\alpha_0 g_3 + g_4\right)}{4\alpha_0^3}. \tag{44}$$

Then, the solution corresponding to (1) is a singular soliton with $a_2 > 0$

$$\psi(x,t) = \left\{ \frac{2a_2 - \left(4g_3 \pm \sqrt{(4g_3 - 2a_2)^2 - 36g_2 g_4}\right)}{6g_2} \mp \frac{a_3 \alpha_0\left(3\alpha_0^2 g_2 + 4\alpha_0 g_3 + 3g_4\right)}{2\left(2\alpha_0^2 g_2 + 2\alpha_0 g_3 + g_4\right)} \right.$$
$$\left. \times \frac{1}{\sqrt{a_2 a_4}} \operatorname{csch}[\sqrt{a_2}(x - vt)] \right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{45}$$

**Case (2):** If we set $a_0 = a_1 = 0$ and $a_2 = \frac{a_3^2}{4a_4}$, we obtain

$$\alpha_0 = \frac{\Delta_1 + \sqrt{2\left(-\Delta_1 g_3 + g_3^2 - 3g_2 g_4\right)} - g_3}{2g_2}, \quad \alpha_1 = \pm\sqrt{-\frac{4\Delta_1 a_4 g_4}{g_2\left(\Delta g_3 + g_3^2 - 2g_2 g_4\right)}}, \quad \beta_1 = 0,$$
$$\Delta_1 = \sqrt{g_3^2 - 2g_2 g_4}, \quad g_1 = \frac{g_2\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)}{4\Delta_1 g_4}, \quad g_5 = \frac{g_4\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)}{4\Delta_1 g_2},$$
$$a_3 = -\frac{\alpha_1\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)\sqrt{-\Delta_1 g_3 + g_3^2 - 3g_2 g_4}}{\sqrt{2}\Delta_1 g_4}. \tag{46}$$

Then, the solution corresponding to (1) are dark soliton solutions with $a_2 > 0$.

$$\psi(x,t) = \left\{\frac{\Delta_1 + \sqrt{2\left(-\Delta_1 g_3 + g_3^2 - 3g_2 g_4\right)} - g_3}{2g_2} \pm \sqrt{-\frac{a_2\Delta_1 g_4}{g_2\left(\Delta_1 g_3 + g_3^2 - 2g_2 g_4\right)}}\right.$$
$$\left.\times\left(1 + \tanh\left[\frac{1}{2}\sqrt{a_2}(x - vt)\right]\right)\right\}^{\frac{1}{2}} e^{i(\kappa x - \omega t + \phi(k(x - vt)))}. \tag{47}$$

## 4. Conclusions

The paper recovers a full spectrum of perturbed 1-soliton solutions to the NLSE with Hamiltonian perturbation terms and the parabolic law of SPM. The solitons appear with parameter constraints that are also listed. Additional solutions that are in terms of singular periodic functions and Jacobi's elliptic functions have additionally emerged from the integration scheme, namely, the extended tanh function scheme. These results are now ready to be explored, further along. Additional integration schemes would lead to a soliton solution when the perturbation term is considered with maximum intensity or with full nonlinearity. This would be with the application of the semi-inverse variational principle. Moreover, the soliton parameter dynamics can also be obtained with the application of the variational principle, moment method, or the collective variables approach.

The integration approach implemented in this paper can be expanded to address the fourth-order nonlinear Schrödinger's equation as well. Recently, nonlocal integrable nonlinear Schrödinger's equations are presented via the group reductions of matrix spectral problems [17]. The multidimensional version of the model that is studied in the paper can be applied to DWDM topology or to the case of twisted fibers where matrix version of the model appears. This would be another rich avenue to venture into with such an integrability approach in multi-dimensions, and to recover novel results that would be applicable to such optoelectronic devices.

Other avenues of expansion with this model would be to look at the variation of parameters for the corresponding chirped solitons that can be recovered with the usage of the variational principle, moment method, soliton perturbation theory, collective variables approach, and several others. These would lead to the variation of the soliton parameters, including the phase constant, which cannot be recovered using the commonly studied approaches such as soliton perturbation theory. However, the soliton perturbation theory would lead to the effect of optical soliton cooling, an important feature that is needed for soliton transmission across intercontinental distances.

Apart from these, one is slated to have a look at the supercontinuum generation for the model. One of the most important perturbation terms, namely the intrapulse Raman scattering, was tacitly omitted in the current work since the main focus of the paper is the integrability of the model. This being a non-Hamiltonian perturbation term, would lead to the perturbed model being non-integrable. However, the inclusion of this effect would give a broader perspective to the model, such as the computation of the soliton frequency downshift.

While the perturbed version of the model is considered in this paper, it must be borne in mind that the perturbation terms are strong. If however, these would have been weak perturbation terms, then the integrability would lead to quasi-stationary solitons, rather than exact soliton solutions, in which case the Raman scattering effect could be included. These quasi-stationary solitons are recoverable with the usage of a multiple-scales perturbation procedure, in which case non-Hamiltonian perturbation terms can be rendered to be integrable as well.

Apart from the deterministic perturbation terms, it is of paramount importance to take into effect the perturbation terms that are of stochastic type. After all, the effect of randomness is always present in soliton propagation dynamics. Therefore, it is absolutely necessary to address these effects with the inclusion of a random perturbation term, in both additive and multiplicative formats. For the multiplicative perturbation term, the white noise effect can be addressed with the aid of Ito Calculus. On the other hand, the additive stochastic perturbation term can be handled with the formulation of the Langevin equation, which would lead to the mean free velocity of the soliton. This is therefore an open project that is on the table to be taken up.

On another note, it is important to check out the numerical studies, in addition to the plethora of analytical approaches that are enumerated. The model and its similar counterparts are to be addressed numerically using several forms of computational approaches. A few such approaches are the improved Adomian decomposition scheme, the Laplace-Adomian decomposition scheme, the variational iteration method, the finite differences approach, the finite element method, the boundary element method, and many other similar approaches. These would express the soliton dynamics numerically with a visual perspective.

Finally, experimental approaches are on the table too. It is imperative to take a look at the soliton perturbations from an oscilloscope. The eye diagrams would give a rounded study to the model that would be the final touch at the lab before the rubber meets the road! Thus, the current paper is just a drop in the ocean, and consequently, a lot of work lies ahead of us. The results are currently awaited and will be disseminated sooner rather than later.

**Author Contributions:** Writing—original draft preparation, A.H.A.; Methodology, I.S.; Conceptualization, A.B.; Writing—review and editing, Software, O.G.-G.; Project administration, Investigation, L.M.; Investigation, C.I.; Validation, S.M.; Data Curation, A.A.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** This work was supported by the project "DINAMIC", Contract no. 12PFE/2021.162. The authors are extremely thankful for this.

**Conflicts of Interest:** The authors declare no conflict of interest.

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